Modified Theory of Laminar Boundary-Layer Flow by Natural Convection on a Vertical Hot Plate

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Abstract: - Scale-invariant forms of conservation equations in chemically -reactive fields are described and the modified equation of motion is solved for the classical problem of laminar flow by natural convection on a vertical hot plate. The results are found to be in good agreement with the experimental observations of *Schmidt* and *Beckmann* as well as the classical theories of *Pohlhausen* and *Ostrach*.

Key-Words: - Theory of laminar flow by natural-convection on vertical hot plates.

1 Introduction

A scale-invariant model of statistical mechanics and its applications to thermodynamics [4] and derivation of invariant forms of conservation equations [5, 6] was recently described. The exact solutions of the modified equation of motion for the classical problems of laminar [8] and turbulent [9] axi-symmetric and two-dimensional jets have also been reported. In the present study, following the classical investigations of *Pohlhausen* [10] and *Ostrach* [11], the solution of the modified equation of motion for the classical problem of laminar boundary-layer flow by natural convection on a vertical hot plate is considered. The resulting analytical solutions are found to be in good agreement with the observations of *Schmidt* and *Beckmann* [12] as well as the classical theories.

2 Scale-Invariant Form of the Conservation Equations for Reactive Fields

Following the classical methods [1-3], the invariant definitions of the density ρ_β, and the velocity of *atom* $\mathbf{u}_\beta$, *element* $\mathbf{v}_\beta$, and *system* $\mathbf{w}_\beta$ at the scale β are given as [4]

$$\rho_\beta = n_\beta m_\beta = m_\beta \int f_\beta du_\beta \quad , \quad \mathbf{u}_\beta = \mathbf{v}_{\beta-1} \tag{1}$$

$$\mathbf{v}_\beta = \rho_\beta^{-1} m_\beta \int \mathbf{u}_\beta f_\beta d\mathbf{u}_\beta \quad , \quad \mathbf{w}_\beta = \mathbf{v}_{\beta+1} \tag{2}$$

Also, the invariant definitions of the peculiar and the diffusion velocities are given as [4]

$$\mathbf{V}'_\beta = \mathbf{u}_\beta - \mathbf{v}_\beta \quad , \quad \mathbf{V}_\beta = \mathbf{v}_\beta - \mathbf{w}_\beta = \mathbf{V}'_{\beta+1} \tag{3}$$

Next, following the classical methods [1-3], the scale-invariant forms of mass, thermal energy, and linear momentum conservation equations at scale β are given as [5, 6]

$$\frac{\partial \rho_\beta}{\partial t} + \nabla.\left(\rho_\beta \mathbf{v}_\beta\right) = \Omega_\beta \tag{4}$$

$$\frac{\partial \varepsilon_\beta}{\partial t} + \nabla.\left(\varepsilon_\beta \mathbf{v}_\beta\right) = 0 \tag{5}$$

$$\frac{\partial \mathbf{p}_\beta}{\partial t} + \nabla.\left(\mathbf{p}_\beta \mathbf{v}_\beta\right) = 0 \tag{6}$$

involving the *volumetric density* of thermal energy $\varepsilon_\beta = \rho_\beta h_\beta$ and linear momentum $\mathbf{p}_\beta = \rho_\beta \mathbf{v}_\beta$. Also, Ω_β is the chemical reaction rate and h_β is the absolute enthalpy.

The local velocity $\mathbf{v}_\beta$ in (4)-(6) is expressed in terms of convective $\mathbf{w}_\beta = \langle \mathbf{v}_\beta \rangle$ and diffusive $\mathbf{V}_\beta$ velocities as [5]

$$\mathbf{v}_\beta = \mathbf{w}_\beta + \mathbf{V}_{\beta g} \quad , \quad \mathbf{V}_{\beta g} = -D_\beta \nabla \ln(\rho_\beta) \tag{7a}$$

$$\mathbf{v}_\beta = \mathbf{w}_\beta + \mathbf{V}_{\beta tg} \quad , \quad \mathbf{V}_{\beta tg} = -\alpha_\beta \nabla \ln(\varepsilon_\beta) \tag{7b}$$

