An Ordering Method For Intuitionistic Fuzzy Numbers

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Abstract: - In this paper first we review two ranking methods for intuitionistic fuzzy numbers (IF numbers), then we proposed a new ordering method for IF numbers in which we consider two characteristic values of membership and non-membership for an IF number.

Key-Words: - Intuitionistic Fuzzy Number, Ranking Function Methods, Characteristic Value

1 Introduction

Ranking fuzzy numbers is one of the fundamental problems of fuzzy arithmetic and fuzzy decision making. It is due to the fact that fuzzy numbers are not linearly ordered. This problem is also important in the case of intuitionistic fuzzy numbers. In this paper the characteristic value for fuzzy number introduced by Chiao [3] is generalized in the case of intuitionistic fuzzy numbers.

2 Preliminaries

Let X be the universal set. Then an intuitionistic fuzzy set (IF set) [1,2] A in X is a set of ordered on triples

$$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\},$$

where the functions $\mu_A, \nu_A : X \to [0, 1]$ are functions such that

$$0 \le \mu(x) + \nu(x) \le 1 \qquad \forall x \in X.$$

For each x the numbers $\mu_A(x)$ and $\nu_A(x)$ represent the degree of membership and degree of nonmembership of the element $x \in X$ to $A \subset X$, respectively. For each element $x \in X$ we can compute the so-called, the intuitionistic fuzzy index of x in A defined as follows

$$\Pi_A(x) = 1 - \mu(x) - \nu(x)$$

Atanassov has also defined two kinds of α-cut for intuitionistic fuzzy sets. Namely

$$A_\alpha = \{x \in I\!R | \mu_A(x) \ge \alpha\},$$

$$A^\alpha = \{x \in I\!R | \nu_A(x) \le \alpha\}.$$

Since from now on we will restrict our consideration to intuitionistic fuzzy numbers, hence our universe of discourse would be the real line, i.e. $X = I\!R$. We define an intuitionistic fuzzy number (IF number) as follows:

Definition 1 An IF set $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in I\!R\}$ of the real line is called an intuitionistic fuzzy number(IF number) if
a) A is IF-normal, i.e. there exist at least two points $x_0,\ x_1 \in X$ such that $\mu_A(x_0) = 1$, and $\nu_A(x_1) = 1$,
b) A is IF-convex, i.e. its membership function μ is fuzzy convex and its nonmembership function ν is fuzzy concave,
c) μ_A is upper semicontinuous and ν_A is lower semicontinuous,
d) *supp* $A = \{x \in X | \nu_A(x) < 1\}$ is bounded.

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From the definition given above we get at once that for any IF number A there exist eight numbers $a_1,\ a_2,\ a_3,\ a_4,\ b_1,\ b_2,\ b_3,\ b_4 \in I\!R$ such that $b_1 \leq a_1 \leq b_2 \leq a_2 \leq a_3 \leq b_3 \leq a_4 \leq b_4$ and four functions $f_A, g_A, h_A.k_A : I\!R :\rightarrow [0,1]$, called the sides of a fuzzy number, where f_A and k_A are nondecreasing and g_A and h_A are nonincreasing, such that we can describe a membership function μ_A in form

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < a_1, \\ f_A(x) & \text{if} \quad a_1 \leq x \leq a_2, \\ 1 & \text{if} \quad a_2 \leq x \leq a_3, \\ g_A(x) & \text{if} \quad a_3 \leq x \leq a_4, \\ 0 & \text{if} \quad a_4 < x, \end{cases}$$

while a nonmembership function ν_A has a following form

$$\nu_A(x) = \begin{cases} 0 & \text{if} \quad x < b_1, \\ h_A(x) & \text{if} \quad b_1 \leq x \leq b_2, \\ 1 & \text{if} \quad b_2 \leq x \leq b_3, \\ k_A(x) & \text{if} \quad b_3 \leq x \leq b_4, \\ 0 & \text{if} \quad b_4 < x. \end{cases}$$

