Continuous and Digital Nonlinear Chaotic systems: Energy-metric Approach, Simulation and Implementation

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Abstract: - In the contribution a new approach to problems of nonlinear system analysis, synthesis and design, including generation of different classes of deterministic, chaotic, pseudo-chaotic and hyper-chaotic signals is presented. The proposed method insists on a new abstract formulation of the energy conservation principle, based on the idea of introducing a proper state-space metric as measure of total energy, accumulated in the state space of a system representation. The fundamental idea is illustrated by examples.

Key-Words: Signal power, state energy, nonlinearity, equivalence, conservativity, dissipativity, chaoticity

1 Introduction

It seems to be obviously true that chaos is all around us. The word *chaos* has been invoked with gay abandon across a broad range of fields in recent years [1],[2],[3]. At the beginning, it is of crucial importance to realize that the concept of the *chaoticity* must not be *confused* with the term of *randomness* in the sense of mathematical statistics and probability theory. The *new science of chaos theory* [3] has provoked reactions ranging from a belief that it will solve all problems of nonlinear dynamics and help us to understand the underlying nature of the universe. Before we start with our presentation at least three fundamental questions should be put: 1. What is chaos? 2. Why should chaotic systems be investigated? 3. What is the relation between well known stochastic processes and processes generated by real-world chaotic, pseudo-chaotic and hyper-chaotic systems? We start with the last question.

2 Chaoticity versus randomness

When we see any *irregularity* we cling to *randomness and disorder* for explanations. Why should this be so? Why is it that when the ubiquitous irregularity of engineering, physical, biological, and other systems are studied, it is *assumed to be random* and the whole vast machinery of probability and statistics is applied? Rather recently, however, we have begun to realize that the tools of deterministic chaos theory can be applied toward the understanding, manipulation, and control of a variety of systems, with many of the practical applications [3], [4]. Historically, the study of chaos started in mathematics and physics [5], [6]. It then expanded into high technology [3], such as communication [3], medicine diagnostics, and recently also into social sciences, and many other fields.

Generally, the boundary between deterministic chaos and probabilistic random processes may not always be clear since *seemingly random systems* could involve *deterministic underlying rules* yet to be found. As a simple example of a *chaotic but not probabilistic random system* in computer science is a *pseudo-random number generator.* The underlying rule in this case is a *simple deterministic* formula (e.g., $x(n+1) = c\, x(n) \bmod m$). However, the resulting generated and *observed signals or solutions*, such as the pseudo-random numbers are very *irregular* and *unpredictable* (the more unpredictable, the closer to random). We also note that a *small change in the initial condition* can yield a *significantly different sequences of random numbers*. These pseudo-random number *generators are chaotic* but *also periodic* with certain (very long) periods. It means that from *finite time observation point of view* such *pseudo-random* signals *seem to be random,* but from the *generation point of view* they are often considered as *chaotic,* although *in principle* they are obviously *"long- term periodic",* and thus only *pseudo-chaotic*.

3 Chaotic system in nature

While this pseudo-random number generator is an *artificial chaotic system*, there are numerous *real-world chaotic systems* in nature. While *engineers typically shun chaos* and irregularity, *nature* may indeed *treasure and exploit it.* For example, it is known that *normal brain activity* may be *chaotic*, and *pathological order* may indeed be the cause of such diseases as epilepsy. It is even suspected that *biological systems* exploit deterministic chaos to *store, encode, and decode information.*

To understand if, and *why this hypothesis should be true*, and how it can be *effectively utilized* in practice, one must try to *understand how chaotic systems behave, and why*. It seems reasonable to *start with a working knowledge of chaos generation mechanism, profoundly, but sometimes subtly different from the behavior of random systems.* Although the history of chaotic systems research is not new, the computer revolution gave life to most of their *practical applications*. Hence one key aspect of *chaos generation* is the power of easily accessible computer hardware for its *speed and memory capacity.* There are known numerous types of commercial and industrial applications, based on different attributes of chaotic systems [3]. It means that there are *serious practical as well theoretical reasons* for intensive research work in *the field of chaos generating systems* [4], [6].

