A Method for Solving Linear Programming Problems with Fuzzy Parameters Based on Multiobjective Linear Programming Technique

M. ZANGIABADI, H. R. MALEKI, M. MASHINCHI
Faculty of Mathematics and Computer Sciences
Kerman University
Kerman, IRAN

Abstract: - In the real-world optimization problems, coefficients of the objective function are not known precisely and can be interpreted as fuzzy numbers. In this paper we define the concepts of optimality for linear programming problems with fuzzy parameters (FLP). Then by using the concept of comparison of fuzzy numbers we transform FLP problem into a multiobjective linear programming (MOLP) problem. To this end, we propose several theorems which are used to obtain optimal solutions of FLP. Finally an example is given to illustrate the proposed method of solving linear programming problem with fuzzy parameters (FLP).

Key-Words: - Linear optimization; Fuzzy number; Ranking function; Multiobjective linear programming.

1 Introduction

Fuzzy linear programming (FLP) was first proposed by Tanaka et al. [11] and Zimmermann [16]. To solve FLP problems numerous methods have been developed by different authors [15]. In some of them, authors define a classic linear programming model associated to the FLP problem and then apply linear programming techniques to obtain optimal solutions of the FLP problem [4,7,10,14]. One of the most convenient methods is based on the concept of comparison of fuzzy numbers by using ranking function, [5,12]. However it is clear that using a single ranking function will produce too broad a summary of the aforementioned information (as in probability theory when one uses only the average value to represent a certain probability distribution). Therefore, a description based on more than one characteristic seems more appropriate [3,8]. In this paper, to remove the shortcoming in applying ranking functions we associate a k-dimentional vector of ranking functions to a fuzzy number, where the components of this are selected on the basis of the decision maker's preferences.

On the other hand, Maeda [7] formulated the FLP problem as a two-objective linear programming problem and Zhang et al. [14] formulated it as a four-objective linear programming problem to solve FLP. The aim of this paper is to extend the Zhang et al. method by using a vector of ranking functions. In fact we solve linear programming problem with fuzzy parameters based on multiobjective linear programming techniques.

The paper has the following structure. In section 2, we present comparison of fuzzy numbers by using ranking functions and review the concept of optimality for MOLP. In section 3, we apply a vector of ranking functions to convert FLP problem to a multiobjective linear programming problem. Also an example is presented.

2 Preliminaries

2.1 Vector ranking function

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)'$, $\mathbf{y} = (y_1, y_2, \ldots, y_n)' \in \mathbb{R}^n$ be two vectors, where "$'$" denotes transpose of the vector. Then we write $\mathbf{x} \geq \mathbf{y}$ if and only if $x_i \geq y_i$, for all i belong to $N = \{1, 2, \ldots, n\}$; $\mathbf{x} > \mathbf{y}$ if and only if $x_i > y_i$, for all $i \in N$; $\mathbf{x} \neq \mathbf{y}$ if and only if $x_i \neq y_i$, for some $i \in N$.

Definition 2.1 [3] *A fuzzy set $\tilde{a}$ on $\mathbb{R}$ is called a fuzzy number if it holds:*
1) Its membership function is upper semi continuous.
2) There exist three interval $[a, b]$, $[b, c]$, $[c, d]$ such that $\tilde{a}$ is increasing on $[a, b]$, equal to 1 on $[b, c]$,decreasing on $[c, d]$ and equal to 0 anywhere else.

We denote the set of all fuzzy numbers by $F(\mathbb{R})$. A simple method for ordering the elements of $F(\mathbb{R})$ consists in the defining of a ranking function $R : F(\mathbb{R}) \to \mathbb{R}$ which maps each fuzzy number into a real number, where a natural order exists. It is obvious that more than one ranking function can be defined [2,3].

Based on the decision maker's Preferences, assume there exist k important attributes associated to fuzzy number $\tilde{a}$ such that the "i" th of them can be characterized by the ranking function $R_i : F(\mathbb{R}) \longrightarrow \mathbb{R}$. In this case, we associate a crisp k-dimensional vector, $\mathbf{R}(\tilde{a})$, to $\tilde{a}$ as follows:

$$\mathbf{R}(\tilde{a}) = (R_1(\tilde{a}), R_2(\tilde{a}), \ldots, R_k(\tilde{a}))'.$$

Definition 2.2 *The vector function $\mathbf{R}(.)$, defined as above, is called a vector of ranking functions. Moreover, let $\tilde{a}$ and $\tilde{b}$ belong to $F(\mathbb{R})$, then :*

