Fractional Differential Operators and its Applications

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Abstract:- Fractional differential operators represent an interesting extension to the usual operators. They are analogous, but of different characteristics and they allow new applications. They can be implemented exactly through disctrete filters in the fractional splines functions' context.
In this paper, the principal properties of these operators are exposed, thogether with the numerical methods for its implementation. An application to the Radon Transform inversion is exposed.

Key-Words:- Fractional differential operator - Fractional B-spline functions - Ram Lak filter - Radon transform.

1 Introduction

If $s \in L^2(R)$, its generalized derivative of $n \in N$ order can be defined through the symbolic formula

$$(\widehat{D^n s})(\omega) = (\imath\omega)^n \hat{s}(\omega)$$

understood in the distribution sense. In the particular case of the centered **B**-spline functions or m order:

$$\hat{Q}^m(\omega) = \left(\frac{\text{sen}\omega/2}{\omega/2}\right)^{m+1}$$

these derivatives can be expressed, directly in the time domain, by centered differences operators. More precisely:

$$\begin{aligned} DQ^m(x) &= \Delta Q^{m-1}(x) \\ &= Q^{m-1}(x+1/2) - Q^{m-1}(x-1/2) \end{aligned}$$

This clearly gives a significant advantage to make the differential calculus in the spline functions space' context.

The differential operators D_*^α proposed by Unser and Blu in [1]:

$$(\widehat{D_*^\alpha s})(\omega) = |\omega|^\alpha \hat{s}(\omega)$$

take a generalization of the previous scheme of differences.

It is important to point out that this operation in the frequency domain equals to the action of a filtering operator that generalizes the classical *Ram-Lak* filter, used in the image tomographic reconstruction and other applications, [2-4].

We will expose here the definitions, results and the numerical methods that allow us to make the fractional differential calculus, in the signals and images process' context. Consequently, we will present an application to the implementation of the inverse Radon transform.

2 Centered Differences Operator

For sheer motivation, we will consier the centered differences of $n \in N_0$ order defined as:

$$\Delta_0^n s(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} s(x - k + n/2)$$

or equally, in the Fourier domain:

$$(\widehat{\Delta_0^n s})(\omega) = (1 - e^{-i\omega})^n e^{i\omega n/2} \hat{s}(\omega)$$

We can denote that $\Delta_0 = \Delta_0^1$, so
$\Delta_0^{n+1} = \Delta_0 \Delta_0^n$ and clearly, $\Delta_0^0 = 1$. Thus, for $m \in N_0$ the Q^m functions are defined through their transform:

$$\begin{aligned} \widehat{Q}^m(\omega) &= \left(\frac{1 - e^{-i\omega}}{i\omega}\right)^{m+1} e^{i\omega\frac{m+1}{2}} \\ &= \left(\frac{\text{sen}\omega/2}{\omega/2}\right)^{m+1} \end{aligned}$$

So we have:

$$Q^m(x) = \chi_{[-1/2,1/2]} * \cdots * \chi_{[-1/2,1/2]}(x)$$

on the other hand,

$$(i\omega)\widehat{Q}^m(\omega) = (1 - e^{-i\omega})e^{i\omega/2}\widehat{Q}^{m-1}(\omega)$$

This is:

$$\begin{aligned} DQ^m(x) &= \Delta_0 Q^{m-1}(x) \\ D^p Q^m(x) &= \Delta_0^p Q^{m-p}(x), \text{ if } p \in Z \end{aligned}$$

where $0 \leq p \leq m$. Moreover, it can be deduced from the previous formulas that:

$$Q^m(x) = \frac{\Delta_0^{m+1}}{m!}(x - k + \frac{m+1}{2})_+^m$$

where $f_+(x) = f(x)$ if $x \geq 0$ y $f_+(x) = 0$ in another case. The Q^m functions are polynomial splines functions of m order with knots in Z if m is odd and in $Z + 1/2$ if m is even.

Besides, the Δ_0 difference operator makes the D differential operator in these functions.

Nevertheless, the advantage is attenuated by the incongruence between the succeeding knot nets as the application of the operator Δ_0 modifies them in half integer.

On the other hand, as it has been said , the D_* differential operator represents the application in the transform domain, the antisymmetrical and complex filter $|\omega|$.

These results and considerations lead to the following development.

