Optimization Over Pseudo-Boolean Lattices

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Abstract

In this paper, first we will find the solution of the system $A * X \leq b$, where A, b are the known suitable matrices and X is the unknown matrix over a pseudo-Boolean lattice. Then its application to find the solution of some fuzzy linear systems as well as finding the solution of the optimization problem $Z = max\{C*X|A*X \leq b\}$ is discussed.

Key-words: pseudo-Boolean lattice, linear programming, fuzzy linear systems, optimization.

1 Introduction and preliminaries

Linear and combinatorial optimization have been studied by many authors [5]. Optimization over residuted, lattice-ordered commutative monoid is studied in [5]. On the other hand in many applications, one need to find the solution of fuzzy linear systems of equations and inequalities over a bounded chain in [3]. In this paper we replace a bounded chain by any pseudo-Boolean lattice R, which is recently studied in [1]. Then by using the approach in [5] we will solve linear system $A * X \leq b$, over R. The method given here is very easy to be applied for solving the fuzzy linear systems studied in [3].

Definition 1.1. [1,5] A bounded lattice $(L, \leq)$ is called pseudo-Boolean if for all $a, b \in L$, there exists $c \in L$ such that

$$a \wedge x \leq b \Leftrightarrow x \leq c \quad \forall x \in L.$$

If such element c exists, then it is unique and will be denoted by $b : a$.

For the following remark see [5].

Remark 1.2. (i) Every finite distributive lattice is pseudo-Boolean.

(ii) In a Boolean lattice B, one can see that $b : a = b \vee a^*$. Hence B is pseudo-Boolean.

(iii) In general, a pseudo-Boolean lattice may not be Boolean. For example consider a bounded chain $(L, \leq)$. Then $a \wedge b = min(a, b)$, $a \vee b = max(a, b)$ and

$$b : a = \begin{cases} 1 & if \quad b \geq a \\ b & if \quad b < a \end{cases}$$

for all $a, b \in L$. But L is not Boolean, since for any a; $0 < a < 1$, we have

$$a \vee (0 : a) = a \vee 0 = a < 1$$

.

For the following theorem see Proposition 1.17 of [5] and Theorem I.6.11 of [2].

Theorem 1.3. Let L be a lattice.
(i) If L is pseudo-Boolean lattice, then it is distributive.
(ii) If L is complete infinitely distributive lattice, then it is pseudo-Boolean.

Definition 1.4. A non-empty set H with binary operation $* : H \times H \longrightarrow H$ is called a semigroup if

$$a * (b * c) = (a * b) * c \quad \forall a, b, c \in H.$$

A semigroup H is called a monoid if it contains an element $e \in H$ such that for all $a \in H$,

$$e * a = a * e = a.$$

Definition 1.5. Let H be a commutative semigroup with a reflexive and transitive order $\leq$ on it. Then $(H, \leq)$ is called an ordered commutative semigroup if
$a \leq b \Longrightarrow a * c \leq b * c \quad \forall a, b, c \in H.$

Definition 1.6. Let $(H, \leq)$ be an ordered commutative semigroup. Then H is called residuated semigroup, if for all $a, b \in H$, there exists $c \in H$ such that:

$$a * x \leq b \Leftrightarrow x \leq c \quad \forall x \in H.$$

If such an element $c \in H$ exists for all $a, b \in H$, then it is called a residual of b with respect to a.

Remark 1.7. If the order relation $\leq$ on H is antisymmetric, then a residual $c \in H$ is uniquely determined and denoted by $b : a$.

Definition 1.8. Let $(H, \leq)$ be an ordered commutative semigroup(monoid). If the order $\leq$ on H is a partial order then $(H, \leq)$ is called a lattice-ordered commutative semigroup(monoid).

Example 1.9. Every lattice $(H, \leq)$ is a lattice-ordered commutative semigroup, by letting $* = \wedge$. Clearly a bounded lattice is a lattice-ordered commutative monoid in this way.

Through this paper suprimum and infimum over the empty set ϕ are taken to be 0 and 1; respectively.

2 Solutions of $A * X \leq b$

If R is a residuated semigroup, then for all $a, b \in R$, there exists $c \in R$ such that

$$a * x \leq b \Leftrightarrow x \leq c \quad \forall x \in R.$$

This result can be extended to matrix inequalities. Let $A = (a_{ij})_{m \times n}$, $X = (x_j)_{n \times 1}$, $b = (b_i)_{m \times 1}$, be matrices over

R, i.e. $a_{ij}, x_j, b_i \in R$, for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$. Define

$$A * X = ((\bigvee_{j=1}^{m} a_{ij} * x_j)_i)_{m\times 1}. \qquad (1)$$

For the following theorem see Proposition 10.7 of [5].

