A Robust Frequency Estimator of Noisy Exponential Sinusoidal Signal

ABDELHAK HAJJARI[ϒ], M'HAMMED HAJJARI[*]
[ϒ]111 Lafayette Road, Syracuse, NY 13205, USA
[*]Universite Sidi M[ed] ben Abdellah, Facultè des Sciences Dhar Mahraz, Fès, Maroc

Abstract: This paper is concerned with the problem of estimating the frequencies of exponential sinusoids. A new method based on the autocorrelation samples for estimating the frequency of noisy exponential sinusoids is introduced. The proposed method uses the approach of estimating frequencies from phase differences. Properties of the conventional frequency estimator, which uses the raw data, and of the proposed method, which uses the autocorrelation samples, are derived analytically and their performances are investigated using numerical simulation. Accuracy and robustness of the proposed method is statistically assessed by Monte Carlo simulations. The results obtained show that the proposed method out performs the conventional approach in terms of accuracy and convergence ratio especially for low signal to noise ratio.

Key words: Exponential Sinusoid, Frequency Estimation, Autocorrelation Samples, Phase Difference.

1 Introduction

The estimation of the parameters of exponential sinusoids embedded in additive white Gaussian noise is one of the classical topics addressed in the signal processing literature. In particular, the estimation of frequencies is of great interest for several applications such as communications, sonar, radar, and biomedical signal processing [1,2]. The problem of frequency estimation has been widely explored. Some of the algorithms that have been previously proposed for frequency estimation include those based on linear prediction estimation [3,4], and those based on adaptive notch filters [5]. However, most of these algorithms do not have good estimation performance at low signal to noise ratio (*SNR*).

In this paper, we propose the use of an algorithm, which employs the approach of estimating frequencies from phase differences [6,10,11]. This algorithm has low computational complexity. Later, we propose the use of the autocorrelation samples to run the algorithm. A preliminary work in which we have addressed this problem, was published in [12]. This new approach lead to better results than the conventional one, which uses raw data samples directly.

The outline of this paper is as follows. Section 2 states the problem and presents the algorithm proposed to estimate the instantaneous phases of the components of the signal. In section 3, the problem of estimating the frequencies from phase differences is addressed. In section 4, we derive the autocorrelation-based method. In section 5, we investigate the performance of the proposed method via numerical simulation. A comparison with the conventional approach is also conducted. Finally, section 6 concludes the paper.

2 Problem Statement and Algorithm derivation

Consider a complex sinusoidal signal, which has been corrupted by noise in receiver. The received signal, $y(k)$, is expressed as follows,

$$\begin{aligned} y(k) &= x(k) + v(k), \qquad k = 1,\ldots,N \\ x(k) &= \sum_{i=1}^{p} \alpha_i \exp(j2\pi f_i k + j\varphi_i) \end{aligned} \tag{1}$$

where N is the number of samples received, and p is the number of sinusoids present in the signal . f_i, α_i and φ_i are, respectively, the frequency, the amplitude and the initial phase of the i^{th} sinusoid. The additive noise $v(k)$ is assumed to be complex white Gaussian noise with zero mean and variance σ^2. The additive noise $v(k)$ is assumed to be independent of the signal $x(k)$. It is also assumed that the real and imaginary parts of the noise are independents. These assumptions can be formulated as follows,

$$\begin{aligned} &E\{x(k)v(l)\} = 0 \quad \forall k,l \\ &E\{v(k)\} = 0 \\ &E\{v(k)v(l)\} = 0 \quad \forall k,l \\ &E\{v(k)v^*(k)\} = \sigma^2\ \delta_{k,l} \end{aligned} \tag{2}$$

where the symbol $E\{\}$ stands for statistical expectation, $\delta_{k,l}$ is the kronecker symbol, and * is the complex conjugate.

The problem addressed here is how to estimate the instantaneous phase of each sinusoid of the signal. For the moment, let us assume that the frequencies are known. Let the following notations,

$$A_{i,k} = A_{i,0}\exp(j\omega_i k) \tag{3}$$

$$\Phi_{i,k} = \arg(A_{i,k}) \tag{4}$$

where $A_{i,0} = \alpha_i \exp(j\varphi_i)$. $A_{i,k}$ and $\Phi_{i,k}$ are, respectively the instantaneous complex amplitude and phase of the i^{th} sinusoid at time k.