As with many terms in science, there is no standard definition of chaos. There lies a behavior between rigid regularity and randomness based on pure chance. The main *attributes* of any chaotic system include:

- *Nonlinearity.* If it is linear, it cannot be chaotic.
- *Determinism.* It has *deterministic* (rather than probabilistic) underlying rules.
- *Conditioned instability.* Any chaotic system contains at least one potentially instable subsystem
- *Sensitivity to initial state. S*mall changes in initial state cause radically different effects in future states.
- *Sustained irregularity.* Caused by hidden modes, including a *large or infinite number of unstable periodic motions.*
- *Long-term prediction is mostly impossible* due to *sensitivity* to initial state, known only to a *finite degree of precision.*
- *Long-term control and stabilization of chaotic behavior is in principle possible.*

4 Energy-metric approach to linear and non-linear causal systems

Instability and/or unwanted chaos are often considered as the *most important phenomena*, which should be investigated before any other aspect of reality will be attacked. *Two typical situations* should be distinguished in causal systems theory. The first one arises if the *energy function* of a given system is *known* in mathematical form and can be explicitly used to describe the *time evolution of internal system energy E[x(t)]*. In such situations some form of the energy non-increasing test can bring useful information about the system state evolution.

Unfortunately, *in most real-world situations some form of energy conservation law is known to play a crucial role, but any mathematical expression for the system energy is not available.* One standard way to overcome this difficulty is to make some additional restrictive assumptions, such as linearity and time-invariance, and try to use *explicit knowledge of the solution* of differential or difference *equations describing state trajectories* of the system, combined with *computer simulations*, *laboratory experiments*, and with sophisticated *mathematical methods such as, for instance, the bifurcation analysis.*

As an alternative a new conceptually different approach to nonlinear phenomena, based on an original idea of [7], [8], has recently been proposed in [12, [13], and called the Signal Energy-Metric approach. The *crucial idea* is that, in fact, it is *not the physical energy* by itself, but only *a measure of distance* from the system equilibrium x^* to the actual state *x(t)*, what is *needed for analysis of the internal behavior.*

Thus, instead of the concept of energy a *state space metric* $\rho[x(t), x^*]$ has been defined in a proper way, and the *essence of the energy-metric approach* has then formally been expressed by:

$$E(x) = \frac{1}{2}\rho^2\left[x(t),\ x^*\right] \tag{1}$$

Within the *state space paradigm* the proposed concept of the *abstract energy E(x)* seems to be one of the most natural means *describing the internal system topology*. A *measure of distance* of the *actual state* from an *invariant set* can be thought as a *measure of energy accumulated in the state space* of the given system. To avoid confusion the concepts of the *signal power* and that of the *signal energy* for a class of state space system representations have been introduced [9], [10]. In case of zero input $u(t)=0,\ \forall t \geq t_0$ *energy accumulated in the system state x(.)* in time t_0 must be *equal to the amount of energy dissipated on the interval* $[t_0;\infty)$ *by the output y(t), (chaotic or non-chaotic)*, and we get:

$$E(t_0) = \int_{t_0}^{\infty} \|y(t)\|^2\, dt \tag{2}$$

Thus the resulting *structure* of *physically correct system representations is* given by the *algebraic structure*:

$$\begin{aligned} \Re\{S\}:\ \dot{x}(t) &= f[x(t)] + Bu(t),\ x(t_0) = x^0, \\ y(t) &= Cx(t),\quad f[x(t)] = A[x(t)]x(t), \end{aligned} \tag{3}$$

$$A = \begin{pmatrix} -\alpha_1 & \alpha_2 & 0 & 0 & \cdots & 0 & 0 \\ -\alpha_2 & 0 & \alpha_3 & 0 & \cdots & 0 & 0 \\ 0 & -\alpha_3 & 0 & \alpha_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & -\alpha_{n-1} & 0 & \alpha_n \\ 0 & 0 & 0 & 0 & \cdots 0 & -\alpha_n & 0 \end{pmatrix},\ C^T = \begin{bmatrix} \gamma \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},\ B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix} \tag{4}$$

5 Chaotic behavior and related non-linear phenomena

When leaving the idealized mathematical domain and looking around the natural world, one certainly finds a very interesting and realistic phenomenon - there is almost *nothing that is linear* [11].

