Dissipation Normal Form, Conservativity, Instability and Chaotic Behavior of Continuous-time Strictly Causal Systems

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Abstract: - The paper deals with structural properties of a class of strictly causal systems. It is shown that a special physically correct internal structure of a given system representation caled dissipation normal form can be derived as a natural consequence of strict causality, dissipativity, minimality and asymptotic stability requirements. A proper generalization of classic Tellegen's theorem together with a concept of bi-orthonormal basis of the state velocity space have been used as basic ingrediences expressing the signal energy conservation law for abstract system state space representations. It is demonstrated by examples that in continuous-time version the resulting structure represent a unifying tool for analysis and synthesis of a relatively general class of linear as well as nonlinear causal systems.

Key-Words: energy-metric function, bi-orthonormal basis, dissipation normal form, Tellegen's principle, non-linear phenomena, instability, chaotic behavior

1 Introduction

Almost in any field of science and technology some sort of stability problem can appear. Instability and chaos are certainly the most important phenomena which should be treated before any other aspect of reality will be attacked. Hence it is not very surprising that a broad variety of approaches to the problem of stability, instability and analysis of chaotic phenomena exists. Many of the most popular techniques in the field of stability and chaos are in a certain sense related to the work of A.M.Liapunov. For instance the well known Lyapunov exponents in chaos theory or Lyapunov functions in stability theory [1, 2] can be mentioned as typical examples which seem to be *energy oriented*.

Tellegen's theorem is one of the well known and appropriate forms of energy conservation statement in the field of electrical engineering [3, 4]. The most important feature of Tellegen's approach is the fact that the energy conservation principle holds without any regard to physical nature of constituent network elements.This is the key idea of the proposed approach to problems of stability, dissipativity and chaos.

2 Abstract form of energy conservation

Certainly, any realizable system has to fulfill some *causality* and *energy conservation* requirements. Recall that *existence of an abstract state space representation is necessary* for a system to be *causal*. On the other hand causality does not imply energy conservation. In the field of electrical engineering *Kirchhoff's laws are necessary and sufficient for physical correctness of* any electrical network from energy conservation point of view. Tellegen's theorem, which is known to be one of the most powerful tools *of system analysis and synthesis* in electrical network theory, can be seen as a very elegant abstract form of *energy conservation principle* for a class *physically correct* system state space representations, in which voltages and currents have been chosen as state variables.

Let us briefly summarize the *essential features* of the *original version* of Tellegen's theorem [4]. Assume that an *arbitrary connected electrical network* of b components is given. Let us *disregard* the specific nature of the *network components* and represent the *network structure* by an *oriented graph* with n *vertices and* b *branches*. Let the *set of Kirchhoff law constraints* be given in a form

$$Ai = 0 \qquad Bv = 0 \tag{1}$$

where A is a *node incidence matrix*, B is *loop incidence matrix*, and vectors i and v are defined

$$i = [i_1, i_2, \dots, i_n]^T \quad v = [v_1, v_2, \dots v_n]^T \tag{2}$$

Let J be the set of all vectors i and V be the set of all vectors v such that i and v satisfy (1). Both the vectors of currents and voltages are elements of a b-dimensional *vector space with the inner product.* Then the Tellegen's principle follows from:

Theorem 1. (Classical Tellegen's theorem - CTT)

If $i \in J$ and $v \in V$ then it holds

$$\forall t: \quad \langle i(t), v(t) \rangle = 0 \tag{3}$$

That is to say J and V are *orthogonal subspaces* of the Euclidean space E_b. Furthermore J and V together *span* the vector space E_b.

It is obvious fact, following directly from the *definition of inner product*, that relation (3) *is just a form of constant energy statement* for a class of representations in which *elements* of a *set* of *voltages and currents* have been *chosen* as *state variables*, as well as components of a *gradient vector of a scalar field in the state space.*

Let $\Re\{S\}$ is a *continuous-time* finite dimensional time-invariant strictly causal nonlinear system state space representation given by:

$$\begin{aligned} \Re\{S\}: \ & \dot{x}(t) = f[x(t)] + B\,u(t), \qquad x(t_0) = x^0, \\ & y(t) = C\,x(t), \end{aligned} \tag{4}$$