3 Fractional Difference Operator

For each $\alpha \in R_{>0}$, the centered fractional differences of α order can be defined from the following formula:

$$(\widehat{\Delta_*^\alpha s})(\omega) = |1 - e^{-i\omega}|^\alpha \, \hat{s}(\omega)$$

Let's observe previously that if $\alpha = 2n$:

$$(\widehat{\Delta_*^{2n} s})(\omega) = (-1)^n (\widehat{\Delta_0^{2n} s})(\omega)$$

On the other hand, if $\alpha \neq 2n$, considering that:

$$|1 - e^{-i\omega}|^\alpha = \sum_{k \in Z} (-1)^k \begin{vmatrix} \alpha \\ k \end{vmatrix} e^{-i\omega k}$$

where:

$$\begin{vmatrix} \alpha \\ k \end{vmatrix} = \frac{\Gamma(\alpha + 1)}{\Gamma(1 + \alpha/2 + k)\, \Gamma(1 + \alpha/2 - k)}$$

we finally have:

$$\begin{aligned} \Delta_*^\alpha s(x) &= \sum_{k \in Z} (-1)^k \begin{vmatrix} \alpha \\ k \end{vmatrix} s(x - k) \text{ si } \alpha \neq 2n \\ \Delta_*^{2n} s(x) &= \sum_{k=-n}^{n} (-1)^k \begin{vmatrix} 2n \\ k \end{vmatrix} s(x - k) \end{aligned}$$

Starting from the previous definitions, it is easy to verify that:

$$\Delta_*^\alpha s * \Delta_*^\eta s\ (x) = \Delta_*^{\alpha+\eta} s(x)$$

The Q_*^α fractional spline functions are defined from:

$$\hat{Q}_*^\alpha(\omega) = \left| \frac{1 - e^{-i\omega}}{\omega} \right|^{\alpha+1} = \left| \frac{\text{sen}\omega/2}{\omega/2} \right|^{\alpha+1}$$

Now, if $0 \leq \eta \leq \alpha$ we have:

$$\begin{aligned} |\omega|^\eta \, \hat{Q}_*^\alpha(\omega) &= \left|1 - e^{-i\omega}\right|^\eta \left| \frac{\text{sen}\omega/2}{\omega/2} \right|^{\alpha-\eta} \\ &= (\widehat{\Delta_*^\eta \, Q_*^{\alpha-\eta}})(\omega) \end{aligned}$$

and naturally, the fractional differential operator of η order is characterized from:

$$D_*^\eta \, Q_*^\alpha(x) = \Delta_*^\eta \, Q_*^{\alpha-\eta}(x) \quad \text{if } 0 \leq \eta \leq \alpha$$

This definition extends over the subespace of $L^2(R)$ generated by Q_*^α.

In order to simplify, we denote d_k^α to the coefficients of Δ_*^α operator. It can be demostrated that the sequence $d^\alpha = (d_k^\alpha)_{k \in Z}$ is absolutely summable.

Let's define V^α to subspaces of $L^2(R)$ generated by integer traslations of $Q_*^\alpha(x)$ fractional spline functions of α order and square summable.

Then, if $s(x) \in V^\alpha$:

$$s(x) = \sum_{n \in Z} s_n Q_*^\alpha(x - n)$$

with the sequence of the coefficients $s = (s_n)_{n \in N}$, of de square summable. Thus,

$$\begin{aligned} (\widehat{D_*^\eta s})(\omega) &= |\omega|^\eta \, \hat{s}(\omega) \\ &= |\omega|^\eta \, (\sum_n s_n e^{-i\omega n}) \, \hat{Q}_*^\alpha(\omega) \\ &= (\sum_n s_n e^{-i\omega n}) \, \widehat{D_*^\eta Q_*^\alpha}(\omega) \\ &= (\sum_n s_n e^{-i\omega n}) \, \widehat{\Delta_*^\eta \, Q_*^{\alpha-\eta}}(\omega) \end{aligned}$$

That is to say:

$$\begin{aligned} D_*^\eta s(x) &= \sum_n s_n \sum_k d_k^\eta Q_*^{\alpha-\eta}(x-k-n) \\ &= \sum_p (\sum_n s_n d_{p-n}^\eta)\, Q_*^{\alpha-\eta}(x-p) \\ &= \sum_p (s * d^\eta)(p)\, Q_*^{\alpha-\eta}(x-p) \end{aligned}$$

This is, summing up:

$$\tilde{s} \quad = \quad (s * d^\eta)$$

where $\tilde{s}$ represents the coefficients of the fractional derivatives in the $Q_*^{\alpha-\eta}(x-p)$ functions.

The fractional derivatives are completely characterized in V^α.