Example 1.: Let a simple example *nonlinear* system representation with *one nonlinearity N(x) and with two adjustable parameters* α_4, $x_3(0)$ is given

$$\begin{aligned}\dot{x}_1 &= -\alpha_1 x_1 + \alpha_2 x_2\\ \dot{x}_2 &= -\alpha_2 x_1 + \alpha_3 x_3\\ \dot{x}_3 &= -\alpha_3 x_2 + \alpha_4 x_4\\ \dot{x}_4 &= -\alpha_4 x_3\end{aligned} \qquad \alpha = \begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix} = \begin{bmatrix}-1+N(x)\\ 1\\ 1\\ \textit{frequency}\end{bmatrix} \quad x(0) = \begin{bmatrix}0\\ 0\\ \textit{init}\\ 0\end{bmatrix} \tag{5}$$

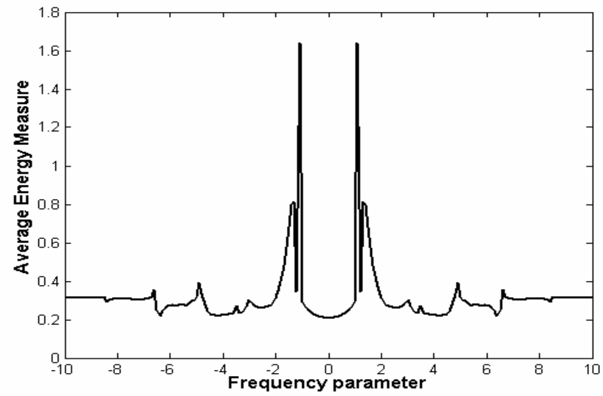

Fig. 1. State energy-frequency plot, *N(x)=10x₂²*

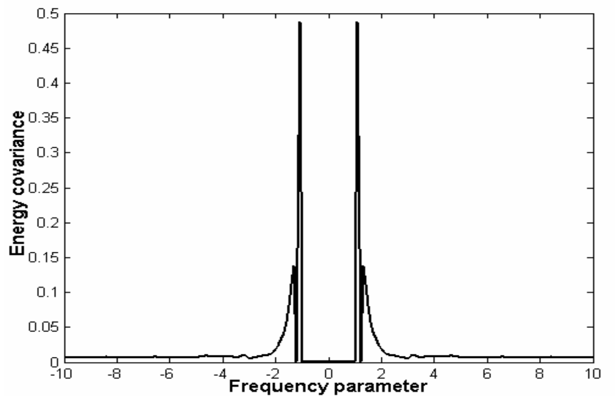