The *arbitrariness* in the choice of state coordinates motivates introducing a *group of state- and feedback-transformations* on which the *generalization of classical Tellegen's principle* has been proposed in [4].

$$\begin{aligned} & \exists \varphi, \ \exists T, \ T^{-1}: \ \bar{x} = T(x), \ \ \bar{u} = \varphi(u, \bar{x}): \\ & \left\langle f, (grad \quad E)^T \right\rangle = 0 \ \Leftrightarrow \ \left\langle \dot{\bar{x}}, \bar{x} \right\rangle = 0 \\ & \Leftrightarrow \forall t: \ \bar{E}[\bar{x}(t)] = E[x(t)] \end{aligned} \tag{5}$$

For a class of discrete-time finite dimensional *internal system representations* $\Re\{S\}$ given by

$$\begin{aligned} & x(k+1) = f\left[x(k)\right] + w(k), \\ & w(k) = B\,u(k), \quad y(k) = C x(k) \end{aligned} \tag{6}$$

Similarly as in the case of continuous-time systems, a new *discrete-time generalization of Tellegen's principle* has been introduced in [4]. If any input *u(k)* and any state value *x(k)* will be chosen then the next state value *x(k+1)* is given, and the *state difference vector Δx(k)* can be defined as

$$\Delta x(k) = x(k+1) - x(k) \equiv \Delta x_k, \quad k \in \{0,1,2,..\} \tag{7}$$

together with a row "*gradient vector" η(k)* defined by:

$$\eta(\mathrm{k}) = \frac{1}{2}[x(k+1) + x(k)]^T \equiv \eta_k, \ k \in \{\ 0,1,2,\dots\ \} \tag{8}$$

Interpretation of the vector η_k as a natural *discrete-time energy function gradient vector* is obvious, and the *discrete-time generalization of Tellegen's principle* is then given by the *inner product*:

$$\forall t \equiv k, k \in \{0,1,2,\dots\}: \ \left\langle \Delta x_k \quad , \quad \eta_k{}^T \right\rangle = 0 \tag{9}$$

For deeper understanding a *geometric interpretation* of the *generalized Tellegen's principle* is visualized at the Fig.1. with continuous-time version as a limit of the discrete-time case.

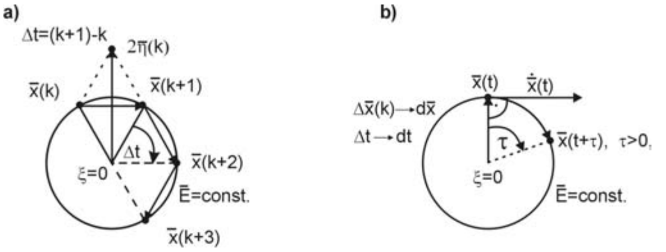

Fig.1. Geometric interpretation of the generalized Tellegen's principle a) discrete-time b) continuous-time, (for n=2)

3 Dissipativity and Stability

Let us consider the class of continuous-time nonlinear time-varying strictly causal systems given by the state space representation

$$R(S): \quad \dot{x}(t) = f[t; x(t), u(t)] \tag{10}$$

$$y(t) = h[t; x(t)] \tag{11}$$

with t as continuous time variable,

$x_1, x_2, \dots, x_n$ as the state space coordinates,

$\dot{x}_1, \dot{x}_2, \dots, \dot{x}_n$ as coordinates of the state velocity,

$u_1, u_2, \dots, u_r$ as the input signals, and with

$y_1, y_2, \dots, y_p$ as the observed output signals

Recall that according to Liouville's theorem of vector analysis, *dissipative systems* have the important property that any volume of the state space strictly decreases under the action of the system flow. For nonlinear system representations $\Re\{S\}$ with the state velocity given by a nonlinear vector field f the property of *dissipativity* is defined by using the *operation of divergence* as follows [2].

Definition 1: *(Dissipativity of a vector field)*

The representation $\Re\{S\}$ with the state velocity vector field f is *dissipative* if it holds

$$\operatorname{div} f(x) \ = \ \sum_{i=1}^{n} \frac{\partial f_i(x)}{\partial x_i} < 0 \tag{12}$$

Let us now define a *constituent set* of finite number of *non-interacting elementary subsystems*

$$\begin{array}{ll} \dot{x}_1 = f_1(t, x_1, u_1), & y_1 = h_1(t, x_1) \\ \dot{x}_2 = f_2(t, x_2, u_2), & y_2 = h_2(t, x_2) \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ \dot{x}_n = f_n(t, x_n, u_n), & y_n = h_n(t, x_n) \end{array} \tag{13}$$

It follows that the *constituent set* (29) is *dissipative* if *at least one* of the elementary subsystems is dissipative.

