State Feedback Controller Design via Takagi-Sugeno fuzzy Model : A Linear Matrix Inequalities Approach

Farid KHABER, Abdelaziz HAMZAOUI, Khaled ZEHAR
QUERE Laboratory, Automatic Department, Setif University, 19000 SETIF, ALGERIA

Abstract: - In this paper, we introduce a robust state feedback controller design using Linear Matrix Inequalities (LMIs) and guaranteed cost approach for Takagi-Sugeno fuzzy systems. The purpose on this work is to establish a systematic method to design controllers for a class of uncertain linear and non linear systems. Our approach utilizes a certain type of fuzzy systems that are based on Takagi-Sugeno fuzzy models to approximate nonlinear systems. We use a robust control methodology to design controllers. This method not only guarantees stability, but also minimizes an upper bound on a linear quadratic performance measure. A simulation example is presented to show the effectiveness of this method.

Key-Words: - Takagi-Sugeno fuzzy model, State feedback, Linear Matrix Inequalities, Robust stability, Guaranteed cost.

1 Introduction

Recently there has been a great deal of interest in using dynamic Takagi-Sugeno fuzzy models to approximate nonlinear systems. This interest relies on the fact that dynamic T-S models are easily obtained by linearization of the nonlinear plant around different operating points. Once the T-S fuzzy models are obtained, linear control methodology can be used to design local state feedback controllers for each linear model. Aggregation of the fuzzy rules results in a generally nonlinear model, but in a very special form, which is exactly the same as a time varying and nonlinear system described by a set of Polytopic Linear Inclusions (PLDI) [1].

Since powerful convex optimization algorithms exist for dealing for these kind of systems, it is natural to use these algorithms for design of stabilizing T-S fuzzy controllers [2,3]. Sufficient conditions for the stability of T-S systems was first proposed in [4]. These sufficient conditions required the existence of a common positive definite matrix P which would satisfy a set of Lyapunov inequalities. Although looking for a common positive definite solution of the Linear Matrix Inequalities (LMI), which can be efficiently solved in polynomial time using the recently developed interior point method [5]. The stability of these systems have been discussed in detail in [2,6], but there have been few results that have gone beyond stability, and have considered performance. Authors in [3] have added the degree of stability, and have shown that controller design with guaranteed degree of stability can be transformed into Generalized Eigen Value Problem (GEVP) [1]. Recently the authors in [6,7] have added an LMI condition that guarantees the control action is norm-bounded, and therefore would not exceed a certain pre-defined limit.

In this paper, we generalize these results to the problem of minimizing the expected value of a quadratic performance measure with respect to randomized initial conditions, with zero mean and a covariance equal to the identity. Using the guaranteed cost approach [8,9], we minimize an upper bound on an LQ measure representing the control effort and the regulation error. We show that this problem can be transformed into a trace minimization problem, which can be solved using any of the available convex optimization software package (for example the Matlab LMI Control Toolbox [10]).

The structure of this paper is as follows: In section 2, we present an overview of dynamic Takagi Sugeno systems and their LMI formulation. Section 3 deals with the robust guaranteed cost performance problem and upper bound on the performance measure and it's formulation as LMIs. Simulation example is presented in section 4. Finally conclusions and some future work are discussed in section 5.

2 Takagi-Sugeno Fuzzy Model

A dynamic T-S fuzzy model is described by a set of fuzzy "IF … THEN" rules with fuzzy sets in the antecedents and dynamic linear time-invariant systems in the consequents. A generic T-S plant rule can be written as follows [11]:

***i*^th^ Plant Rule:** IF $x_1(t)$ *is* M_{i1} *and* $\ldots, x_n(t)$ *is* M_{in}

THEN $\dot{x} = A_i x + B_i u$

where $x \in \mathbb{R}^{n\times 1}$ is the state vector, $i = \{1,\ldots,n\}$, r is the number of rules, M_{ij} are input fuzzy sets, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, and $u \in \mathbb{R}^{m\times 1}$