- $\tilde{a} \underset{\mathbf{R}}{\geq} \tilde{b}$ *if and only if* $\mathbf{R}(\tilde{a}) \geq \mathbf{R}(\tilde{b})$.
- $\tilde{a} \underset{\mathbf{R}}{>} \tilde{b}$ *if and only if* $\mathbf{R}(\tilde{a}) > \mathbf{R}(\tilde{b})$.
- $\tilde{a} \underset{\mathbf{R}}{=} \tilde{b}$ *if and only if* $\mathbf{R}(\tilde{a}) = \mathbf{R}(\tilde{b})$.
- $\tilde{a} \underset{\mathbf{R}}{\neq} \tilde{b}$ *if and only if* $\mathbf{R}(\tilde{a}) \neq \mathbf{R}(\tilde{b})$.

Also we write $\tilde{a} \underset{\mathbf{R}}{\leq} \tilde{b}$ if and only if $\tilde{b} \underset{\mathbf{R}}{\geq} \tilde{a}$; $\tilde{a} \underset{\mathbf{R}}{<} \tilde{b}$ if and only if $\tilde{b} \underset{\mathbf{R}}{>} \tilde{a}$.

Example 2.3 *Let $\tilde{a}$ be a fuzzy number.*

a) For $k = 1$, we consider the Roubens ranking function [1] which is defined as:

$$R(\tilde{a}_r) = 1/2 \int_0^1 (inf\tilde{a}_r + sup\tilde{a}_r)dr,$$

where $\tilde{a}_r$ is an r-cut of $\tilde{a}$ $(0 < r \leq 1)$ i.e, $\tilde{a}_r = \{x \in \mathbb{R} \,|\tilde{a}(x) \geq r\}$.

b) For $k = 2$, consider

$$\mathbf{R}(\tilde{a}) = (E(\tilde{a}), -Var(\tilde{a}))',$$

where $E(\tilde{a})$ and $Var(\tilde{a})$ are the expectation and variance of the density function associated with $\tilde{a}$. See [6].

c) For $k = 3$, consider

$$\mathbf{R}(\tilde{a}) = (V(\tilde{a}), A(\tilde{a}), F(\tilde{a}))'$$

where $V(\tilde{a})$, $A(\tilde{a})$ and $F(\tilde{a})$ are value, ambiguity and fuzziness of $\tilde{a}$, respectively, which are defined as:

$$V(\tilde{a}) = \int_0^1 r[L_{\tilde{a}}(r) + R_{\tilde{a}}(r)]dr,$$

$$A(\tilde{a}) = \int_0^1 r[R_{\tilde{a}}(r) - L_{\tilde{a}}(r)]dr,$$

$$F(\tilde{a}) = \int_0^{1/2} [R_{\tilde{a}}(r) - L_{\tilde{a}}(r)]dr + \int_{1/2}^1 [L_{\tilde{a}}(r) - R_{\tilde{a}}(r)]dr,$$

where $L_{\tilde{a}}(.)$ and $R_{\tilde{a}}(.)$ both from $[0, 1]$ to $\mathbb{R}$ defined by

$$L_{\tilde{a}}(r) = \begin{cases} inf\{x|x \in \tilde{a}_r\} & ifr \in (0, 1], \\ inf\{x|x \in Supp(\tilde{a})\} & ifr = 0. \end{cases}$$

$$R_{\tilde{a}}(r) = \begin{cases} sup\{x|x \in \tilde{a}_r\} & ifr \in (0, 1], \\ sup\{x|x \in Supp(\tilde{a})\} & ifr = 0. \end{cases}$$

See [2], [3].

2.2 Multiobjective linear programming

In this subsection we briefly describe multiobjective linear programming problem (MOLP) and the concept of optimality for it.

Multiobjective linear programming problem (MOLP) is defined as follows:

$$\begin{array}{ll} \max & \mathbf{z}(\mathbf{x}) = \mathbf{C}\mathbf{x}, \\ \text{s.t.} & \mathbf{A}\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}. \end{array} \tag{1}$$

where $\mathbf{C}$ is the $k \times n$ matrix of coefficients of the linear objective functions, $\mathbf{A} \in \mathbb{R}^{\mathbf{m}\times\mathbf{n}}$, $\mathbf{b} \in \mathbb{R}^{\mathbf{m}}$ and $\mathbf{x} \in \mathbb{R}^{\mathbf{n}}$.

For the sake of simplicity, we set $\mathbf{X} = \{\mathbf{x} \in \mathbb{R}^{n} | \mathbf{A}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$. Now, we review the concept of optimality for MOLP as usual manner.