Remark 1: It is easy to deduce that the constituent set of non-interacting subsystems with zero input and with unique equilibrium state is *locally asymptotic stable iff each of the elementary subsystems is dissipative.* It means that in general *dissipativity is necessary but not sufficient for asymptotic stability.*

Remark 2 : Nonlinear systems having at an equilibrium state a dissipative approximate linearization are *locally dissipative* in a neighborhood of this equilibrium state, but need not to be *globally dissipative*, i.e. their *region of dissipation* need not be the whole state space.

Remark 3: Recall that systems with

$$\operatorname{div} f(x) = 0 \tag{14}$$

preserve volume along state trajectories; such systems are usually referred to as *conservative.*

Notice that this concept of conservativity is *not always compatible* with the classical meaning of the term *conservative as energy preserving* (or Hamiltonian).

Remark 4: Notice that a *linear time invariant system*

$$\begin{aligned} \Re\{S\}:\ & \dot{x}(t) = Ax(t) + Bu(t),\ x(t_0) = x^0, \\ & y(t) = Cx(t), \end{aligned} \tag{15}$$

is dissipative if and only if its *matrix A has negative trace*, i.e. if it holds

$$Tr\,A < 0 \tag{16}$$

Thus an *asymptotically stable* linear system is *always dissipative,* while the *converse is not true in general*.

4 Minimality of state velocity space

It is challenging to find such a *structure of interactions* between the elements of the *constituent set* that the *intrinsic relations* between fundamental system properties such as dissipativity, conservativity, asymptotic stability, instability, state and parameter minimality and chaoticity will be clearly displayed. In order to achieve the aim, it is reasonable to specify the *minimal dimension of the state velocity space.* We start with a concept of the Hessenberg matrix.

Definition 2: *(Hessenberg structure of a matrix)*

Let A is a n-th order rectangular matrix. We say that the matrix A has the *Hessenberg structure* if it holds

$$1^o \qquad a_{i,j} = 0, \qquad j > i+1 \tag{17}$$

$$2^o \quad a_{i,i+1} \neq 0, \text{ and } \quad sign(a_{i,i+1}) = 1 \tag{18}$$

Definition 3: *(Hessenberg structure of a vector field)*

A vector field f has the *Hessenberg structure* if it holds

$$1^o \quad \frac{\partial f_i}{\partial x_j} = 0, \qquad j > i+1 \tag{19}$$

$$2^o \quad \frac{\partial f_i}{\partial x_{i+1}} \neq 0, \qquad sign\left(\frac{\partial f_i}{\partial x_{i+1}}\right) = 1 \tag{20}$$

Let a n-th order system representation is given

$$\begin{aligned} \Re\{S\}:\ & \dot{x}(t) = f[x(t)] + B\,u(t), \qquad x(t_0) = x^0, \\ & y(t) = C\,x(t), \end{aligned} \tag{21}$$

and the matrices B and C have the form

$$C = [\ c_1, 0, \dots, 0\ ],\ \ B^T = [\ 0, 0, \dots,\ b_n\ ] \tag{22}$$

Definition 4: *(Hessenberg structure of a system)*

We say that a *system representation* (15),(16) has the *Generalized Hessenberg structure if* vector field f has the *Hessenberg structure*

$$1^o \quad \frac{\partial f_i}{\partial x_j} = 0, \qquad j > i+1 \tag{23}$$

$$2^o \quad \frac{\partial f_i}{\partial x_{i+1}} \neq 0, \qquad sign\left(\frac{\partial f_i}{\partial x_{i+1}}\right) = 1 \tag{24}$$

and in addition if it holds

$$3^o \quad c_1 \triangleq \frac{\partial h_1}{\partial x_1} \neq 0, \qquad sign\left(\frac{\partial h_1}{\partial x_1}\right) = 1 \tag{25}$$

$$4^o \quad b_n \triangleq \frac{\partial f_n}{\partial u_n} \neq 0, \qquad sign\left(\frac{\partial f_n}{\partial u_n}\right) = 1 \tag{26}$$

Remark 5: It is worthwhile to notice that each of the Jacobian matrices $J_x(f)$, $J_u(f)$, $J_x(h)$ has a *properly defined structure* motivated by the *system structure* corresponding to *the cascade connection of the elementary subsystems* according to the Fig .2.