Fig. 2. State energy covariance-frequency plot

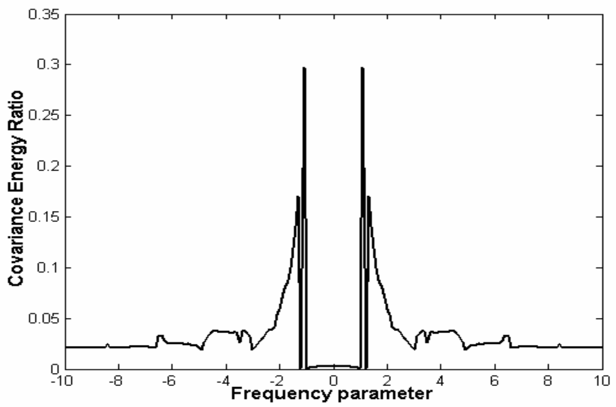

Fig. 3. State energy covariance/ state energy ratio plot

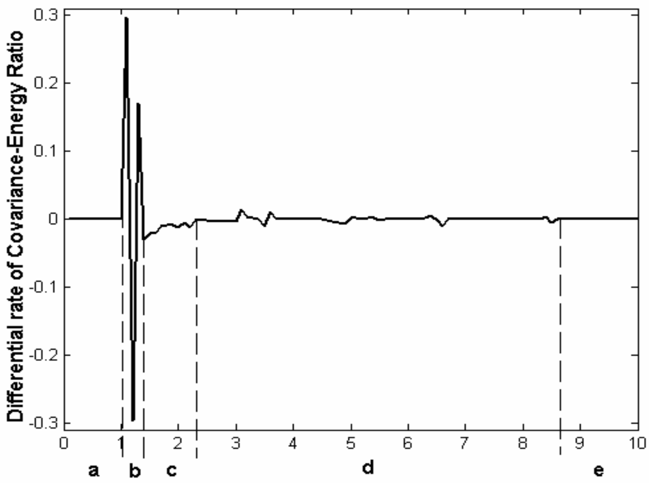

Fig. 4. Derivative of energy covariance/energy ratio for *N(x)=10x₂²*

a - regular behavior, b - nonlinear resonance, c - chaotic behavior, d - mixed , e - reg. modulation

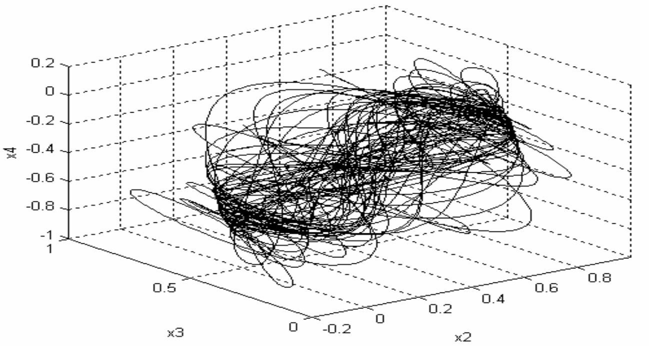

Fig. 5. 3-D Projection of chaos $\alpha_4 = 2.0, x_3(0) = 0.5$

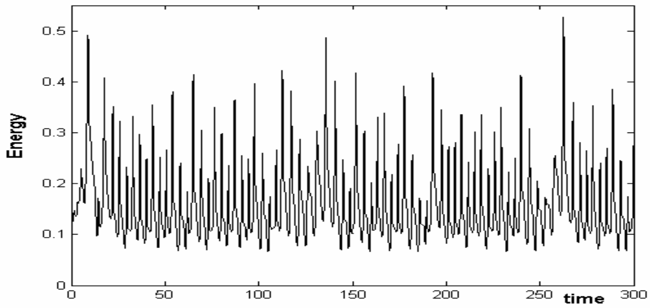

Fig. 6. State energy-time evolution $\alpha_4 = 2.0, x_3(0) = 0.5$

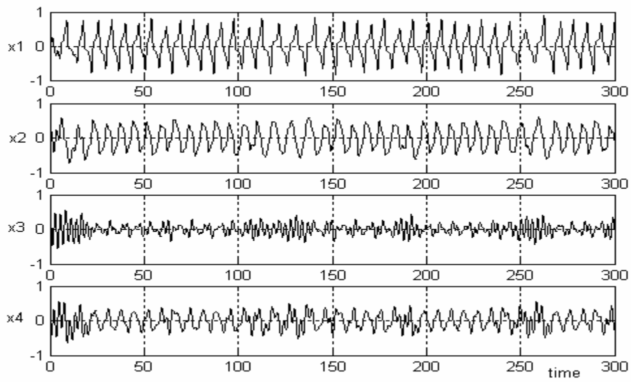

Fig. 7. State variables-time evolution for $N(x)=10x_2^2$, and for $\alpha_4 = 2.0, x_3(0) = 0.5$