Using singleton fuzzifier, max-product inference and center average defuzzifier, we can write the aggregated fuzzy model as:

$$\dot{x} = \frac{\sum_{i=1}^{r}\omega_i(x)(A_i x + B_i u)}{\sum_{i=1}^{r}\omega_i(x)} \tag{1}$$

where ω_i is defined as:

$$\omega_i(x) = \prod_{j=1}^{n}\mu_{ij(x_j)} \tag{2}$$

where μ_{ij} is the membership function of the *j*th fuzzy set in the *i*th rule. Defining

$$\alpha_i = \frac{\omega_i}{\sum_{i=1}^{r}\omega_i} \tag{3}$$

we can write (1) as:

$$\dot{x} = \sum_{i=1}^{r}\alpha_i(x)(A_i x + B_i u) \quad i = 1,\ldots,r \tag{4}$$

where $\alpha_i > 0$ and $\sum_{i=1}^{r}\alpha_i = 1$

Using the same method for generating T-S fuzzy rules for the controller, we have

***i*^th^ Controller Rule**: IF $x_1(t)$ *is* M_{i1} *and* $\ldots x_n(t)$ *is* M_{in}

THEN $u = -K_i x$

The over all controller would be

$$u = -\sum_{i=1}^{r}\alpha_i(x)K_i x \tag{5}$$

Replacing (5) in (4), we obtain the following equation for the closed loop system:

$$\dot{x} = \sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i(x)\alpha_j(x)(A_i - B_i K_i)x \tag{6}$$

We have the following theorem for stability.

Theorem 1 [2]: *The closed fuzzy system (6) is globally asymptotically stable if there exist a common positive definite matrix P which satisfies the following Lyapunov inequalities:*

$$\begin{aligned}&(A_i - B_i K_i)^T P + P(A_i - B_i K_i) < 0 \quad i = 1,\ldots,r\\ &G_{ij}^T P + P G_{ij} < 0 \quad i < j \le r\end{aligned} \tag{7}$$

where G_{ij} *is defined as*

$$G_{ij} = A_i - B_i K_j + A_j - B_j K_i \tag{8}$$

Pre-multiplying and post-multiplying both sides of inequalities in (7) by P^{-1} and using the following change of variables

$$\begin{aligned}&Y = P^{-1}\\ &X_i = K_i Y\end{aligned} \tag{9}$$

we obtain the following LMIs [1]:

$$\begin{aligned}&YA_i^T + A_i Y - B_i X_i - X_i^T B_i^T < 0 \quad i = 1,\ldots,r\\ &Y(A_i + A_j)^T + (A_i + A_j)Y - (B_i X_j + B_j X_i)\\ &-(B_i X_j + B_j X_i)^T < 0 \quad i < j \le r\end{aligned} \tag{10}$$

If the above LMIs have acommon positive definite solution, stability is guaranteed, but in most practical problems stability by itself in not enough, and the controller needs to satisfy certain design objectives. This will be discussed in the next section.

3 Robust Performance

In this section we try to achieve a certain level of performance for the uncertain system (6) using a guaranteed-cost approach [8]. It is a well known result from LQR theory that the problem of minimizing the cost function,

$$J = \int_0^\infty (x^T Q x + u^T R u)dt \tag{11}$$

Subject to

$$\dot{x} = Ax + Bu \quad ; \quad u = -Kx \tag{12}$$

reduces to finding a positive definite solution $P > 0$ of the following Lyapunov equation:

$$(A - BK)^T P + P(A - BK) + Q + K^T RK = 0 \tag{13}$$

where $Q \geq 0$ and $R > 0$. We can write the minimum cost of *J* as [8]: $\min\{J\} = x(0)^T Px(0)$

If we write the Lyapunov equation (13) as a matrix inequality instead of an equality, the solution of the inequality will be an upper bound on the performance measure J, and we can reach $\min\{J\}$ by minimizing that upper bound. While this result holds for a single LTI system, we can extend it to the case of equation (6). To avoid the dependency of the cost function J on initial conditions, we assume the initial conditions randomized with zero mean and identity covariance, i.e.,

$$\mathrm{E}\{x(0)\} = 0$$
$$\mathrm{E}\{x(0)x(0)^T\} = 0 \qquad (14)$$

where E is the expectation operator.