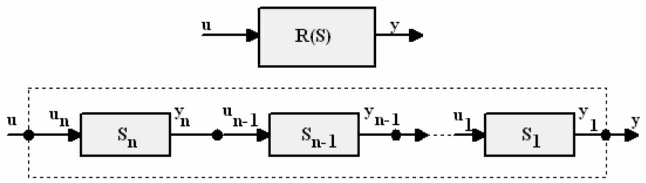

Fig.2. Generalized Hessenberg structure

For the *internal structure of subsystems* S_k see Fig.3.

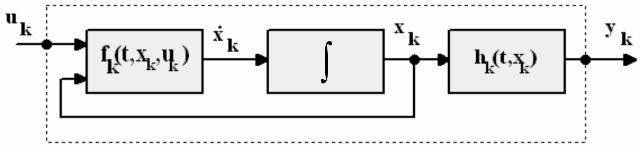

Fig.3. Internal structure of the elementary subsystem S_k

The resulting system representation in *Generalized Hessenberg structure* is obviously *always controllable and observable,* i.e. *minimal* and is explicitly described by

$$\begin{aligned} \dot{x}_1 &= f_1(\,t\,,\,x_1\,,\,x_2\,)\,, \\ \dot{x}_2 &= f_2(\,t\,,\,x_2\,,\,x_3\,), \\ \dot{x}_3 &= f_3(\,t\,,\,x_3\,,\,x_4), \\ & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \dot{x}_{n-1} &= f_{n-1}(\,t\,,\,x_{n-1}\,,\,x_n), \\ \dot{x}_n &= f_n(\,t\,,\,x_n\,)\ +\ u \end{aligned} \tag{27}$$

$$y(t) = h[t; x(t)] = x_1(t) \tag{28}$$

where the *set of external interactions* is given by

$$u(t) = u_n(t), \qquad y(t) = x_1(t) \tag{29}$$

and the *set of internal interactions* is expressed by

$$u_i = x_{i+1}\,,\ i=1,2,...,n-1 \tag{30}$$

$$y_i = x_i\,,\quad i=1,2,...,n \tag{31}$$

5 Bi-orthonormal basis of velocity space

In order to specify the *physically correct internal system structure* in the sense of *energy conservation principle validity* [4], we introduce a *structural representation*

$$\Re^*\{S\}: \quad Q.\dot{x}(t) = A^*.x(t) + B^*.u(t) \qquad (32)$$
$$y(t) = C^*.x(t)$$

Let us assume that *each elementary subsystem* S_k of the constituent set *is dissipative, i.e. it holds*

$$\forall i: \quad \frac{\partial f_i}{\partial x_i} < 0, \qquad i = 1,2,.....n \qquad (33)$$

Then the *simpliest form of the structural matrix* A^* *in the Generalized Hessenberg representation* reads

$$A^* = \begin{bmatrix} -1 & 1 & 0 & \ldots & 0 & 0 & \\ 0 & -1 & 1 & \ldots & 0 & 0 & 0 \\ 0 & 0 & -1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & ::: & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & -1 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 & -1 \end{bmatrix} \qquad (34)$$

Now, let the *structural matrices* Q, B^*, C^* be given by

$$Q = \begin{bmatrix} 1 & 0 & & 0 & 0 \\ -1 & 1 & & 0 & 0 \\ 1 & -1 & & . & . \\ -1 & 1 & \ldots & . & . \\ . & -1 & . & 0 & 0 \\ . & . & . & 1 & 0 \\ . & . & . & -1 & 1 \end{bmatrix}, B^* = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ 1 \end{bmatrix}, (C^*)^T = \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \\ 0 \end{bmatrix} \qquad (35)$$

where the columns $\boldsymbol{q_1}, \boldsymbol{q_2}, \ldots, \boldsymbol{q_n}$ of the matrix Q form a *biorthonormal basis* in the *state velocity space* given by

$$\boldsymbol{q_k} + \boldsymbol{q_{k+1}} = \boldsymbol{e_k}, \; k = 1,2, \ldots, n-1, \; \boldsymbol{q_n} = \boldsymbol{e_n} \qquad (36)$$