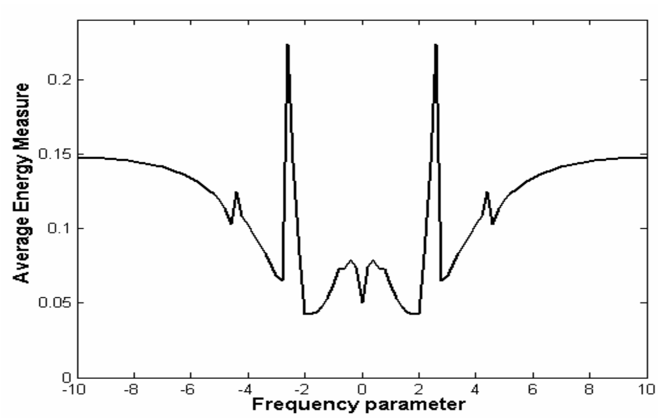

Fig. 8. State energy-frequency plot, $N(x)=20x_1^2$

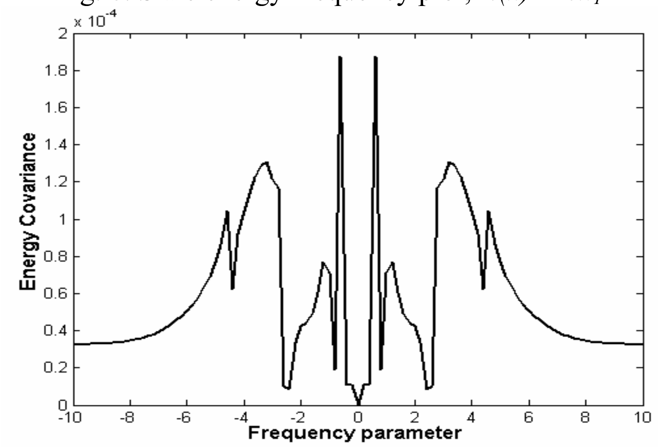

Fig.9. State energy covariance-frequency plot

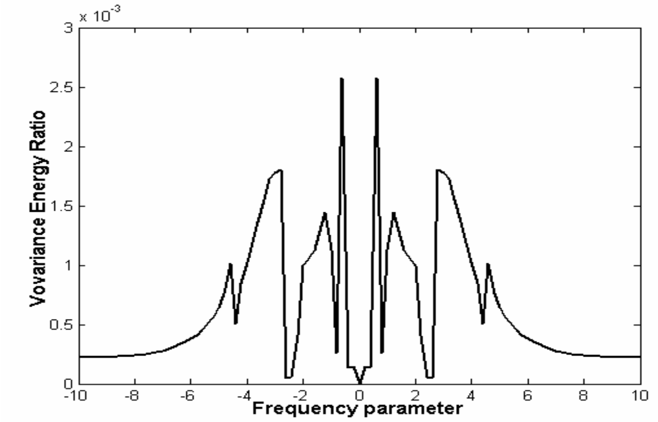

Fig.10. State energy covariance/ state energy ratio plot

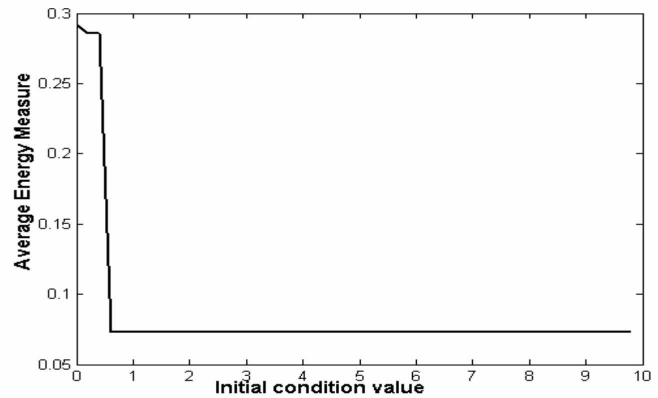

Fig. 11. Average state energy as a function of the initial state energy for $N(x)=20x_1^2$

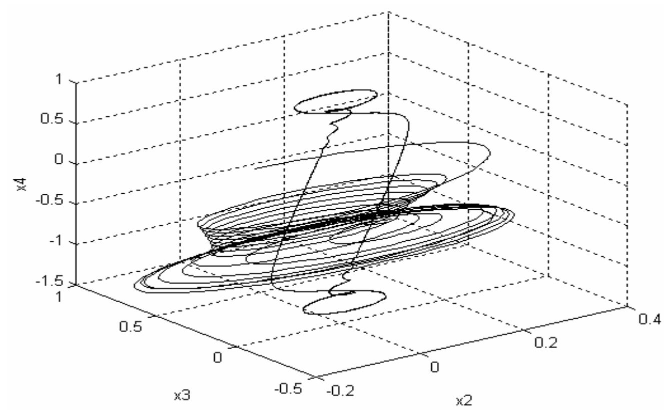

Fig. 12. 3-D Projection of state for $N(x)=20x_1^2$, $\alpha_4=2.0$, $x_3(0)=0.535562$