The estimation of the vector of the complex amplitudes, $A_k = (A_{1,k},\ldots,A_{m,k})^T$, is given by minimizing the criterion V_k [6],

$$V_k = \sum_{n=0}^{k} \lambda^{k-n} \left|\varepsilon_k(n)\right|^2 \tag{5}$$

where λ is the forgetting factor, and where the prediction error $\varepsilon_k(n)$ is defined as below,

$$\varepsilon_k(n) = y(n) - \hat{y}_k(n/n-1) \tag{6}$$

where $\hat{y}_k(n/n-1)$ is the prediction of $y(n)$ based on the previous estimates. If we admit that the signal's parameters vary slowly in time, $\hat{y}_k(n/n-1)$ can be modeled in a neighborhood n of k by the following equation,

$$\hat{y}_k(n/n-1) = \sum_{i=1}^{p} \hat{A}_{i,k}\exp(j2\pi f_i(n-k)) \tag{7}$$

where $\hat{A}_{i,k}$ is the estimate of the instantaneous complex amplitude of the i^{th} sinusoid at time k.

Since the criterion V_k is quadratic, it can be minimized analytically for sufficiently large n ($n > p$) by,

$$\hat{A}_k = G_k^{-1}\theta_k \tag{8}$$

where,

$$G_k = (g_{i,l}^k)_{i,l=1}^p \tag{9}$$

$$\theta_k = (\theta_i^k)_{i=1}^p \tag{10}$$

and,

$$g_{i,l}^k = \sum_{n=0}^{k} \lambda^{k-n}\exp(j2\pi(f_i - f_l)(k-n)) \tag{11}$$

$$\theta_i^k = \sum_{n=0}^{k} y(n)\lambda^{k-n}\exp(j2\pi f_i(k-n)) \tag{12}$$

It can be seen that the last equation can be rewritten into recursive fashion,

$$\theta_i^k = \lambda\exp(j2\pi f_i)\theta_i^{k-1} + y(k),\, i = 1,\ldots,p \tag{13}$$

Finally, the estimate of the of instantaneous phase vector, $\Phi_k = \{\Phi_{1,k},\ldots,\Phi_{p,k}\}$, is calculated according to the relation given by equation (4).

The estimation of the instantaneous angle phases is based on the assumption that the frequencies, $\{f_i\}_{i=1}^p$, are known. However, in most applications the frequencies are unknown. In such cases, the algorithm described above can be used together with a recursive update scheme for estimating the frequencies $\{f_i\}_{i=1}^p$. A derivation of such a scheme is presented in the following section.

3 Frequency Estimation from Phase Differences

The approach of estimating frequencies from phase differences has been developed by Kay [10]. This approach is well known by its computational simplicity and its good performance. A new derivation of this method, providing its adaptive version, has been introduced by Lang et al [6]. This algorithm adaptively updates the signal frequencies with each new phase observation according to [6, 9].

$$\hat{f}_i(k) = \hat{f}_i(k-1) + \frac{1}{2\pi}\hat{\tau}_i(k) \tag{14}$$

$$\hat{\tau}_i(k) = \rho\,\hat{\tau}_i(k-1) + (1-\rho)^2 \Delta\hat{\Phi}_i(k) \tag{15}$$

where the phase differences are given by,

$$\Delta\hat{\Phi}_i(k) = \hat{\Phi}_i(k) - \hat{\Phi}_i(k-1),\, i = 1,\ldots,p \tag{16}$$

ρ is user-chosen tuning variable in the interval $]0,1[$.

The characteristics of the algorithm, including its tracking and its suppressing noise ability, depends on the forgetting factors λ and ρ. Thus both of them have to be chosen as a compromise between tracking sensitivity and accuracy. It is also important to know the optimum relation between them so as to minimize the total estimation error. This optimal relation is given in [9] by,

$$\rho = \frac{2\sqrt{\lambda}}{1+\sqrt{\lambda}} \tag{17}$$

4 Autocorrelation-based Approach

Let $\{r_k\}$ denote the theoretical autocorrelation function of $y(t)$,

$$r_k = E\{y(t)y^*(t-k)\}, \quad k = 0,1,\ldots,M \qquad (18)$$

Under the assumption made in equation (2), The autocorrelation function of $y(t)$ is given by:

$$r_k = \sum_{i=1}^{p} \alpha_i^2 \exp(j2\pi f_i k) + \sigma^2 \delta_{k,0} \qquad (19)$$

It follows from equation (19) that the sequence of autocorrelation samples $\{\hat{r}_1, \hat{r}_2, \ldots, \hat{r}_M\}$ ($M \le N$) is a noise corrupted sinusoidal signal with the same frequencies as the raw data (1). This observation suggests that the frequency estimator can be determined by fitting a sinusoidal model to the autocorrelation sequence. It should be remarked here that the idea of using sample autocorrelations in lieu of raw data to run frequency estimator enhances the SNR. This idea has been proposed in [7,8] for the estimation of the parameters of auto-regressive mean-average model. In the next section, we demonstrate the performance of this approach via numerical examples.