Because Q is *always invertible*, we have

$$Q^{-1} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & . & \ldots & 1 & 0 \\ 0 & 0 & . & \ldots & 1 & 1 \end{bmatrix} \qquad (37)$$

and a resulting *generic structure of the matrix A* follows

$$A = Q^{-1}A^* = \begin{bmatrix} -1 & 1 & 0 & 0 & \ldots & 0 & 0 \\ -1 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & -1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & ::: & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & 0 & \ldots & -1 & 0 \end{bmatrix} \qquad (38)$$

6 Structure of continuous-time systems in dissipation normal form

Our goal is to specify a class of strictly causal system representations for which a form of energy conservation such as the The Generalized Tellegen's principle holds. We start with the hypothesis that *it is not the physical energy by itself*, but only a *measure of distance from the system equilibrium to the actual state x(t)*, what is needed for this aim. Thus, instead of the physical energy a *metric* ρ[x(t), x*] will be defined in a proper way, and for an *abstract energy* E(x) we then put formally:

$$\mathrm{E(x)} \triangleq \frac{1}{2}\rho^2\left[\mathrm{x(t)},\ \mathrm{x}^*\right] = \frac{1}{2} \| x(t) - x^* \|^2 \qquad (39)$$

It has been shown in [1], [3], that the resulting *state equivalent system representation in dissipation normal form, corresponding to the derived generic structure* (35), (38) is described by a triple of matrices $\{ \widetilde{A}, \widetilde{B}, \widetilde{C} \}$ as follows

$$\tilde{A} = \begin{pmatrix} -\alpha_1 & \alpha_2 & 0 & 0 & \cdots & 0 & 0 \\ -\alpha_2 & 0, & \alpha_3 & 0 & \cdots & 0 & 0 \\ 0, & -\alpha_3 & 0 & \alpha_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots -\alpha_{n-1} & 0 & \alpha_n \\ 0 & 0 & 0 & 0 & \cdots 0 & -\alpha_n & 0 \end{pmatrix} \qquad (40)$$

$$\tilde{C}^T = \begin{bmatrix} \gamma \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix} \qquad (41)$$

It is easy to show that the set of *real basic design parameters* α_i, γ, β_i must satisfy the following fundamental *consistency conditions:*

1. $\forall i, \quad i \in \{1,2,...,n\} : 0 < \alpha_i < \infty \Leftrightarrow$

   for structural asymptotic stability

2. $\forall i, i \in \{2,3,...,n\} : 0 \neq \alpha_i, \gamma \neq 0, \exists i : \beta_i \neq 0 \Leftrightarrow$

   for structural minimality

The *generic internal structure* of an n-th order continuous-time strictly causal system in *dissipation normal form* is shown at the Fig. 4.

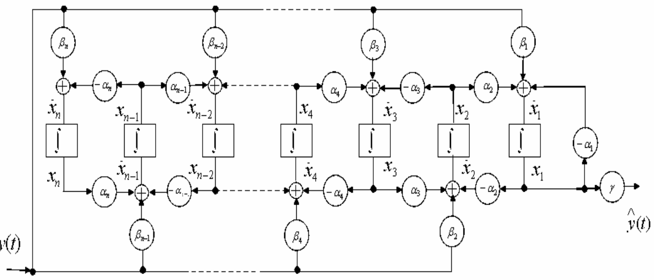

Fig. 4. Internal structure of continuous-time strictly causal system in the dissipation normal form

7 Dissipativity and stability analysis

Example 1. (Stability analysis of a linear system)
Let the n-th order system representation is given by the linear differential equation with constant coefficients

$$y^{(6)} + a_1 y^{(5)} + \ldots + a_4 \ddot{y}(t) + a_5 \dot{y}(t) + a_6 y(t) = 0 \qquad (41)$$

with characteristic polynomial

$$P(s) = s^6 + a_1 s^5 + a_2 s^4 + \ldots + a_4 s^2 + a_5 s + a_6$$

and with *matrix A in the dissipation normal form*

$$A = \begin{bmatrix} -\alpha_1 & \alpha_2 & 0 & 0 & 0 & 0 \\ -\alpha_2 & 0 & \alpha_3 & 0 & 0 & 0 \\ 0 & -\alpha_3 & 0 & \alpha_4 & 0 & 0 \\ 0 & 0 & -\alpha_4 & 0 & \alpha_5 & 0 \\ 0 & 0 & 0 & -\alpha_5 & 0 & \alpha_6 \\ 0 & 0 & 0 & 0 & -\alpha_6 & 0 \end{bmatrix} \qquad (42)$$