It is worth noting that each IF number $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in I\!R\}$ is a conjunction of two fuzzy numbers: A^+ with a membership function $\mu_{A^+}(x) = \mu_A(x)$ and A^- with a membership function $\mu_{A^-}(x) = 1 - \nu_A(x)$. It is seen that *supp* $A^+ \subseteq$ *supp* A^-.
A useful tool for dealing with fuzzy numbers are their α-cuts. Every α-cut of a fuzzy number is a closed interval and a family of such intervals describes completely a fuzzy number under study. In the case of intuitionistic fuzzy numbers it is convenient to distinguish following α-cuts: $(A^+)_\alpha$ and $(A^-)_\alpha$. It is easily seen that

$$(A^+)_\alpha = \{x \in I\!R | \mu_A(x) \geq \alpha\} = A_\alpha,$$

$$\begin{aligned}(A^-)_\alpha &= \{x \in I\!R | 1 - \nu_A(x) \geq \alpha\} \\ &= \{x \in I\!R | \nu_A(x) \leq 1 - \alpha\} = A^{1-\alpha}.\end{aligned}$$

According to the definition it is seen at once that every α-cut, $(A^+)_\alpha$ or $(A^-)_\alpha$ is a closed interval. Hence we have $(A^+)_\alpha = [A^+_L(\alpha), A^+_U(\alpha)]$ and $(A^-)_\alpha = [A^-_L(\alpha), A^-_U(\alpha)]$, respectively, where

$$A^+_L(\alpha) = \inf\{x \in I\!R | \mu_A(x) \geq \alpha\},$$

$$A^+_U(\alpha) = \sup\{x \in I\!R | \mu_A(x) \geq \alpha\},$$

$$A^-_L(\alpha) = \inf\{x \in I\!R | \nu_A(x) \leq 1 - \alpha\},$$

$$A^-_U(\alpha) = \sup\{x \in I\!R | \nu_A(x) \leq 1 - \alpha\}.$$

If the sides of the fuzzy numbers A are strictly monotone then, we my adopt the convention that $f_A^{-1}(\alpha) = A^+_L(\alpha)$, $g_A^{-1}(\alpha) = A^+_U(\alpha)$, $h_A^{-1}(\alpha) = A^-_L(\alpha)$ and $k_A^{-1}(\alpha) = A^-_U(\alpha)$.

3 Existing Ranking Methods for Intuitionistic fuzzy numbers

It is known that there is no unique linear ordering in a family of fuzzy numbers. Thus ranking fuzzy numbers is one of the fundamental problems of fuzzy arithmetics. The same is true in the case of intuitionistic fuzzy numbers. Below we review a method for ranking IF numbers which suggested by Grzegorzewski in [4].

Suppose $\mathcal{A}$ is a subfamily of all IF numbers.

Definition 3.1 An IF number $L(\mathcal{A})$ is called the lower horizon of a given subfamily $\mathcal{A}$ if $\sup(suppL(\mathcal{A})) \leq \inf(suppA)$ for any $A \in \mathcal{A}$. Similarly, an IF number $U(\mathcal{A})$ is called the upper horizon of a given subfamily $\mathcal{A}$ if $\inf(suppL(\mathcal{A})) \geq \sup(suppA)$ for any $A \in \mathcal{A}$.

It is obvious that $\mathcal{A}$ may have one or more horizons.Two following orders proposed by Grzegorzewski in [4]:

Definition 3.2 Let $A, B \in \mathcal{A}$. Moreover, let $H = L(\mathcal{A})$ and let d be a metric in the family of IF numbers. The relation $\succ_L$ in $\mathcal{A} \times \mathcal{A}$ given by

$$A \succ_L B \Leftrightarrow d(A, H) \geq d(B, H)$$

is called the order respect to the lower horizon H.

Definition 3.3 Let $A, B \in \mathcal{A}$. Moreover, let $H = L(\mathcal{A})$ and let d be a metric in the family of IF numbers. The relation $\succ_U$ in $\mathcal{A} \times \mathcal{A}$ given by

$$A \succ_U B \Leftrightarrow d(A, H) \geq d(B, H)$$

is called the order respect to the upper horizon H.

Not that, using different metrics, e.g. d_p or ρ_p given above, we may obtain different orders. It should be noticed that both relations $\succ_L$ and $\succ_U$ are not antisymmetric and hence they are only quasi-ordering relations, not ordering relations.