Definition 2.4 *A point $\mathbf{x}^* \in \mathbf{X}$ is called a complete optimal solution for MOLP if and only if $\mathbf{z}(\mathbf{x}^*) \geq \mathbf{z}(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{X}$.*

Definition 2.5 *A point $\mathbf{x}^* \in \mathbf{X}$ is called a pareto optimal solution for MOLP if and only if there does not exist another $\mathbf{x} \in \mathbf{X}$ such that $\mathbf{z}(\mathbf{x}) \geq \mathbf{z}(\mathbf{x}^*)$ and $\mathbf{z}(\mathbf{x}) \neq \mathbf{z}(\mathbf{x}^*)$.*

Definition 2.6 *A point $\mathbf{x}^* \in \mathbf{X}$ is called a weak pareto optimal solution for MOLP if and only if there does not exist another $\mathbf{x} \in \mathbf{X}$ such that $\mathbf{z}(\mathbf{x}) > \mathbf{z}(\mathbf{x}^*)$.*

Now, let E^c, E^p and E^{wp} be sets of all complete optimal solutions, pareto optimal solution and all weak pareto optimal solutions for MOLP, respectively, then it is easy to show that $E^c \subseteq E^p \subseteq E^{wp}$.

3 linear programming problem with fuzzy parameters

In this section we introduce a linear programming problem with fuzzy parameters, and then we define optimal solutions for it. To this end we suppose that $\mathbf{R}$ be any given vector ranking function.

Definition 3.1 *The model*

$$\begin{array}{ll} \max & \tilde{z} \underset{\mathbf{R}}{=} \tilde{\mathbf{c}}\mathbf{x}, \\ s.t. & \mathbf{A}\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}. \end{array} \tag{2}$$

where $\mathbf{A} = (a_{ij})_{m\times n}$, $\mathbf{b} = (b_1, b_2, \ldots, b_n)'$ and $\tilde{\mathbf{c}} = (\tilde{c_1}, \tilde{c_2}, \ldots, \tilde{c_n}) \in (F(\mathbb{R}))^n$, is called a linear programming problem with fuzzy parameters (FLP).

Definition 3.2 *A point $\mathbf{x}^* \in \mathbf{X}$ is called an $\mathbf{R}$-optimal solution for FLP (2) if and only if $\tilde{\mathbf{c}}\mathbf{x}^* \underset{\mathbf{R}}{\geq} \tilde{\mathbf{c}}\mathbf{x}$ for all $\mathbf{x} \in \mathbf{X}$.*

Definition 3.3 *A point $\mathbf{x}^* \in \mathbf{X}$ is called an $\mathbf{R}$-efficient solution for FLP (2) if and only if there does not exist another $\mathbf{x} \in \mathbf{X}$ such that $\tilde{\mathbf{c}}\mathbf{x} \underset{\mathbf{R}}{\geq} \tilde{\mathbf{c}}\mathbf{x}^*$ and $\tilde{\mathbf{c}}\mathbf{x}^* \underset{\mathbf{R}}{\neq} \tilde{\mathbf{c}}\mathbf{x}$.*

Definition 3.4 *A point $\mathbf{x}^* \in \mathbf{X}$ is called an $\mathbf{R}$-weak efficient solution for FLP (2) if and only if there does not exist another $\mathbf{x} \in \mathbf{X}$ such that $\tilde{\mathbf{c}}\mathbf{x} \underset{\mathbf{R}}{>} \tilde{\mathbf{c}}\mathbf{x}^*$.*

Let X^{RO} be the set of all $\mathbf{R}$-optimal solutions, X^{RE} be the set of all $\mathbf{R}$-efficient solutions and X^{RW} be set of all $\mathbf{R}$-weak efficient solutions for FLP (2). Then by definition, we have $X^{RO} \subseteq X^{RE} \subseteq X^{RW}$.

Now, associated with the model (2), we consider the following MOLP problem:

$$\begin{array}{ll} \max & \mathbf{z}(\mathbf{x}) = (R_1(\tilde{\mathbf{c}}\mathbf{x}), R_2(\tilde{\mathbf{c}}\mathbf{x}), \ldots, R_k(\tilde{\mathbf{c}}\mathbf{x}))', \\ \text{s.t.} & \mathbf{A}\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}. \end{array} \tag{3}$$

In a more compact format, MOLP (3) is written:

$$\max\{\mathbf{z}(\mathbf{x}) = \mathbf{R}(\tilde{\mathbf{c}}\mathbf{x}) | \mathbf{x} \in \mathbf{X}\}, \tag{4}$$

where $\mathbf{R}(.) = (R_1(.), R_2(.), \ldots, R_k(.))'$.