$$\mathbf{v}_\beta = \mathbf{w}_\beta + \mathbf{V}_{\beta hg} \quad , \quad \mathbf{V}_{\beta hg} = -\nu_\beta \nabla \ln(\mathbf{p}_\beta) \tag{7c}$$

where ($\mathbf{V}_{\beta g}$, $\mathbf{V}_{\beta tg}$, $\mathbf{V}_{\beta hg}$) are respectively the diffusive, the thermo-diffusive, the linear hydro-diffusive velocities. For unity *Schmidt* and *Prandtl* numbers, one may express

$$\mathbf{V}_{\beta tg} = \mathbf{V}_{\beta g} + \mathbf{V}_{\beta t} \quad , \quad \mathbf{V}_{\beta t} = -\alpha_\beta \nabla \ln(h_\beta) \tag{8a}$$

$$\mathbf{V}_{\beta hg} = \mathbf{V}_{\beta g} + \mathbf{V}_{\beta h} \quad , \quad \mathbf{V}_{\beta h} = -\nu_\beta \nabla \ln(\mathbf{v}_\beta) \tag{8b}$$

that involve the thermal $\mathbf{V}_{\beta t}$, and linear hydrodynamic $\mathbf{V}_{\beta h}$ diffusion velocities [5]. Since for an ideal gas $h_\beta = c_{p\beta} T_\beta$, when $c_{p\beta}$ is constant and $T = T_\beta$, Eq.(8a) reduces to the *Fourier* law of heat conduction

$$\mathbf{q}_\beta = \rho_\beta h_\beta \mathbf{V}_{\beta t} = -\kappa_\beta \nabla T \tag{9}$$

where κ_β and $\alpha_\beta = \kappa_\beta / (\rho_\beta c_{p\beta})$ are the thermal conductivity and diffusivity. Similarly, (8b) may be identified as the shear stress associated with diffusional flux of linear momentum and expressed by the generalized *Newton* law of viscosity [5]

$$\boldsymbol{\tau}_{ij\beta} = \rho_\beta \mathbf{v}_{j\beta} \mathbf{V}_{ij\beta h} = -\mu_\beta \partial \mathbf{v}_{j\beta} / \partial \mathbf{x}_i \tag{10}$$

Substitutions from (7a)-(7c) into (4)-(6), neglecting cross-diffusion terms and assuming constant transport coefficients with $Sc_\beta = Pr_\beta = 1$, result in [5, 6]

$$\frac{\partial \rho_\beta}{\partial t} + \mathbf{w}_\beta . \nabla \rho_\beta - D_\beta \nabla^2 \rho_\beta = \Omega_\beta \tag{11}$$

$$h_\beta \left[\frac{\partial \rho_\beta}{\partial t} + \mathbf{w}_\beta . \nabla \rho_\beta - D_\beta \nabla^2 \rho_\beta \right] + \rho_\beta \left[\frac{\partial h_\beta}{\partial t} + \mathbf{w}_\beta . \nabla h_\beta - \alpha_\beta \nabla^2 h_\beta \right] = 0 \tag{12}$$

$$\mathbf{v}_\beta \left[\frac{\partial \rho_\beta}{\partial t} + \mathbf{w}_\beta . \nabla \rho_\beta - D_\beta \nabla^2 \rho_\beta \right] + \rho_\beta \left[\frac{\partial \mathbf{v}_\beta}{\partial t} + \mathbf{w}_\beta . \nabla \mathbf{v}_\beta - \nu_\beta \nabla^2 \mathbf{v}_\beta \right] = 0 \tag{13}$$