Let's define now:

$$\begin{aligned} x_*^\alpha &= \frac{|x|^\alpha}{-2\mathrm{sen}(\pi/2\alpha)} \quad \alpha \neq 2n \\ x_*^{2n} &= \frac{x^{2n}\,\log|x|}{(-1)^{1+n}\pi} \quad \alpha = 2n \end{aligned}$$

It can be demostrated (see [1]), that:

$$\begin{aligned} Q_*^\alpha(x) &= \frac{\Delta_*^{\alpha+1}\, x_*^\alpha}{\Gamma(\alpha+1)} \\ &= \frac{1}{2\mathrm{sen}(\pi/2\alpha)\,\Gamma(\alpha+1)} \cdot \\ &\quad \cdot \sum_k (-1)^{k+1} \left| \begin{array}{c} \alpha+1 \\ k \end{array} \right| |x-k|^\alpha, \end{aligned}$$

if $\alpha \neq 2n$

and:

$$\begin{aligned} Q_*^{2n}(x) &= \frac{\Delta_*^{2n+1}\, x_*^{2n}}{2n!} \\ &= \frac{(-1)^n}{2n!\pi} \sum_k (-1)^{k+1} \left| \begin{array}{c} 2n+1 \\ k \end{array} \right| \cdot \\ &\quad \cdot |x-k|^{2n}\, \log|x-k| \end{aligned}$$

4 Computation of the Coefficients of the Δ_*^α Differences

Let's denote $d_k^\alpha = (-1)^k \left| \begin{array}{c} \alpha \\ k \end{array} \right|$ to the coefficients that make the difference operation. If $\alpha \neq 2n$ from the previous formula, we get the following recursive relation:

$$\begin{aligned} \left| \begin{array}{c} \alpha \\ 0 \end{array} \right| &= \frac{4\,\Gamma(\alpha)}{\alpha\,(\Gamma(\alpha/2))^2} \\ \left| \begin{array}{c} \alpha \\ k+1 \end{array} \right| &= \left| \begin{array}{c} \alpha \\ k \end{array} \right| \frac{k-\alpha/2}{k+1+\alpha/2} \quad k \geq 0 \\ \left| \begin{array}{c} \alpha \\ -k \end{array} \right| &= \left| \begin{array}{c} \alpha \\ k \end{array} \right| \quad \forall \; k \in Z \end{aligned}$$

Analogously, if $\alpha = 2n$ we obtain the binomial coefficients:

$$\begin{aligned} \left| \begin{array}{c} 2n \\ k \end{array} \right| &= \left(\begin{array}{c} 2n \\ n-k \end{array} \right) \quad 0 \leq k \leq n \\ \left| \begin{array}{c} 2n \\ -k \end{array} \right| &= \left| \begin{array}{c} 2n \\ k \end{array} \right| \quad 0 \leq |k| \leq n \\ \left| \begin{array}{c} 2n \\ k \end{array} \right| &= 0 \quad |k| > n. \end{aligned}$$

These relations allow the efficient calculus of the coefficients, if $\alpha \neq 2n$ we have:

$$\begin{aligned} d_0^\alpha &= \frac{4\,\Gamma(\alpha)}{\alpha\,\Gamma^2(\alpha/2)} \\ d_{k+1}^\alpha &= d_k^\alpha \frac{k-\alpha/2}{k+1+\alpha/2} \quad k \geq 1 \\ d_{-k}^\alpha &= d_k^\alpha \end{aligned}$$

and we observe that $d_0^\alpha < 0$ and $d_k^\alpha > 0$ if $k > \lfloor 1+\alpha/2 \rfloor$. Particularly if $k < 1$ all the coefficients, but d_0^α, are positive. This conservation of the sign is particularly useful in the convolutions' calculus.

On the other hand, it can be deduced that if k is big enough:

$$|d_{k+p}^\alpha| \cong |d_k^\alpha| [\frac{k}{k+p}]$$

that is to say, d_k^α decay as $(\frac{1}{k})^{1+\alpha}$. It can be concluded that the coefficients d_k^α are absolutely summable.

From the previous results, we know that:

$$d^\alpha * d^\eta \quad = \quad d^{\alpha+\eta}$$

In particular if $\alpha + \eta$ is *even*, $d^{\alpha+\eta}$ can be represented through a finite vector.

5 Fractional Derivative of Functions in $L^2([0,T])$

The differential fractional calculus is trivial in the case of the T-periodic funtions. In effect, if:

$$s(x) = \sum_{k \in Z} c_k \, e^{ik\omega_0 x}$$

with $\omega_0 = 2\pi/T$, the impulse train results:

$$\hat{s}(\omega) = \sum_{k \in Z} c_k \, \delta(\omega - k\omega_0)$$

So, from $|\omega|^\alpha \, \hat{s}(\omega)$ the fractional derivatives are obtained:

$$D_*^\alpha s(x) = \sum_{k \in Z} (|k|\omega_0)^\alpha \, c_k \, e^{ik\omega_0 x}$$

This operation illustrates the fractional derivatives' sense, in this class of periodic functions. It equals, thus, to a ramp filter. This one increases the effect of high frequencies, keeping the phase of each elementary function. This is the main difference as regards the usual derivatives in the integer order case.

On the other hand, the convergence properties of the series are analogous to the ordinary derivation case. Moreover,if the coefficients c_k decay as $\frac{1}{|k|^r}$ the derivative $D_*^\alpha s$ is a function of $L^2([0,T])$, with $r - \alpha > 1/2$.