The relationship between the optimal solutions of the MOLP (4) and the model (2) can be characterized by the following theorems.

Theorem 3.5 *A point $\mathbf{x}^* \in \mathbf{X}$ is an $\mathbf{R}$-optimal solution for the model (2) if and only if $\mathbf{x}^*$ is a complete optimal solution for MOLP (4).*

Theorem 3.6 *A point $\mathbf{x}^* \in \mathbf{X}$ is an $\mathbf{R}$-efficient solution for model (2) if and only if $\mathbf{x}^*$ is a pareto optimal solution for MOLP (4).*

Theorem 3.7 *A point $\mathbf{x}^* \in \mathbf{X}$ is an $\mathbf{R}$-weak efficient solution for model (2) if and only if $\mathbf{x}^*$ is a weak pareto optimal solution for MOLP (4).*

A classic method to generate a pareto (weak pareto) optimal solution of MOLP (4) is to use the weighted sums of objective functions, i.e., to consider the solutions of the following weighted problem:

$$Max\{\mathbf{wR}(\tilde{\mathbf{c}}\mathbf{x})|\mathbf{x} \in \mathbf{X}\} \quad (5)$$

where $\mathbf{w} = (w_1, w_2, ..., w_k) \geq \mathbf{0}$ and $\mathbf{w} \neq \mathbf{0}$. Now, for finding $\mathbf{R}$-efficient solution or $\mathbf{R}$-weak efficient solutions of the model (2), it suffices to use the following theorems.

Theorem 3.8 *Let a point $\mathbf{x}^* \in \mathbf{X}$ be an optimal solution of weighted problem (5) for some $\mathbf{w} > \mathbf{0}$, then $\mathbf{x}^*$ is an $\mathbf{R}$-efficient solution for model (2).*

Theorem 3.9 *Let a point $\mathbf{x}^* \in \mathbf{X}$ be an $\mathbf{R}$-efficient solution for model (2) , then $\mathbf{x}^*$ is an optimal solution of weighted problem (5) for some $\mathbf{w} > \mathbf{0}$.*

Theorem 3.10 *Let $\mathbf{x}^*$ be an optimal solution of weighted problem (5) for some $\mathbf{w} \geq \mathbf{0}$ and $\mathbf{w} \neq \mathbf{0}$, then $\mathbf{x}^*$ is an $\mathbf{R}$-weak efficient solution for model (2).*

Before closing this section, we shall give a numerical example for illustrating the method.

Example 3.11 *Consider the following FLP problem:*

$$\begin{aligned} &max\ z(\mathbf{x}) = \tilde{c}_1 x_1 + \tilde{c}_1 x_1, \\ &s.t.\ x_1 + 4x_2 \leq 14, \\ &\quad 4x_1 + 10x_2 \leq 38, \\ &\quad 28x_1 - 5x_2 \leq 14, \\ &\quad \mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0}. \end{aligned} \quad (6)$$

where the membership functions of $\tilde{c}_1$ and $\tilde{c}_2$ are

$$\tilde{c}_1(x) = \begin{cases} o & x < 5, \\ x - 5 & 5 \leq x < 6, \\ 1 & 6 \leq x \leq 7, \\ (20 - x)/13 & 7 < x \leq 20, \\ 0 & 20 < x. \end{cases}$$

and

$$\tilde{c}_2(x) = \begin{cases} o & x < 16, \\ x - 16 & 16 \leq x < 17, \\ 1 & 17 \leq x \leq 18, \\ (40 - x)/22 & 18 < x \leq 40, \\ 0 & 40 < x. \end{cases}$$

Let, based on the decision maker's preferences, we consider $K = 3$ and

$$\mathbf{R}(\tilde{a}) = (V(\tilde{a}), A(\tilde{a}), F(\tilde{a}))'$$

where $V(\tilde{a})$, $A(\tilde{a})$ and $F(\tilde{a})$ are value, ambiguity and fuzziness of $\tilde{a}$, respectively, which are defined in previous section.
Note that

$$V(\tilde{c_1}) = 8.5, V(\tilde{c_2}) = 21,$$
$$A(\tilde{c_1}) = 17/6, A(\tilde{c_2}) = 2/3,$$
$$F(\tilde{c_1}) = 3.5, F(\tilde{c_2}) = 11.5.$$

So associated with problem (6), we have the following MOLP :

$$\begin{aligned} \max\ \mathbf{z}(\mathbf{x}) = (&8.5x_1 + 21x_2, 17/6x_1 - 2/3x_2, \\ &3.5x_1 + 11.5x_2)', \\ s.t.\quad &x_1 + 4x_2 \leq 14, \\ &4x_1 + 10x_2 \leq 38, \\ &28x_1 - 5x_2 \leq 14, \\ &\mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0}. \end{aligned} \quad (7)$$