In the first and second parts of Eqs.(12)-(13), the *gravitational* versus the *inertial* contributions to the change in energy and momentum density are apparent. Substitutions from (11) into (12)-(13) result in the invariant forms of conservation equations [6]

$$\frac{\partial \rho_\beta}{\partial t} + \mathbf{w}_\beta . \nabla \rho_\beta - D_\beta \nabla^2 \rho_\beta = \Omega_\beta \tag{14}$$

$$\frac{\partial T_\beta}{\partial t} + \mathbf{w}_\beta . \nabla T_\beta - \alpha_\beta \nabla^2 T_\beta = -h_\beta \Omega_\beta / (\rho_\beta c_{p\beta}) \tag{15}$$

$$\frac{\partial \mathbf{v}_\beta}{\partial t} + \mathbf{w}_\beta . \nabla \mathbf{v}_\beta - \nu_\beta \nabla^2 \mathbf{v}_\beta = -\mathbf{v}_\beta \Omega_\beta / \rho_\beta \tag{16}$$

An important feature of the modified equation of motion (16) is that it involves a convective velocity $\mathbf{w}_\beta$ that is different from the local fluid velocity $\mathbf{v}_\beta$. Because the convective velocity $\mathbf{w}_\beta$ is not *locally-defined* it cannot occur in *differential form* within the conservation equations [5]. This is because one cannot differentiate a function that is not locally, i.e. differentially, defined. To determine $\mathbf{w}_\beta$, one needs to go to the next higher scale (β+1) where $\mathbf{w}_\beta = \mathbf{v}_{\beta+1}$ becomes a local velocity. However, at this new scale one encounters yet another convective velocity $\mathbf{w}_{\beta+1}$ which is not known, requiring consideration of the higher scale (β+2). This unending chain constitutes the *closure problem* of the statistical theory of turbulence discussed earlier [5].

3 Connection Between the Modified Form of the Equation of Motion and the Navier-Stokes Equation

The original form of the *Navier-Stokes* equation with constant coefficients is given as [1, 2]

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} . \nabla \mathbf{v} = -\nabla P + \mu \nabla^2 \mathbf{v} + \frac{1}{3} \mu \nabla (\nabla . \mathbf{v}) \tag{17}$$

Since thermodynamic pressure P_t is an isotropic scalar, P in (17) is not P_t. Rather, the pressure P is generally identified as the *mechanical pressure* that is defined in terms of the total stress tensor $T_{ij} = -P_t \delta_{ij} + \tau_{ij}$ as [7]

$$P_m = -(1/3) T_{ii} = P_t - (1/3) \tau_{ii} \tag{18}$$

The normal viscous stress is given by (10) as $(1/3)\tau_{ii} = (1/3)\rho \mathbf{v}_i \mathbf{V}_{ii} = -(1/3)\mu \nabla . \mathbf{v}$ and since $\nabla P_t \approx 0$ because of isotropic nature of P_t, the gradient of (18) becomes

$$\nabla P = \nabla P_m = \nabla \frac{1}{3}\mu(\nabla.\mathbf{v}) = \frac{1}{3}\mu\nabla(\nabla.\mathbf{v}) \tag{19}$$

Substituting from (19) into (17), the *Navier-Stokes* equation assumes the form

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}.\nabla\mathbf{v} - \nu\nabla^2\mathbf{v} = 0 \tag{20}$$

that is almost identical to (16) with $\Omega_\beta = 0$ except that in (16) the convective velocity $\mathbf{w}_\beta$ is different from the local velocity $\mathbf{v}_\beta$. However, because (20) includes a diffusion term and the $\mathbf{w}_\beta$ and $\mathbf{v}_\beta$ are related by $\mathbf{v}_\beta = \mathbf{w}_\beta + \mathbf{V}_\beta$, it is clear that (20) should in fact be written as (16).