5 Simulation Results

In this section, we demonstrate the performance of the proposed method with numerical examples. For comparison purposes, we also show the performance of the raw data-based approach.

Example 1: In this example, we illustrate the performances of the two approaches for different *SNR*. For this purpose, we consider a data set of $N = 600$ that consists of a single complex sinusoid with additive noise,

$$y(k) = \alpha_1 \exp(j2\pi f_1 k + j\varphi_1) + v(k), k = 1,\ldots,N \quad (20)$$

where $\alpha_1 = 1$, $f_1 = 0.13$ and φ_1 is randomly chosen in the interval $[0, 2\pi]$. The variance σ^2 of the complex white Gaussian noise with zero mean is chosen to yield a given *SNR*, which is defined by,

$$SNR = 10 \log_{10} \frac{\alpha_1^2}{\sigma^2} \qquad (21)$$

In the runs, the values of λ and ρ are given respectively according to:

$$\lambda(k) = \lambda_i \ \lambda(k-1) + (1-\lambda_i)\lambda_e \qquad (22)$$

$$\rho(k) = \frac{2\sqrt{\lambda(k)}}{1+\sqrt{\lambda(k)}} \qquad (23)$$

where $\lambda_i = 0.995$, $\lambda_e = 0.9995$ and $\lambda(0) = 0.95$.

The results are presented in Fig.1, Fig.2 and Fig. 3 for *SNR* equal to -10dB, 5dB and 25dB, respectively. These figures depict the estimate frequencies obtained by the two approaches. Fig.1 shows clearly the superiority of the autocorrelation-based approach (convergence rate and accuracy) for an *SNR* equal to –10dB. Fig.2 and Fig.3 show that even for higher *SNR*, the proposed method still performs the conventional one.

Example 2: In order to discus the frequency estimation accuracy for the two approaches, in more detail, we propose the use of the Sum Square Error (*SSE*) as a performance metric for the two approaches versus *SNR*. A Monte Carlo simulation is carried out in this example. The *SSE* is calculated from a 10000 trials, and it is evaluated as,

$$SSE = \frac{1}{10000} \sum_{t=1}^{10000} \left(\frac{1}{250} \sum_{k=351}^{600} (\hat{f}_t(k) - f_1)^2 \right) \qquad (24)$$

$\hat{f}_t(k)$ is the estimate of the frequency at time k in the t^{th} realization of Monte Carlo loop.

The results obtained by the two approaches are presented in Fig.4. These results show that the autocorrelation-based approach out performs significantly the raw data-based one especially at low *SNR*, notice a gain of more than 12dB for an *SNR* of –10dB. This result demonstrates the robustness of the proposed approach against the noise effect.

6 Conclusion

A new method, based on using autocorrelation samples to run the phase differences approach, for estimating the frequencies of a complex sinusoidal signal has been proposed. The performance of this method is investigated by comparison with the data-based approach via numerical examples. It is shown that the proposed approach yield better performance for low *SNR*.