To solve the above problem, we consider the following weighted problem:

$$\begin{aligned} \max\ &w_1(8.5x_1 + 21x_2) \\ &+w_2(17/6x_1 - 2/3x_2) \\ &+w_3(3.5x_1 + 11.5x_2), \\ \text{s.t.}\ &x_1 + 4x_2 \leq 14, \\ &4x_1 + 10x_2 \leq 38, \\ &28x_1 - 5x_2 \leq 14, \\ &\mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0}. \end{aligned} \quad (8)$$

From Theorem 3.8, if $\mathbf{x}^*$ is an optimal solution to the weighted problem (8) for some $\mathbf{w} > \mathbf{0}$, then $\mathbf{x}^*$ is an $\mathbf{R}$-efficient solution for model (6). The solution of the problem depends on the choice of the weights in problem (8). For example, if we set $w_1 = 0.5, w_2 = 0.25, w_3 = 0.25$, then the solution is $(x_1^*, x_2^*) = (1.0769, 3.2308)$.

4 conclusion

In this paper we consider a linear programming problem with fuzzy parameters in objective function. There are several approaches for solving this problem which use different ranking function. To improve the draw back of using a single characteristic, we associated a k-dimensional vector ranking function to a fuzzy number. Our aim is solving FLP based on multiobjective linear programming techniques, as a continuation of the Zhang et al. method by using the vector ranking function.

References:

[1] G. Bortolan and R. Degani, A review of some methods for ranking fuzzy numbers, *Fuzzy Sets and Systems*, Vol.15, 1985, pp. 1-19.

[2] M. Delgado, M.A. Vila and W. Voxman, A fuzziness measure for fuzzy numbers: applications, *Fuzzy Sets and Systems*, Vol.94, 1998, pp. 205-216.

[3] M. Delgado, M.A. Vila and W. Voxman, On a canonical reperesentation of fuzzy numbers, *Fuzzy Sets and Systems*, Vol.93, 1998, pp. 125-135.

[4] M.K. Luhandjura, Linear programming with a possibilistic objective function, *European Journal of Operational Research*, Vol.13, 1987, pp. 137-145.

[5] H.R. Maleki, M. Tata and M. Mashinchi, Linear programming with fuzzy variables, *Fuzzy Sets and Systems*, Vol.109, 2000, pp. 21-33.

[6] H.R. Maleki, H. Mishmast Nehi and M. Mashinchi, Fuzzy number linear programming: a probabilistic approach, *Soft Computing - AFSS 2002, Springer-Verlag Press, Berlin Heidelberg*, 2002, pp. 491-496.

[7] T. Meada, Fuzzy linear programming problems as bi-criteria optimization problems, *Applied mathematics and computation*, Vol.120, 2001, pp. 109-121.

[8] H. Mishmast Nehi, H.R. Maleki, Solving fuzzy number linear prigramming problem by lexicographic ranking function, *Italian journal of pure and applied mathematics*, Vol.15, 2004, pp. 9-20.

[9] R. Steuer, *Multiple criteria optimization: Theory, Computations, and Application*, Wiley , 1986.

[10] H. Tanaka, H. Ichihashi, K. Asai, A formulation of fuzzy linear programming problem based on comparison of fuzzy numbers, *Control and Cybernetics*, Vol.3, 1991, pp. 185-194.

[11] H. Tanaka, T. Okauda and K. Asai, On fuzzy mathematical programming, *J. Cybernetic*, Vol.3, 1974, pp. 37-46.

[12] E. Yazdani Peraei, H.R. Maleki and M. Mashinchi, A method for solving a fuzzy linear programming, *Korean J. Comput. and Appl. Math.*, Vol.8 , No.2, 2001, pp. 347-356.

[13] K.P. Yoon, A probabilistic approach to rank complex fuzzy numbers, *Fuzzy Sets and Systems*, Vol.80, 1996, pp. 167-176.

[14] G. Zhang, Y-H. Wu, M. Remias, J. Lu, Formulation of fuzzy linear programming problems as four-objective constrained optimization problems, *Applied Mathematics and Computation* , Vol.139, 2003, pp. 383-399.

[15] H.J. Zimmermann, Applications of fuzzy sets theory to mathematical programming, *Information Science*, Vol.36, 1985, pp. 29-58.

[16] H.J. Zimmermann, Optimization in fuzzy invironment, *Presented at 21 st Int. TIMS and 46st ORSA Conference, Sun Juan, Puerto Rico*, 1974.