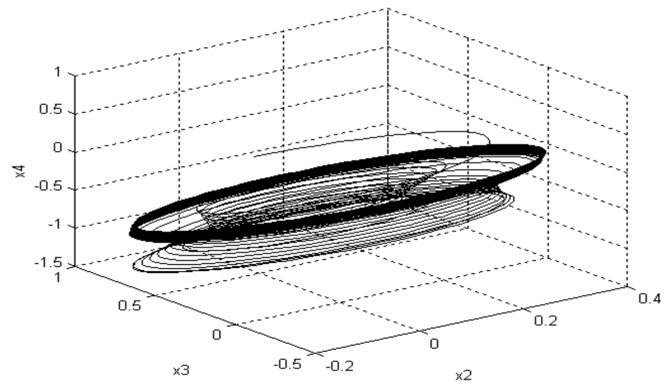

Fig. 13. 3-D Projection of state for $N(x)=20x_1^2$, $\alpha_4=2.0$, $x_3(0)=0.535563$

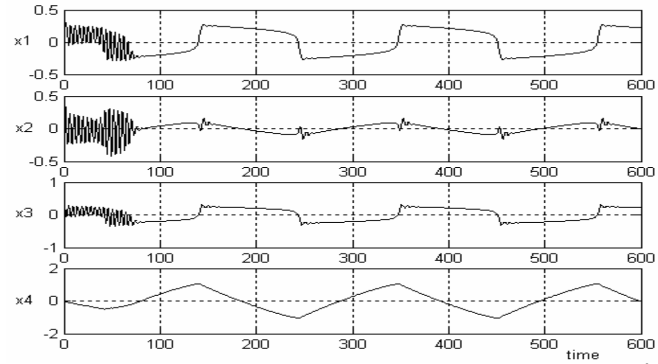

Fig.14. State variables-time evolution for $N(x)=20x_1^2$ and for $\alpha_4=2.00$, $x_3(0)=0.535562$

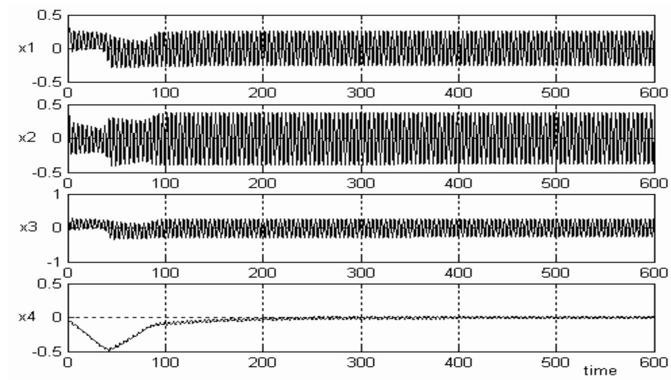

Fig. 15. State variables-time evolution for $N(x)=20x_1^2$ and for $\alpha_4=2.00$, $x_3(0)=0.535563$

6 Simple continuous chaotic systems

In 1963, Lorenz published a paper in which he showed that what we call chaos can occur in systems of autonomous ordinary differential equations (ODEs) with as few as three variables and two quadratics nonlinearities. Later, other scientist found another types of equations and posed the question "What is the simplest ODE and nonlinear function that gives chaos?". It was found [14] that equation (6) with its variants appears to be the simplest function that exhibits chaos:

$$\dddot{x} = -2.017\ddot{x} + \dot{x}^2 - x \qquad (6)$$

One class of simple circuits that leads to chaotic behaviour is [14]:

$$\dddot{x} = -A\ddot{x} - \dot{x} + D(x) - \alpha \qquad (7)$$

where x represents the voltage at particular node in the corresponding circuit, A and α are constants and $D(x)$ is nonlinear function that characterizes the nonlinearity in the circuit. Some nonlinear function that can be performed with operational amplifiers (OA) and diodes are shown in Fig. 16.