Our objective is to minimize the expected value of the performance index J with respect to all possible initial conditions with zero mean and covariance equal to the identity.

Lemma 1: *For random initial conditions with zero mean and covariance equal the identity, we have:*

$$\mathrm{E}_x(0)\{x(0)^T Px(0)\} = tr(P) \qquad (15)$$

where $tr(\bullet)$ denotes the trace of the matrix.

Using the above lemma, we can state the following theorem:

Theorem 2: *Consider the closed loop fuzzy system (6). The following bound on the performance objective J,*

$$J = \mathrm{E}_{x(0)} \int_0^\infty (x^T Qx + u^T Ru)dt < tr(P) \qquad (16)$$

where P is the solution of the following inequalities:

$$(A_i - B_i K_i)^T P + P(A_i - B_i K_i) + Q + \sum_{i=1}^{r} K_i^T RK_i < 0 \quad i = 1,\dots,r$$

$$G_{ij}^T P + PG_{ij} + Q + \sum_{i=1}^{r} K_i^T RK_i < 0 \quad i < j \leq r \qquad (17)$$

and the control law u is defined as in equation (5).

Proof: We already know that $J < tr(P)$ where P satisfies the following inequalities:

$$(A_i - B_i K_i)^T P + P(A_i - B_i K_i) + Q + \sum_{i=1}^{r} K_i^T RK_i < 0$$

$$G_{ij}^T P + PG_{ij} + Q + \sum_{i=1}^{r} K_i^T RK_i < 0 \begin{cases} i = 1,\dots,r \\ i < j \leq r \end{cases}$$

We just need to show that

$$(\sum_{i=1}^{r} \alpha_i K_i) R (\sum_{i=1}^{r} \alpha_i K_i^T) < \sum_{i=1}^{r} K_i^T RK_i \qquad (18)$$

For simplicity, we will show that the above inequality is true for the case where we only have two rules for the controller, the extension to more than two rules can be done via induction. We need to show that:

$$(\alpha_1 K_1 + \alpha_2 K_2)^T R(\alpha_1 K_1 + \alpha_2 K_2) < K_1^T RK_1 + K_2^T RK_2 \qquad (19)$$

To illustrate this, we rewrite the left hand side of the above equation as the following quadratic form:

$$\begin{bmatrix} K_1^T R^{1/2} & K_2^T R^{1/2} \end{bmatrix} \begin{bmatrix} \alpha_1^2 & \alpha_1\alpha_2 \\ \alpha_1\alpha_2 & \alpha_2^2 \end{bmatrix} \begin{bmatrix} R^{1/2} K_1 & R^{1/2} K_2 \end{bmatrix} \qquad (20)$$

The right hand side of the equation can be written as:

$$\begin{bmatrix} K_1^T R^{1/2} & K_2^T R^{1/2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R^{1/2} K_1 & R^{1/2} K_2 \end{bmatrix} \qquad (21)$$

To prove the theorem we have to show that:

$$\begin{bmatrix} \alpha_1^2 & \alpha_1\alpha_2 \\ \alpha_1\alpha_2 & \alpha_2^2 \end{bmatrix} < \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (22)$$

This is already satisfied since the difference of the two matrices is positive definite, i.e., we have the following

$$\begin{bmatrix} 1-\alpha_1^2 & \alpha_1\alpha_2 \\ \alpha_1\alpha_2 & 1-\alpha_2^2 \end{bmatrix} > 0 \qquad (23)$$