Hence the parameters a_i, $i \in \{1, 2, \ldots, 6\}$ are given by

$$a_1 = \alpha_1$$
$$a_2 = \alpha_2^2 + \alpha_3^2 + \alpha_4^2 + \alpha_5^2 + \alpha_6^2$$
$$a_3 = \alpha_1(\alpha_3^2 + \alpha_4^2 + \alpha_5^2 + \alpha_6^2)$$
$$a_4 = \alpha_2^2(\alpha_4^2 + \alpha_5^2 + \alpha_6^2) + \alpha_3^2(\alpha_5^2 + \alpha_6^2) + \alpha_4^2\alpha_6^2$$
$$a_5 = \alpha_1\alpha_3^2(\alpha_5^2 + \alpha_6^2) + \alpha_1\alpha_4^2\alpha_6^2$$
$$a_6 = \alpha_2^2\alpha_4^2\alpha_6^2$$

Recall that the *necessary and sufficient condition for existence of the unique equilibrium state* $x^* = 0$ is

$$\det A = a_6 = \alpha_2^2\alpha_4^2\alpha_6^2 \neq 0 \qquad (43)$$

From the existence of a unique equilibrium state point of view, the *dissipation parameter* α_1, as well as *interaction parameters* α_3, α_5 can be chosen *arbitrary.*
Now, let all the *parameters* $a_1, a_2, \ldots, a_n$ of $P_n(s)$ be considered as *unknown,* and let us specify the *region of asymptotic stability in the parameter space.*
The *representation in dissipation normal form* reads

$$\begin{aligned} \Re(S):\quad & \dot{x}_1(t) = -\alpha_1 x_1(t) + \alpha_2 x_2(t) \\ & \dot{x}_2(t) = -\alpha_2 x_1(t) + \alpha_3 x_3(t) \\ & \dot{x}_3(t) = -\alpha_3 x_2(t) + \alpha_4 x_4(t) \\ & \dot{x}_4(t) = -\alpha_4 x_3(t) + \alpha_5 x_5(t) \\ & \dot{x}_5(t) = -\alpha_5 x_4(t) + \alpha_6 x_6(t) \\ & \dot{x}_6(t) = -\alpha_6 x_5(t) \\ & y(t) = \gamma x_1(t) \end{aligned} \qquad (44)$$

and for the Eucleidian metric $\rho = \rho_2$

$$E[x(t)] = \tfrac{1}{2}\rho^2[x(t), 0] = \tfrac{1}{2}\|x(t)\|^2 = \tfrac{1}{2}\sum_{i=1}^{n} x_i^2(t), \qquad (45)$$

it holds

1^o $E(x) = 0 \Leftrightarrow x(t) = x^*$, $(x^* = 0)$

2^o $x_i(t) \in R \Leftrightarrow x_i^2(t) \geq 0 \Rightarrow E(x) > 0 \Leftrightarrow x(t) \neq x^*$

For the *derivative of the signal energy function E(x) along the system representation (44)* we get

$$\left.\frac{\mathrm{d}E(t)}{\mathrm{d}t}\right|_{\Re\{s\}} = -\alpha_1 x_1^2(t) = -\frac{\alpha_1}{\gamma^2} \cdot y^2(t) \qquad (46)$$

where γ is a *real output scaling parameter*

$$0 < \gamma < \infty \qquad (47)$$

Thus, for *non-zero output dissipation power* $y^2(t)$ the *signal energy conservation principle* holds if and only if: $P(t) = y^2(t) \Leftrightarrow \alpha_1 = \gamma^2 > 0$ (48)

Remark 2: Notice that the *dissipation parameter* α_1 is *the only element* of the matrix *A,* which *sign separates* the *system dissipativity* from its *anti-dissipativity.* The *critical value* of $\alpha_1 = 0$, corresponds to the *system conservativity* and *separates stability* of the *equilibrium state* from its *anti-stability*.