4 A Modified Theory of Laminar Flow by Free-Convection on a Vertical Hot Plate

Introducing the conventional boundary layer assumption $\partial^2/\partial x^2 \ll \partial^2/\partial y^2$, and neglecting transverse convective velocity $\mathbf{w}_y \ll \mathbf{w}_x$, the steady forms of the conservation equations (14)-(16) for momentum, energy, and mass in the absence of reactions Ω =0 but in the presence of buoyancy effects become

$$w_x \frac{\partial v_x}{\partial x} = \nu\frac{\partial^2 v_x}{\partial y^2} + g\left(\frac{T_w - T_\infty}{T_\infty}\right)\theta \tag{21}$$

$$w_x \frac{\partial \theta}{\partial x} = \alpha\frac{\partial^2 \theta}{\partial y^2} \tag{22}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{23}$$

that are subject to the boundary conditions

$$y = 0 \qquad \theta - 1 = v_x = 0 \tag{24a}$$

$$y = \infty \qquad \theta = v_x = 0 \tag{24b}$$

The dimensionless temperature is defined as

$$\theta = \frac{T - T_\infty}{T_w - T_\infty} \tag{25}$$

and T_w and T_∞ denote the constant temperature of the plate and the stationary fluid far away from the plate (Fig.1).

Following the classical studies [2, 10, 11] one introduces the similarity variable η and the stream function Ψ as

$$\eta = \frac{cy}{x^{1/4}} \quad , \qquad \Psi = 4\nu c x^{3/4} f(\eta) \tag{26}$$

where the parameter c is defined as

$$c = \left(\frac{g(T_w - T_\infty)}{4\nu^2 T_\infty}\right)^{1/4} \tag{27}$$

From (26) the axial and transverse velocities are obtained as

$$v_x = 4\nu c^2 x^{1/2} f' \tag{28a}$$

$$v_y = \nu c x^{-1/4}(\eta f' - 3f) \tag{28b}$$

The similarity variable η suggests that the local momentum boundary layer thickness varies as [2]

$$\delta \propto x^{1/4} \tag{29}$$

where the symbol $(\propto)$ denotes proportionality. Since the transverse diffusion of axial momentum and hence the local hydrodynamic boundary layer thickness $\delta(x)$ is governed by the diffusion length

$$\delta^2 = 2\nu t = 2\nu(x / w_x) \tag{30}$$

With the local time given by

$$t = x/w_x \tag{31}$$

(30) and (31) lead to the axial convective velocity

$$w_x \propto x^{1/2} \tag{32}$$

In view of the local axial velocity given in (28a), the axial convective velocity (32) that is independent of the transverse coordinate y and hence of $f(\eta)$ is expressed as

$$w_x = 4\nu c^2 x^{1/2} \tag{33}$$

With the definition of Ψ the continuity equation (23) is identically satisfied and substitutions from (26)-(28), and (33) in equations (21)-(22) lead to

$$\theta'' + \Pr \eta \theta' = 0 \tag{34}$$

$$f''' + \eta f'' - 2f' + \theta = 0 \tag{35}$$

With the definitions

$$\xi = \eta / \sqrt{2} \qquad , \qquad \phi = f' \tag{36}$$

and for unity *Prandtl* number Pr = 1, one obtains

$$\theta'' + 2\xi\theta' = 0 \tag{37}$$

$$\phi'' + 2\xi\phi' - 4\phi + 2\theta = 0 \tag{38}$$

that are subject to the boundary conditions

$$\xi = 0 \qquad \theta - 1 = \phi = 0 \tag{39a}$$
$$\xi = \infty \qquad \theta = \phi = 0 \tag{39b}$$

The temperature field is readily obtained from the solution of (37) and (39) as

$$\theta = 1 - \mathrm{erf}(\xi) \tag{40}$$

and the resulting temperature profile shown in Fig.1a will be compared with the classical results [2, 10-12] in the following.

Next, from coupling of (37) and (38) one obtains

$$R'' + 2\xi R' - 4R = 0 \qquad , \qquad R = \phi - \frac{\theta}{2} \tag{41}$$

By taking the first derivative of (41) and defining

$$G = R' = \frac{dR}{d\xi} \tag{42}$$

and going back to the original variable $\eta = \sqrt{2}\xi$ one obtains

$$G'' + \eta G' - G = 0 \tag{43}$$

that has the general solution

$$G = A\eta + B\left[\exp(-\eta^2/2) + \eta \int \exp(-\eta^2/2) d\eta\right] \tag{44}$$