Now, using the same change of variables as in (9), and multiplying both sides of equation (17) by P^{-1} and also using theorem 1, we can write (17) as the following inequalities:

$$N_i + YQY + \sum_{i=1}^{r} X_i^T RX_i < 0 \quad i = 1,\dots,r$$

$$T_{ij} + YQY + \sum_{i=1}^{r} X_i^T RX_i < 0 \quad i < j \leq r \qquad (24)$$

where

$$N_i = YA_i^T + A_iY - B_iX_i - X_i^TB_i^T \quad (25)$$

$$S_{ij} = B_iX_j + B_jX_i \quad (26)$$

$$T_{ij} = Y(A_i + A_j)^T + (A_i + A_j)Y - S_{ij} - S_{ij}^T \quad (27)$$

Using the LMI lemma [8], we can write the above inequalities as follows:

$$\begin{bmatrix} N_i & YQ^{1/2} & X_1^TR^{1/2} & \cdots & X_r^TR^{1/2} \\ Q^{1/2}Y & -I_n & 0 & \cdots & 0 \\ R^{1/2}X_1 & 0 & -I_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R^{1/2}X_r & 0 & 0 & \cdots & -I_m \end{bmatrix} < 0$$

$$\begin{bmatrix} T_{ij} & YQ^{1/2} & X_1^TR^{1/2} & \cdots & X_r^TR^{1/2} \\ Q^{1/2}Y & -I_n & 0 & \cdots & 0 \\ R^{1/2}X_1 & 0 & -I_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R^{1/2}X_r & 0 & 0 & \cdots & -I_m \end{bmatrix} < 0$$

$i = 1, \ldots, r$ and $i < j \le r$ (28)

To obtain the least possible upper bound using a quadratic Lyapunov function, we have the following optimization problem:

$$\begin{cases} \textbf{Min } tr(Y^{-1}) \\ \textbf{Subject to } \text{LMIs in (28)} \end{cases} \quad (29)$$

This is a convex optimization problem which can be solved in polynomial time [5] using one of the available LMI toolboxes. To make it possible to use *Matlab LMI Toolbox* [10], we introduce an artificial variable Z as an upper bound on Y^{-1}, and minimize $tr(Z)$ instead, i.e, we recast the problem in the following form:

$$\begin{cases} \textbf{Min } tr(Z) \\ \textbf{Subject to } \text{LMIs in (28)} \end{cases} \text{ and } \begin{bmatrix} Z & I_n \\ I_n & Y \end{bmatrix} > 0$$

(30)

If the above LMIs are feasible, we can calculate the controller gains as

$$K_i = X_iY^{-1} \quad (31)$$

and u as in (5), i.e., we can write u as any convex combination of controller gains K_i , $i = 1, \ldots, n$.

4 Simulation example

To illustrate this design approach, consider the problem of balancing an inverted pendulum on a cart. We use the same model as in [2]. The equations for the motion of the pendulum are:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{g\sin(x_1) - amlx_2^2 s\text{in}(2x_1)/2 - a\cos(x_1)u}{4l/3 - aml\cos^2(x_1)}$$

where x_1 denotes the angle of the pendulum (in radians) from the vertical axis, and x_2 is the angular velocity of the pendulum, $g = 9.8m/s^2$ is the gravity constant, *m* is the mass of the pendulum, *M* is the mass of the cart, *2l* is the length of the pendulum, *u* is the force applied to the cart in Newtons, and $a = 1/(m+M)$. The simulations values are $m = 2\,kg$, $M = 8\,kg$ and $2l = 1\,m$. We approximate the nonlinear plant by two Takagi-Sugeno fuzzy rules. Note that the plant is not controllable for $x_1 = \pm\pi/2$, therefore we linearize the system around 80° instead. The plant rules are:

Plant rule (1): *If* x_1 *is close to zero Then*

$$\dot{x} = A_1x + B_1u$$

Plant rule (2): *If* x_1 *is close to* $\pm\pi/2$ *Then*

$$\dot{x} = A_2x + B_2u$$

where *close to zero* and *close to* $\pm\pi/2$ are the input fuzzy sets defined by the membership functions $\mu_1(x_1) = 1 - \frac{2}{\pi}|x_1|$ and $\mu_2(x_1) = \frac{2}{\pi}|x_1|$ respectively, depicted in figure 1, and A_1, A_2, B_1, B_2 are given as follows:

$$A_1 = \begin{bmatrix} 0 & 1 \\ 17.3 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ -0.177 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ 10.58 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ -0.03 \end{bmatrix}$$

Controller rule (1): *If* x_1 *is close to zero Then*

$$u = -K_1x$$

Controller rule (2): *If* x_1 *is close to* $\pm\pi/2$ *Then*

$$u = -K_2x$$

```
% Model parameters
global A; global B;
A1=[0 1;17.3 0]; B1=[0;-0.177];
A2=[0 1;9.45 0]; B2=[0;-0.03];
Q=[3 0;0 0]; QQ=sqrt(Q);
R=2; RR=sqrt(R);
A3=A1+A2;
% Calculus of P, K1, K2.
setlmis([]);
X1=lmivar(2,[1 2]) ;
X2=lmivar(2,[1 2]) ;
Y=lmivar(1,[2 1]) ;
Z=lmivar(1,[2 1]);
lmiterm([1 1 1],1,A1','s');
lmiterm([1 1 1 X1],B1,-1,'s');
lmiterm([1 2 1 Y],QQ,1);
lmiterm([1 2 2 0],-1);
lmiterm([1 3 1 X1],RR,1);
lmiterm([1 3 3 0],-1);
lmiterm([1 4 1 X2],RR,1);
lmiterm([1 4 4 0],-1);
lmiterm([2 1 1 Y],1,A2','s');
lmiterm([2 1 1 X2],B2,-1,'s');
lmiterm([2 2 1 Y],QQ,1);
lmiterm([2 2 2 0],-1);
lmiterm([2 3 1 X1],RR,1);
lmiterm([2 3 3 0],-1);
lmiterm([2 4 1 X2],RR,1);
lmiterm([2 4 4 0],-1);
lmiterm([3 1 1 Y],1,A3','s');
lmiterm([3 1 1 X1],B2,-1,'s');
lmiterm([3 1 1 X2],B1,-1,'s');
lmiterm([3 2 1 Y],QQ,1);
lmiterm([3 2 2 0],-1);
lmiterm([3 3 1 X1],RR,1);
lmiterm([3 3 3 0],-1);
lmiterm([3 4 1 X2],RR,1);
lmiterm([3 4 4 0],-1);
lmiterm([-4 1 1 Z],1,1);
lmiterm([-4 2 1 0],1);
lmiterm([-4 2 2 Y],1,1);
lmiterm([-5 1 1 Y],1,1);
lmis=getlmis;
[tmin,xfeas]=feasp(lmis)
x1=dec2mat(lmis,xfeas,X1)
x2=dec2mat(lmis,xfeas,X2)
y=dec2mat(lmis,xfeas,Y)
z=dec2mat(lmis,xfeas,Z)
P=inv(y)
K1=x1*P
K2=x2*P
```

Table 1. Matlab Program using LMI Control Toolbox

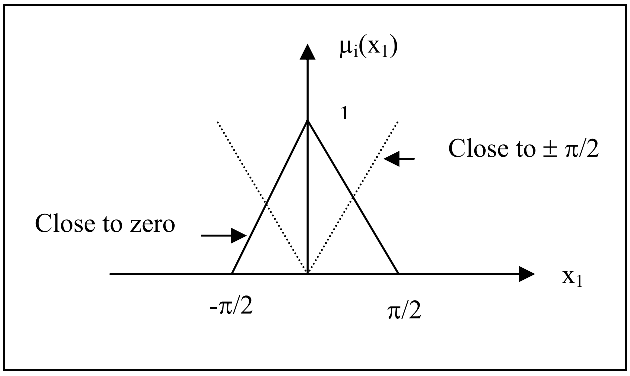

Fig. 1. Membership functions $\mu_i(x_1)$.