Remark 3: Notice, that if we put $\alpha_5 = 0$, then the state variables x_i, $i = 5,6$ become *unobservable by the output y* ; thus *only the first isolated subsystem* with the state variables x_i, $i = 1,2,3,4,$ which is *observable, will be asymptotic stable*, while the *second one will oscilate on the constant energy level*, *(see Fig.3.c for energy evolution)*. Similarly, if we put $\alpha_3 = 0$, then the state variables x_i, $i = 3,4,5,6$ become *unobservable by the output y, and only the observable subsystem*

$$\begin{aligned} \dot{x}_1(t) &= -\alpha_1 x_1(t) + \alpha_2 x_2(t) \\ \dot{x}_2(t) &= -\alpha_2 x_2(t) \\ y(t) &= \gamma x_1(t) \end{aligned} \qquad (49)$$

will be asymptotic stable (see Fig.3b)

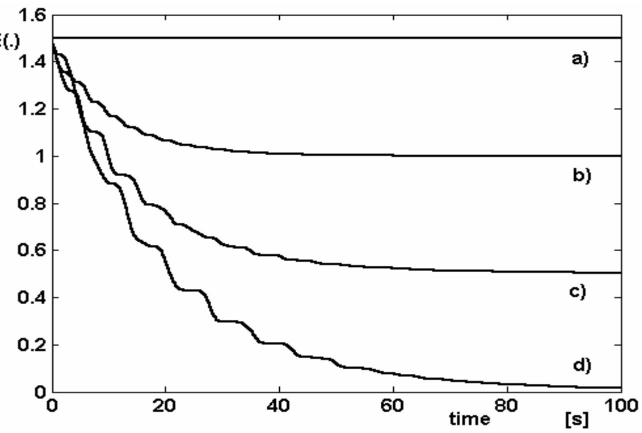

Fig. 5. Time evolution of the signal energy *E[x(t)]*

a) conservative case $\alpha_1 = 0$, α_k -arbitrary for k = 2,3,…,n
b) stability $\alpha_1 > 0$, $\alpha_3 = 0$, c) stability $\alpha_1 > 0$, $\alpha_5 = 0$,
d) asymptotic stability $\alpha_1 > 0$, $\alpha_k \neq 0$, for k = 2,3,…,n

8 Dissipativity and nonlinear phenomena

Example 2. (Non-linear stability analysis)
Let us consider a simple *non-linear system given* by

$$\ddot{y}(t) + \varepsilon\left[\alpha - \beta y^2(t)\right]\dot{y}(t) + a_2 y(t) = 0 \qquad (50)$$

If C is defined by $C = [\gamma, 0]$, and $A(x)$ is defined by the *non-linear dissipation normal form*

$$A(x_1, x_2) = \begin{bmatrix} -\varepsilon\left[\alpha - \frac{1}{3}\beta x_1^2\right], & \sqrt{a_2} \\ -\sqrt{a_2}, & 0 \end{bmatrix} \qquad (51)$$

then the system representation is *locally observable if*

$$\gamma \neq 0, \ a_2 > 0 \qquad (52)$$

and the *signal energy conservation principle* gives

$$\left.\frac{\mathrm{d}E(t)}{\mathrm{d}t}\right|_{\Re(s)} = -P \leq 0, \ P = \varepsilon\left[\alpha - \tfrac{1}{3}\beta x_1^2\right]x_1^2 \qquad (53)$$

It follows that the *unique equilibrium state* $x^* = 0$ is *asymptotically stable in the region* $D \subset X \subset R^2$

$$D = \left\{ x_1, x_2 : |x_1| < \sqrt{\frac{3\alpha}{\beta}} \text{ and } x_1^2 + x_2^2 < \frac{3\alpha}{\beta} \right\} \qquad (54)$$

if $\varepsilon > 0, \ \alpha > 0, \ \beta > 0, \ a_2 > 0$.