Conversely, possible singularities cause significant artifact. For this reason, the fractional derivative cannot fulfill itself in general, using the FFT.

6 Applications

The fractional derivative, as it has been pointed out, is equal to the application of a filter of the $|\omega|^\alpha$ type. This action attenuates low frequencies and tends to enhance the detail components of a signal. This makes it feasible to be used in the synthesis or border detections procedures. For this reason, it is proposed in [3-4] to be put into practice in the inverse of the Radon transform in the 2D image procedure, especially in the biomedical field.

This transform is defined in $L^2(R^2)$ as:

$$\begin{aligned} \mathcal{R}[s](\theta, t) &= \int_{R^2} s(x,y). \\ &\quad \delta(t - (x\cos(\theta) + y\mathrm{sen}(\theta))dxdy \end{aligned}$$

where δ is Dirac' delta. The inverse transform is obtained from the identity:

$$s(x,y) \quad = \quad \mathcal{R}^*[q * \mathcal{R}[s]](x,y)$$

where, q truly represents the ramp filter whose transform is $\widehat{q}(\omega) = \frac{|\omega|}{2\pi}$ and $\mathcal{R}^*$ is the adjoint operator of $\mathcal{R}$:

$$\mathcal{R}^*[p](x,y) \quad = \quad \int_0^\pi p(\theta, x\cos(\theta) + y\mathrm{sen}(\theta)) \, d\theta$$

For the reconstruction of medical images from the (θ_h, t_l) data recollected physically, which is assumed to correspond to a Radon transform of 2D objet. Discrete techniques are used, derived from the previous identity.

In this frame, the use of the fractional derivative is proposed in [3-4], to make filtering operations. For details, the cited bibliography has been proposed.

An implementation is presented here, developing for that purpose, a briefly explained and appropriate computational technique.

What is intented is to reconstruct the image represented by a (x_i, y_j) matrix, from the transform (θ_h, t_l) data. Let' s denote:

$$\rho_h(t) \quad = \quad \mathcal{R}[s](\theta_h, t)$$

to the functions with continuous argument t or *synograms*, for each argument θ_h. And, analogously, we denote:

$$\tilde{\rho}_h(t) \quad = \quad (q * \mathcal{R}[s])(\theta_h, t)$$

to the following filter synograms. The discrete reconstruction formula consist in an approximation to the integral expression:

$$s(x_i, y_j) \quad \cong \quad \sum_h w_h \tilde{\rho}_h (x_i \cos(\theta_h) + y_j \mathrm{sen}(\theta_h))$$

where w_h are appropriate integration weights. Let's observe that the synograms must be filtered from the discrete $\rho_h(t_l)$ versions, and later, interpolate the values in the filtered synograms

$$t = x_i \cos(\theta_h) + y_j \mathrm{sen}(\theta_h)$$

It can be asumed that the synograms are centered spline functions of order $\alpha > 1$, with knots

t_l. From this information the corresponding coefficients r_k can be computerized in the $(Q_*^\alpha(t-p))$ basis.

So as it was said, the application of the ramp filter equals to convolving these coefficients with the coefficients of the differences (d_n^1):

$$\tilde{r} \quad = \quad C_1(r * d^1)$$

with an appropriate C_1 constant, the coefficients of the filtered synogram are obtained in this way, in the basis of spline functions $(Q_*^{\alpha-1}(t-p))$.

From these coefficients, the values can be interpolated:

$$\begin{aligned} \tilde{\rho}(t) &= \sum_p \tilde{r}(p) Q_*^{\alpha-1}(t-p) \\ &= C_1 \sum_p (r * d^1)(p) Q_*^{\alpha-1}(t-p) \end{aligned}$$

and implement the reconstruction sum.

It is interesting to observe that if $\alpha = 2n+1$, integer *odd*, we obtain from the previous results the second formula:

$$\begin{aligned} \tilde{\rho}(t) &= C_2 \sum_p (r * (d^1 * d^{2n+1}))(p)(t-p)_*^{2n} \\ &= C_2 \sum_p (r * d^{2n+2})(p)(t-p)_*^{2n} \end{aligned}$$

for another C_2 constant. In this formula d^{2n+2} is a finite vector, which represents a strong computing advantage. On the other hand, this one makes the operation of interpolation immediately.

7 Conclusion

The fractional derivative operators are noteworthy analytical tools. Beyond its theoretical interest, their properties can be implemented in different applications, particularly in the signal and image processing, in the representation of the spline function' context. The associated calculus, in general, can be solved through discrete convolutions.

For this reasons previously exposed, in our opinion, this subjects deserves to be exploited in order to sharpen and broaden the exposed applications.

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