where A and B are arbitrary constants. After substitution from (41)-(42) into (44) and integration of the resulting equation and the use of the result in (40) and application of the boundary conditions on ϕ in (39) one obtains

$$\phi(\eta) = \frac{\int_0^{\eta} z\left[1 - \mathrm{erf}(z/\sqrt{2})\right] dz}{\sqrt{2}\left\{\int_0^{\infty} z\left[1 - \mathrm{erf}(z/\sqrt{2})\right] dz - 1\right\}} - \frac{\left\{\int_0^{\infty} z\left[1 - \mathrm{erf}(z/\sqrt{2})\right] dz\right\} \mathrm{erf}(\eta/\sqrt{2})}{\sqrt{2}\left\{\int_0^{\infty} z\left[1 - \mathrm{erf}(z/\sqrt{2})\right] dz - 1\right\}} \tag{45}$$

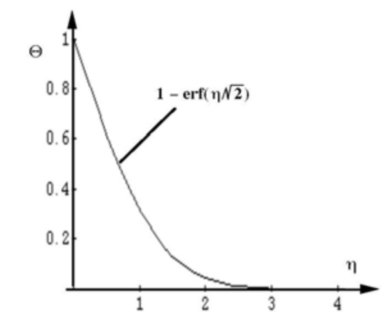
that can also be simplified to

$$\phi(\eta) = \sqrt{2}\left(\mathrm{erf}(\eta/\sqrt{2}) - 2\int_0^{\eta} z\left[1 - \mathrm{erf}(z/\sqrt{2})\right] dz\right) \tag{45a}$$

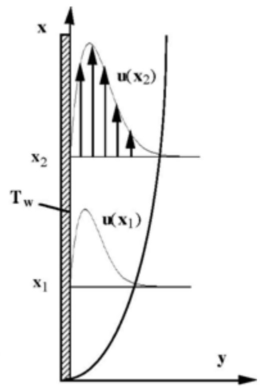
Since $\phi = f'$, the axial velocity (28a) can be expressed as

$$v_x = 4\nu c^2 x^{1/2} \phi \tag{46}$$

The calculated velocity profiles using (45)-(46) as a function of y at two axial locations (x_1, x_2) for parameter c = 1 are shown in Fig.1b and show qualitative agreement with the numerical solutions of the classical theory of *Pohlhausen* [2, 10] and *Ostrach* [11].



(a)



(b)

Fig.1 (a) Predicted temperature profile (b) Predicted axial velocity profiles at two axial locations.

The dimensionless axial velocity from (46) is next calculated from (45a) describing $\phi(\eta)$ as a function of η and shown in Fig.2 for direct comparison with the experimental data of *Schmidt* and *Beckmann* [2, 12]. As shown in Fig.2, for Pr = 1, the predicted velocity profile is only in qualitative agreement with the experimental data. However, in view of (34)-(35), (35a), and (52) the solution of the problem for general Pr may be written as

$$\phi(\eta) = \sqrt{2}\{\mathrm{erf}(\sqrt{\Pr}\eta/\sqrt{2}) \\ -2\Pr\int_0^\eta z\left[1-\mathrm{erf}(\sqrt{\Pr}z/\sqrt{2})\right]dz\} \qquad (45b)$$

Close agreement between the predictions and the data is achieved (Fig.2) with Pr = 0.5 as compared to Pr = 0.73 in the classical theories [2].

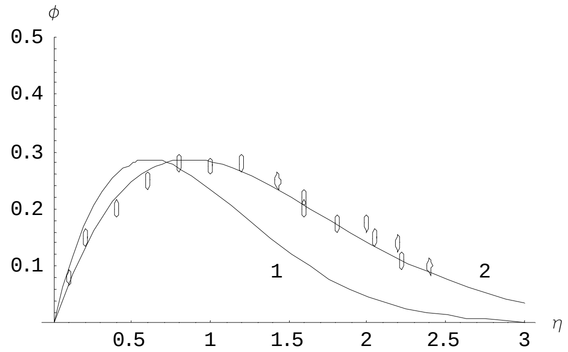


Fig.2 Comparison between the dimensionless velocity $\phi(\eta)$ and the data [12] (1) From (45a) with Pr = 1 (2) From (45b) with Pr = 0.5 .