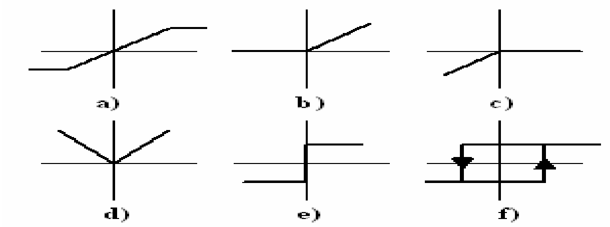

Fig. 16. Some simple nonlinear function that can be performed with operational amplifiers and diodes. a) Amplifier with saturation; b) Ideal diode, $y=N(x)=max(x,0)$; c) Ideal diode, $y=R(x)=min(x,0)$; d) Absolute value, $y=|x|$; e) Signum, $y=sign(x)$ (comparator); f) Hysteresis

Equations of chaotic systems for some types of nonlinearities are given by:

Absolute value:

$$\dddot{x} = -0.6\ddot{x} - \dot{x} \pm (|x| - 1) \qquad (8)$$

Single diode N(x):

$$\dddot{x} = -0.3\ddot{x} - 0.3\dot{x} - N(x) + 1 \qquad (9)$$

Single diode R(x):

$$\dddot{x} = -0.3\ddot{x} - 0.3\dot{x} - R(x) - 1 \qquad (10)$$

Signum function:

$$\dddot{x} = -0.5\ddot{x} - \dot{x} \pm [x - sign(x)] \qquad (11)$$

The nonlinear resistor can be also used, e.g. the Chua well known chaotic generator use nonlinear resistor, which is given by:

$$g(x) = -a\tanh(bx) \qquad (12)$$

where a and b are constant ($a=2, b=0.38$).

The system with diode nonlinearity $N(x)$ (Fig. 16 b)), was simulated and implemented (Eq. 13 and Fig. 17, 18, 19, 20). The equation of this generator is:

$$\dddot{x} = -\left(\frac{R}{R_v}\right)\ddot{x} - \dot{x} + N(x) - \left(\frac{R}{R_0}\right)V_0 \qquad (13)$$

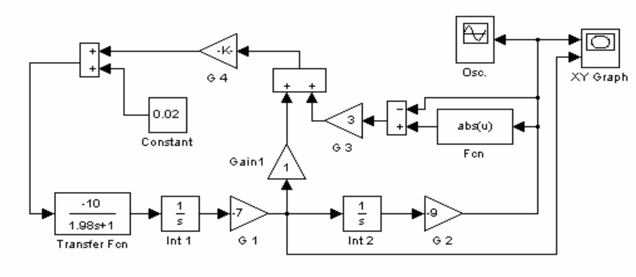

Fig. 17. Simulation of the generator with $N(x)$ nonlinearity.

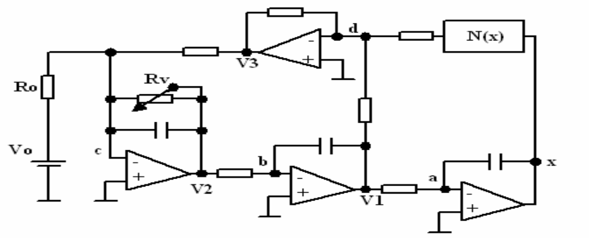

Fig. 18. Implementation of the generator with $N(x)$ nonlinearity. All unsigned resistors values are 100k, capacitors are 4n7.

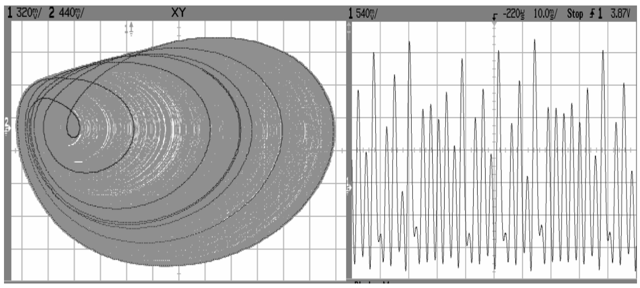

Fig. 19. The generator phase portrait (left) and wave form (right).