We also assume the following values of Q and R :

$$Q = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}, \ R = 2$$

Solving the LMI optimization problem in the previous section using the Matlab LMI Toolbox [10] (Matlab Program of Table 1), we obtain the following values :

$$P = 1.0e+005 \ \times \begin{bmatrix} 1.9821 & 0.6199 \\ 0.6199 & 0.1949 \end{bmatrix},$$

K_1 = [-429.6222 -122.8607],

K_2 = [-913.7748 -284.1638]

The resulting global controller is:

$$u = -\left(\mu_1(x_1)K_1 + \mu_2(x_1)K_2\right)x.$$

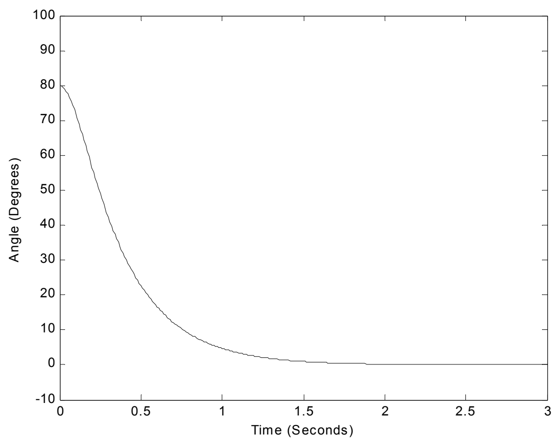

Fig. 2. Response of the pendulum angle.

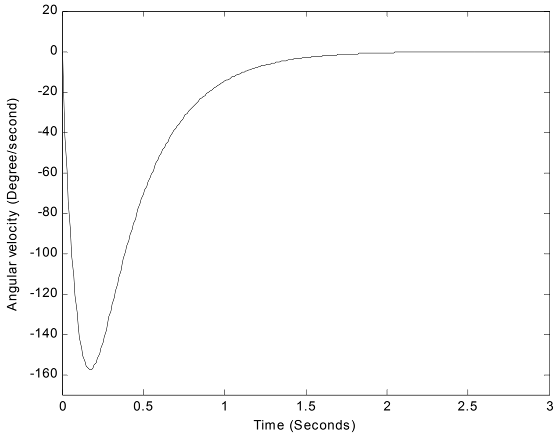

Fig. 3. Response of the pendulum angular velocity.

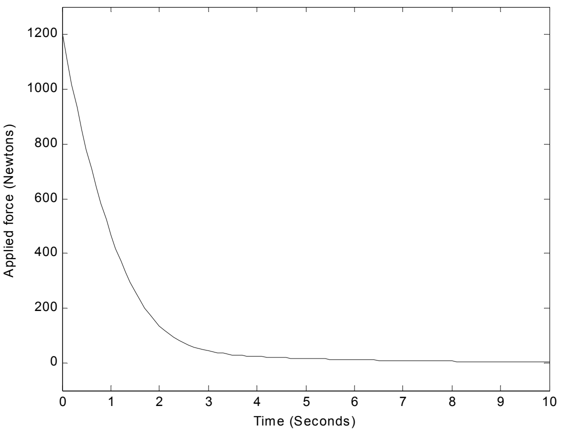

Fig. 4. Applied control.

Simulations indicate the above control law can balance the pendulum for initial conditions between [-80°, 80°].

Results are depicted in figs. 2-4. As it is evident from the simulation results, the controller gains are much smaller than the ones given in [2].

5 Conclusion

The purpose of this paper is was to present a simple design of Takagi-Sugeno fuzzy controllers. We presented a controller which minimizes an upper bound of a linear quadratic performance measure using the guarantees cost approach. The results obtained here can be extended to LQG scheme, using the observer strategy. Using the separation principle, we can design the observer and controller separately, and we still end up with LMIs.

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