It is interesting to examine the effect of the inclusion of the transverse convective velocity w_y that was neglected in the above analysis. The global continuity equation is

$$\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} = 0 \qquad (47)$$

and it is noted that in (47) the convective velocities (w_x, w_y) at scale β should be viewed as the local velocity of the next larger scale i.e. $w_\beta = v_{\beta-1}$, and as such are considered to be locally-defined differentiable functions [6]. After substitution in (47) from (33) one obtains the transverse convection velocity

$$w_y = -\frac{2\nu c^2 y}{x^{1/2}} \qquad (48)$$

The similarity solution being examined herein is considered to be valid only in the range x >>1 away from the plates leading edge. Therefore, the result (48) does indeed indicate that the transverse convective velocity is negligible for large x and thus justifies the assumption made in the previous analysis.

With the convective field (33) and (48) the energy and momentum conservation equations (22) and (21) will now assume the forms

$$\theta'' + 3\Pr\eta\theta' = 0 \qquad (49)$$

$$f''' + 3\eta f'' - 2f' + \theta = 0 \qquad (50)$$

that as compared to (34)-(35), reveal a closer correspondence with the classical theory [2, 10]. For Pr = 1, the coupling of (49)-(50) leads to

$$R'' + 3\eta R' - 2R = 0 \quad , \quad R = \phi - \frac{\theta}{2} \qquad (51)$$

that does not appear to have a simple analytical solution.

The assumption Pr = 1 is only necessary for the coupling between the temperature and the velocity fields (49)-(50). Therefore, in the present theory, the temperature field alone can be determined for arbitrary Prandtl numbers. For example, the solution of (34) and (39) is given by

$$\theta = 1 - \mathrm{erf}(\sqrt{\Pr/2}\eta) \qquad (52)$$

Similarly, the solution of (49) and (39) is given by

$$\theta = 1 - \mathrm{erf}(\sqrt{3\Pr/2}\eta) \qquad (53)$$

A direct comparison between the predicted temperature profiles and the data of *Schmidt* and *Beckmann* [2, 12] is shown in Fig.3.

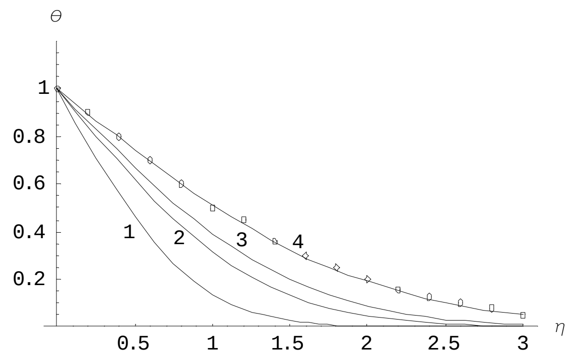


Fig.3 Comparisons of the predicted temperature profiles with the experimental data [12]. (1) Pr = 0.73 from (53) (2) Pr = 0.73 from (52) (3) Pr = 1 from (40) (4) Pr = 0.43 from (52).

The more complex problem of coupled temperature and velocity fields with transverse convection and for arbitrary Pr represented by the system (49)-(50) requires further future considerations.

6 Concluding Remarks

The energy equation and the modified equation of motion were solved for the classical problem of laminar flow by natural convection in the boundary layer adjacent to a hot vertical plate in the presence of gravitation. The predicted velocity profile was found to be in good agreement with the numerical calculations based on the classical theories [2, 10, 11] under the assumption Pr = 0.5 (Pr = 0.73) made in the present (classical) theory. The predicted temperature profiles for various values of Pr were compared with the classical results. For Pr = 0.43, the predicted temperature profile was found to be in excellent agreement with the experimental observations of *Schmidt* and *Beckmann* [12]. as well as the numerical calculations based on the classical theories of *Pohlhausen* [10] and *Ostrach* [11].

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