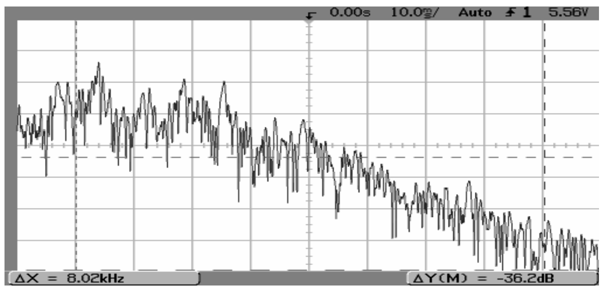

Fig. 21. The generator frequency spectrum.

7 Digital chaotic systems

The discrete systems are a lot easier to handle than continuous systems. They can be chaotic even in less than two dimensions and their solutions doesn't involve solving differential equations. Some simple examples are:

a) The ***Logistic map***. The Logistic equation is:

$$\frac{dx}{dt} = rx(1 - x) \qquad (14)$$

The equation of logistic map is given by:

$$x_{n+1} = rx_n(1 - x_n) \quad (15)$$

with r a positive constant sometimes known as the “biotic potential”.

b) The ***Hénon map***. The Hénon equations are

$$x_{n+1} = 1 - ax_n^2 + y_n, \quad y_{n+1} = bx_n \quad (16)$$

c) The ***Lozi map***. The Lozi map is a simplification of the Hénon map. The quadratic term $-ax_n^2$ is replaced by $-a|x_n|$:

$$x_{n+1} = 1 - a\,|\,x_n\,| + y_n, \quad y_{n+1} = bx_n \quad (17)$$

d) The ***Zaslavski map***. The Zaslavski map is given by the following equations:

$$\begin{aligned} x_{n+1} &= x_n + v + ay_{n+1}(mod1) \\ y_{n+1} &= cos(2\pi x_n) + e^{-r} y_n \end{aligned} \quad (18)$$

The bifurcation diagrams (Fig. 22, 23) are used for digital systems (Eq. 15-18) qualitative behavior visualization.

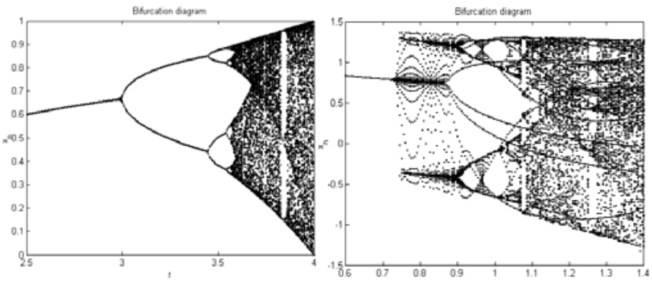

Fig. 22. The Logistic (left) and Hénon (right) bifurcation diagrams.

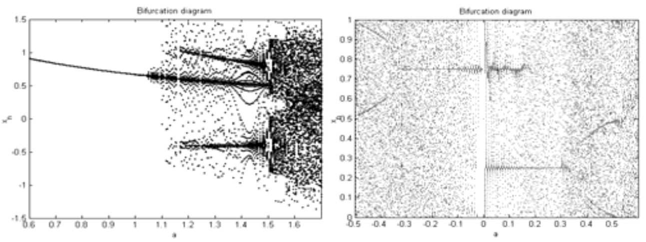

Fig. 23. The Lozi (left) and Zaslavski (right) bifurcation diagrams.

8 Conclusion

In the contribution a new unifying and constructive approach to nonlinear systems, based on a signal-energy-metric concept of the system state space, has been presented. Some typical examples of chaotic and non-chaotic behavior has been presented to make the understanding of the proposed method as easy as possible. The simple continuous and digital examples of chaotic systems were also presented.

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