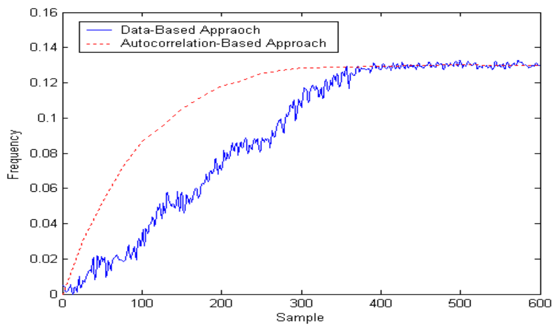

Fig.1: Frequency estimates for $SNR = -10dB$

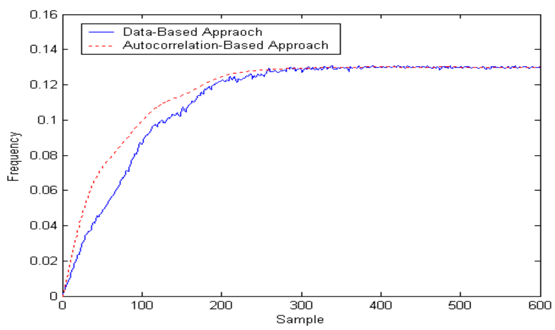

Fig. 2: Frequency estimates for $SNR = 5dB$

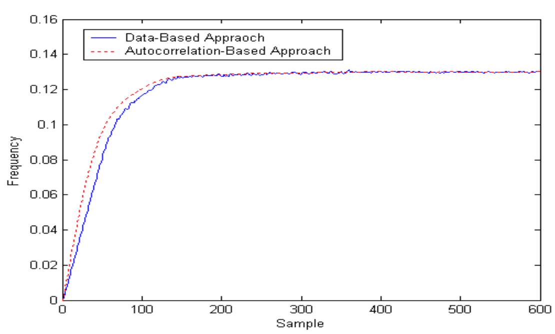

Fig. 3: Frequency estimates for $SNR = 25dB$.

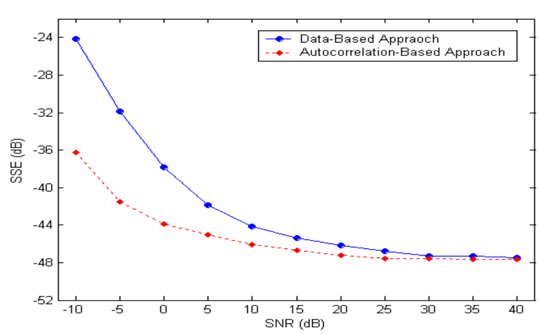

Fig.4: Sum Square Error of the frequency estimation versus *SNR*.

References:

[1] S. Umesh and D. W. Tufts, Estimation of Parameters of Exponentially Sinusoids Using Fast Maximum Likelihood with NMR Spectroscopy Data, *IEEE Trans. on Signal Processing*, Vol.44, No.9, 1994, pp. 2245-2259.

[2] A. Hajjari, M. Benseddik and M. Zouak, Scattering Centers Identification, *Proc. PIERS'97, Progress In Electromagnetic Research Symposium, MIT, Cambridge*, 1997.

[3] R. Kumaresan, L. Scharf and A. Shaw, An Algorithm for Pole-Zero Modeling and Spectral Analysis, *IEEE Trans. on ASSP*, Vol.34, 1986, pp. 637-640.

[4] A. Hajjari, M. Benseddik and A. Bencheqroune, Tracking of Time-Varying Frequency of Sinusoidal Signals, *Signal Processing,* Vol.78, No.2, 1999.

[5] G. Li, A Stable and Efficient Adaptive Notch Filter for Direct Frequency estimation, *IEEE Trans. on Signal Processing*, Vol.45, No.8, 1997, pp. 2001-2009.

[6] S. W. Lang and B. R. Musicus, Frequency Estimation from Phase Differences, *Proc. ICASSP'89, Glasgow*, 1989, pp. 2140-2143.

[7] S. P. Bruzzone and M. Kaveh, Information Tradeoffs in Using the Sample Autocorrelation Function ARMA Parameter Estimation, *IEEE Trans. on ASSP*, Vol.32, 1984, pp. 701-715.

[8] P. Stoica, B. Friendler and T. Sodestrom, Approximate Maximum Likelihood Approach to ARMA Spectral Estimator, *Int. J. Control*, Vol. 45, 1987, pp. 1281-1310.

[9] P. Tchivasky and P. Handel, Efficient Tracking of Multiple Sinusoids with Slowly Varying Parameters, *Proc. ICASSP, Minneapolis, MN, USA*, 1993.

[10] S. Kay, Statistically/Computationally Efficient Frequency Estimation, *Proc. ICASSP'88*, 1988, pp.2292-2295.

[11] A. Source, G. Plantier and J.L. Saumet, Two-dimensional frequency estimation using autocorrelation phase fitting, *Proc. ICASSP '03*, 2003, Vol.3, pp.445-448.

[12] A. Hajjari, and M. Benseddik, Using Samples Autocorrelation for Frequency Estimation of Exponential Sinusoidal Signal from Phase differences, *3rd WSEAS CSCC, Athens, Greece*, 1999.