Also, Grzegorzewski propose an ordering method for IF numbers by using the expected interval of an IF number. The expected interval of an IF number $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in I\!R\}$ is a crisp interval $\widetilde{EI}(A)$ given by (see [4])

$$\widetilde{EI}(A) = [\widetilde{E_*}(A), \widetilde{E^*}(A)]$$

where

$$\widetilde{E_*}(A) = \frac{b_1 + a_2}{2} + \frac{1}{2}\int_{b_1}^{b_2} h_A(x)dx - \frac{1}{2}\int_{a_1}^{a_2} f_A(x)dx,$$

$$\widetilde{E^*}(A) = \frac{a_3 + b_4}{2} + \frac{1}{2}\int_{a_3}^{a_4} g_A(x)dx - \frac{1}{2}\int_{b_3}^{b_4} k_A(x)dx.$$

As in the case of the classical fuzzy sets, the expected value of an IF number define as follows:

Definition 3.3 The expected value $\widetilde{EV}(A)$ of an IF number $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in I\!R\}$ is the center of the expected interval of that IF number, i.e.

$$\widetilde{EV}(A) = \frac{\widetilde{E_*}(A) + \widetilde{E^*}(A)}{2}$$

The following theorem holds:

Theorem 3.4 Let $\succ_L$ and $\succ_U$ denote the quasi-order with respect to the lower and upper horizon, respectively, based on the metric d_1 (i.e. d_p for $p = 1$). Then for any two IF numbers A and B we get

$$A \succ_L B \Leftrightarrow \widetilde{EV}(A) \geq \widetilde{EV}(B)$$

and

$$A \succ_L B \Leftrightarrow \widetilde{EV}(A) \geq \widetilde{EV}(B).$$

4 An ordering method for intuitionistic fuzzy numbers

Based on the characteristic value for a fuzzy number introduced by Chiao in [3], we proposed an ordering method for intuitionistic fuzzy numbers.

Definition 4.1 Let

$$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\},$$

be an IF number. Let $s(r;k) = \frac{(k+1)r^k}{2}$ be a regular reducing function with positive parameter k. Then the characteristic values of membership and non-membership for IF number A with parameter k denoted by $C_\mu^k(A)$, $C_\nu^k(A)$ respectively,are defined by

$$C_\mu^k(A) = \int_0^1 s(r;k)[f_A^{-1}(r) + g_A^{-1}(r)]dr,$$

$$C_\nu^k(A) = \int_0^1 s(r;k)[h_A^{-1}(r) + k_A^{-1}(r)]dr.$$

Simple calculation implies that

$$C_\mu^k(A) = \frac{(k+1)}{2}\int_0^1 r^k[f_A^{-1}(r) + g_A^{-1}(r)]dr,$$

$$C_\nu^k(A) = \frac{(k+1)}{2}\int_0^1 r^k[h_A^{-1}(r) + k_A^{-1}(r)]dr,$$

for $k \in [0, \infty)$.

Various approach to the definition of the characteristic values of membership and non-membership of a IF number is dependent on the parameter k. It has an interesting interpretation. Let $k = 0$, then

$$C_\mu^0(A) = \frac{1}{2}\int_0^1 [f_A^{-1}(r) + g_A^{-1}(r)]dr,$$

$$C_\nu^0(A) = \frac{1}{2}\int_0^1 [h_A^{-1}(r) + k_A^{-1}(r)]dr.$$

Let $k = 1$, then

$$C_\mu^1(A) = \int_0^1 r[f_A^{-1}(r) + g_A^{-1}(r)]dr,$$

$$C_\nu^1(A) = \int_0^1 r[h_A^{-1}(r) + k_A^{-1}(r)]dr.$$

Letting k approach to ∞, it is easily shown that

$$\lim_{k\to\infty} C_\mu^k(A) = \frac{f_A^{-1}(1) + g_A^{-1}(1)}{2} = \frac{a_2 + a_3}{2},$$

$$\lim_{k\to\infty} C_\nu^k(A) = \frac{h_A^{-1}(1) + k_A^{-1}(1)}{2} = \frac{b_2 + b_3}{2},$$

which are the mean values of the intervals $[a_2, a_3]$ and $[b_2, b_3]$ respectively. Note that the larger the value k is , the less influence of the left and right functions of membership and non-membership, is on the characteristic values of the IF number.