Theorem 2.1. Let R be a residuated, lattice-ordered commutative monoid. Let A and B be $m \times m$ and $m \times r$ matrices over R; respectively. Let C as

$$(C)_{jk} = inf\{b_{ik} : a_{ij} | i = 1, 2, ..., m\}.$$

Then for all $X \in M_{n\times r}(R)$, we have:

$$A * X \leq B \Leftrightarrow X \leq C \qquad (2)$$

where $A * X$ is defined in (1). Hence $B : A = C$.

Corollary 2.2. Let $(R, \leq)$ be a bounded chain, $* = \wedge$ and A, b be $m \times n$ and $m \times 1$ matrices over R; respectively. Then the system $A * X \leq b$ is consistent and the greatest solution of this system is X_{gr} such that the $j - th$ component of X_{gr} is $x_j = \bigwedge\{b_i | b_i < a_{ij}\}$. Moreover, all of the solutions of the system $A * X \leq b$ are between 0 and X_{gr}, where 0 is the zero matrix for which all of its elements are the least element of R.

Proof: Since R is a bounded chain, as in Remark 1.2(iii), we have:

$$(b : A)_j = inf\{b_i : a_{ij} | i = 1, 2, ..., m\}$$

$$= inf\{b_i | b_i < a_{ij}\},$$

where infimum over empty set is taken 1. Clearly by (1) $b : A$ is the greatest solution. □

Example 2.3 [3]. Let $R = [0, 1]$ and $* = \wedge$. Consider the fuzzy linear system $A * X \leq b$, where

$$A = \begin{pmatrix} 0.3 & 0.9 & 0.9 & 0.4 & 0.2 \\ 0.7 & 0 & 0.9 & 0.4 & 0.7 \\ 0.6 & 0.1 & 0 & 0.8 & 0.5 \\ 0.8 & 0.7 & 0.4 & 0.2 & 0.7 \\ 0.4 & 0.1 & 0.2 & 0.5 & 0.1 \end{pmatrix},$$

and

$$b = (0.9, 0.9, 0.7, 0.7, 0.5)',$$

where $'$ is the transpose operation.

By Corollary 2.2 we have :

$x_1 = inf\{0.7\} = 0.7$

$x_2 = inf\phi = 1$

$x_3 = inf\phi = 1$

$x_4 = inf\{0.7\} = 0.7$

$x_5 = inf\phi = 1.$

So,

$$X_{gr} = (0.7, 1, 1, 0.7, 1)'.$$

Remark 2.4. Note that the fuzzy linear system in Example 2.3 is solved by a different approach in [3].

Corollary 2.5. Let X be an arbitrary set. Consider the Boolean lattice $(P(X), \subseteq)$, $* = \wedge$ and $\vee = \cup$. Let A and b be $m \times n$ and $m \times 1$ matrices over $P(X)$; respectively. Then the system $A * X \leq b$ is consistent and the greatest solution of system is X_{gr} such that the $j-th$ component of X_{gr} is $x_j = \bigcap_{i=1}^{m}(b_i \cup a_{ij}^c)$.

Proof: It follows from Theorem 2.1 and Remark 1.2(ii).

Example 2.6. Let $R = P(X)$, where X is the chain of non-negative real numbers $R^{\geq 0}$. Consider the system $A*X \leq b$ as follows :

$$\begin{cases} [0,2]\cap x_1 & & & \subseteq & [4,6] \\ [1,2]\cap x_1 & \cup & [1,3]\cap x_2 & \subseteq & [8,10] \\ & & [0,3]\cap x_2 & \subseteq & [2,6], \end{cases}$$

where $* = \wedge$. By using Corollary 2.5, we have:

$x_1 = (2,+\infty)$, $x_2 = (3,+\infty)$. Hence

$$X_{gr} = ((2,+\infty),(3,+\infty))'.$$

3 Optimization problem

Theorem 3.1. Let R be a lattice-ordered commutative monoid, A, X, b, C be $m\times n$, $n\times 1, m\times 1, 1\times n$, matrices over R; respectively. The optimal solution of optimization problem $max\{Z = C*X|A*X \leq b\}$, where $*$ is define in (1), is $X_{gr} = b : A$, which is given in Theorem 2.1.

Example 3.2. Consider the optimization problem $max\{Z = C*X|A*X \leq b\}$, where A, b be as in Example 2.3, and $C = (0.4, 0.5, 0.2, 0, 0.8)$. Then $X_{gr} = (0.7, 1, 1, 0.7, 1)'$ is the optimal solution and $Z^* = 0.8$.

Example 3.3. Consider the optimization problem $max\{Z = C*X|A*X \leq b\}$, where A, b be as in Example 2.6, and $C = ([5,7],[4,8])$. Then $X_{gr} = ((2,+\infty),(3,+\infty))'$ is the optimal solution and $Z^* = [4,8]$.

Remark 3.4. In another paper we will discuss the solution of the linear systems $A*X \geq b$ and $A*X = b$.

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