Example 3. (Generation of Lyapunov functions)
Let the same *non-linear system* be *given*

$$\ddot{y}(t) + \varepsilon\left[\alpha - \beta y^2(t)\right]\dot{y}(t) + a_2 y(t) = 0 \qquad (55)$$

but instead of the *dissipation normal form the* state vector $x(t)$ is defined by

$$x_1 = y, x_2 = dy/dt \qquad (56)$$

Then the corresponding system representation is *structurally observable* with the *observability matrix* $H_o = I$, and from the *signal energy conservation principle*

$$\left.\frac{\mathrm{d}V(t)}{\mathrm{d}t}\right|_{\Re(s)} = -P \leq 0, \ P = \varepsilon\left[\alpha - \tfrac{1}{3}\beta x_1^2\right]x_1^2 \qquad (57)$$

a *unique Lyapunov function* $V(x)$ can be determined by *isometric transformations.*For $\alpha = \beta = a_2 = 1$ we get

$$V(x) = \tfrac{1}{2}\Big[\tfrac{1}{9}\varepsilon^2 x_1^6 - \tfrac{2}{3}\varepsilon^2 x_1^4 + (1+\varepsilon^2)x_1^2 - \\ - \tfrac{2}{3}\varepsilon x_1^3 x_2 + 2\varepsilon x_1 x_2 + x_2^2\Big] \qquad (58)$$

and for *linear conservative case* ($\varepsilon = 0$) it reduces to

$$V(x) = \tfrac{1}{2}\left(x_1^2 + x_2^2\right) \qquad (59)$$

Example 4. (Generation of chaos in a causal system)
Let a 4th order system represented by *dissipation normal form with chaotic state* be given by

$$\begin{aligned} \dot{x}_1 &= -\alpha_1 x_1 + \alpha_2 x_2 \\ \dot{x}_2 &= -\alpha_2 x_1 + \alpha_3 x_3 \\ \dot{x}_3 &= -\alpha_3 x_2 + \alpha_4 x_4 \\ \dot{x}_4 &= -\alpha_4 x_3 \end{aligned} \quad \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} -1+10x_2^2 \\ 1 \\ 1 \\ 2.00 \end{bmatrix}, x(0) = \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ 0 \end{bmatrix} \qquad (60)$$

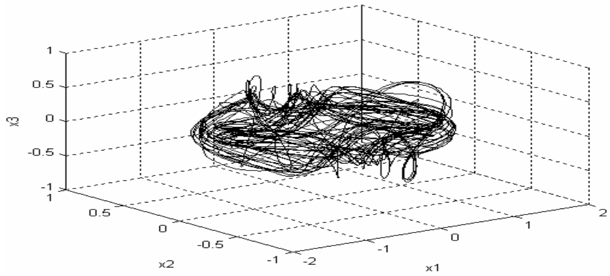

Fig. 6. The 3D-Projection of chaotic state trajectories

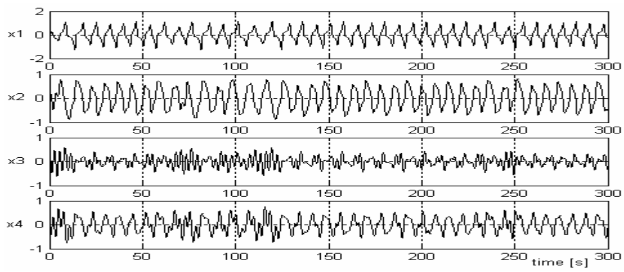

Fig 7. State evolution of the chaotic system

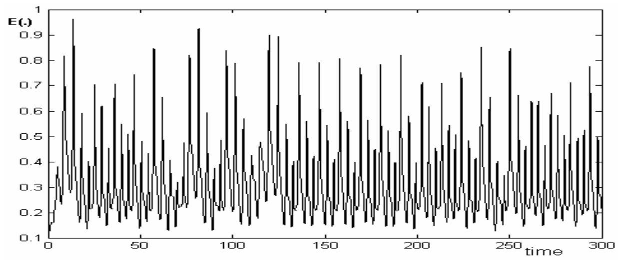

Fig. 8. State energy evolution of the chaotic system

9 Conclusion

In the present paper basic concepts concerning dissipativity, conservativity, state minimality, internal stability, instability and chaos have been examined from a unified structural point of view. Both the linear as well as non-linear state-output system representations are discussed.

Acknowledgements. This work has been supported from Research Project *Diagnostics of interactive phenomena in electrical engineering*, MSM 49777513110 and Department of Applied Electronics and Telecommunication, University of West Bohemia.

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