In particular, let $A = (b_1, a_1, b_2, a_2, a_3, b_3, a_4, b_4)$ be a trapezoidal IF number with membership and non-membership functions,

$$\mu_A(x) = \begin{cases} 0 & \text{if} \quad x < a_1, \\ \frac{x-a_1}{a_2-a_1} & \text{if} \quad a_1 \le x \le a_2, \\ 1 & \text{if} \quad a_2 \le x \le a_3, \\ \frac{x-a_4}{a_3-a_4} & \text{if} \quad a_3 \le x \le a_4, \\ 0 & \text{if} \quad a_4 < x, \end{cases}$$

and

$$\nu_A(x) = \begin{cases} 1 & \text{if} \quad x < b_1, \\ \frac{x-b_2}{b_1-b_2} & \text{if} \quad b_1 \le x \le b_2, \\ 0 & \text{if} \quad b_2 \le x \le b_3, \\ \frac{x-b_3}{b_4-b_3} & \text{if} \quad b_3 \le x \le b_4, \\ 1 & \text{if} \quad b_4 < x. \end{cases}$$

In this case we have $f_A(x) = \frac{x-a_1}{a_2-a_1}$, $g_A(x) = \frac{x-a_4}{a_3-a_4}$, $h_A(x) = \frac{x-b_2}{b_1-b_2}$ and $k_A(x) = \frac{x-b_3}{b_4-b_3}$. The inverses for these shape functions for any $r \in [0,1]$ are

$$f_A^{-1}(r) = a_1 + (a_2 - a_1)r,$$

$$g_A^{-1}(r) = a_4 + (a_3 - a_4)r,$$

$$h_A^{-1}(r) = b_1 + (b_2 - b_1)(1 - r),$$

$$k_A^{-1}(r) = b_4 + (b_3 - b_4)(1 - r).$$

Thus,

$$\begin{aligned} C_\mu^k(A) &= \frac{(k+1)}{2} \int_0^1 r^k [f_A^{-1}(r) + g_A^{-1}(r)] dr \\ &= \frac{(k+1)}{2} \int_0^1 r^k [a_1 + (a_2 - a_1)r + \\ &+ a_4 + (a_3 - a_4)r] dr \\ &= \frac{a_2 + a_3}{2} + \frac{(a_1 - a_2) + (a_4 - a_3)}{2(k+2)}, \end{aligned}$$

and

$$\begin{aligned} C_\nu^k(A) &= \frac{(k+1)}{2} \int_0^1 r^k [h_A^{-1}(r) + k_A^{-1}(r)] dr \\ &= \frac{(k+1)}{2} \int_0^1 r^k [b_1 + (b_2 - b_1)(1 - r) + \\ &+ b_4 + (b_3 - b_4)(1 - r)] dr \\ &= \frac{b_1 + b_4}{2} + \frac{(b_2 - b_1) + (b_3 - b_4)}{2(k+2)}. \end{aligned}$$

Note that if A be a symmetrical trapezoidal IF number, then

$$C_\mu^k(A) = \frac{a_2 + a_3}{2} \text{ and } C_\nu^k(A) = \frac{b_1 + b_4}{2}.$$

Now, an ordering could be given on IF numbers as is shown in the following algorithm:

Algorithm 4.2 As a ranking method, we compare two IF numbers A and B using the following steps:
Step 1) For a given k, compare $C_\mu^k(A)$ and $C_\mu^k(B)$. If they are equal, then go to the step 2. Otherwise rank A and B according to the relative position of $C_\mu^k(A)$ and $C_\mu^k(B)$.
Step 2) Compare $C_\nu^k(A)$ and $C_\nu^k(B)$. If they are equal, then conclude that A and B are equal. Otherwise rank A and B according to the relative position of $-C_\nu^k(A)$ and $-C_\nu^k(B)$.

5 Conclusion

In this paper we have proposed a method of ranking intuitionistic fuzzy numbers based on the characteristic values of membership and non-membership of an IF number. It is worth noting that these results are direct generalization of the results obtained for the classical fuzzy numbers.

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