# Recent Results on Nonlinear, Optimal Regulation and Tracking: Theory and Applications

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*Abstract:* The topic of single or multiple objective, optimal control or design has been a very attractive and desirable feature for many dynamic and static systems such as aerospace, electrical, mechanical and robotics, nuclear and aerospace engineering. In particular, in dynamic optimization of nonlinear systems, recent research interest has been the nonlinear, optimal, feedback control using State-Dependent Riccati Equation (SDRE) arising in regulator and tracking problems. This topic can be thought of as the nonlinear counterpart of the most popular linear, quadratic regulator (LQR) based control design. This paper overviews the recent research results of the authors in the theory of the SDRE for regulation and tracking with applications to many engineering systems under deterministic and stochastic environments. Further, the paper presents some future research directions in the field.

*Key–Words:* Nonlinear control, Optimal control, SDRE

## 1 Introduction

The need to improve performance in controlled systems leads to more and more accurate modeling. However, if a model is a good representation of the real system over a wide range of operating points, it is most often nonlinear. Therefore, the previously used linear control techniques become inadequate and it becomes necessary to use some other nonlinear control techniques [1]. An infinite-horizon, nonlinear optimal regulator with complete attitude dynamics was derived for large maneuvers of asymmetric spacecraft in [2].

The State Dependent Riccati Equation (SDRE) has become a very attractive tool for the systematic design of nonlinear controllers, and it is a very effective algorithm for designing the nonlinear feedback control by allowing nonlinearities in the system states [3].

Moved by the high capability of the algebraic SDRE for regulation and tracking of infinite-horizon nonlinear systems [4, 5], this paper presents the differential SDRE, strictly speaking it could be called state dependent differential Riccati equation (SDDRE), technique for finite-horizon optimal control of nonlinear systems. The SDDRE is based on the substitution algorithm [6], that transforms the differential Riccati equation (DRE) to a linear differential Lyapunov equation (DLE) [7, 8]. At each time step, the coefficients of the linear differential Lyapunov equation are to be calculated and hold from current time to the next time step [9]. Then, during online implementation the resulting Lyapunov equation is to be solved in a closed form at each step. The use of Lyapunov-type equations in solving optimal problems is given in [10]. While the nonlinear, optimal, finite-horizon *regulation* technique was presented in [9], this paper presents the application of a nonlinear, optimal, finite-horizon *regulation* and *tracking* technique developed by [11] to a nonlinear stochastic systems. This is accomplished by integrating of the differential SDRE filter with the finite-horizon SDRE technique. In the application spectrum, a variety of engineering systems such as angle tracking of a gimbaled system in a missile seeker under three engagement scenarios, including fixed target, non-maneuvering target, and maneuvering target are presented to demonstrate the effectiveness of the developed technique. Other applications include regulation and tracking of an inverted pendulum, permanent magnet synchronous motor, a mechanical crane system, solar generator and DC motor, a robotic hand, and wind energy conversion system. A distinguishing feature of the research is the bridging the gap between software simulation and real world applications. Here, the method of hardware in the loop simulation (HILS) is employed by using an experimental setup based on a microcontroller board.

The structure of the paper is as follows: The finite-horizon control via differential SDRE is discussed in Section 2. Section 3 presents a brief overview of using the differential SDRE with incomplete state Information. Section 4 presents the finite-horizon nonlinear control simulation results. Finally, conclusions of this paper are given in Section 5.

# 2 Nonlinear, Finite-Horizon Optimal Control using Differential SDRE

Finite-horizon optimal control of nonlinear systems is a challenging problem in the control field due to the complexity of time-dependency of the Hamilton-Jacobi-Bellman (HJB) differential equation [9]. In finite-horizon optimal nonlinear control problem, the DRE cannot be solved in real time because the DRE arising in the optimal control can only be solved backward in time from its known final value. To overcome this problem, an approximate analytical approach is used [12] to convert the original nonlinear Ricatti equation to a linear DLE that can be solved in closed format at each time step.

## 2.1 Problem Formulation

The nonlinear system considered is assumed to be in the form:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(t), \quad (1)\\ \mathbf{y}(t) &= \mathbf{h}(\mathbf{x}). \quad (2) \end{aligned}$$

That nonlinear system can be expressed in a state-dependent linear-like form

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(\mathbf{x})\mathbf{x}(t) + \mathbf{B}(\mathbf{x})\mathbf{u}(t), \quad (3)\\ \mathbf{y}(t) &= \mathbf{C}(\mathbf{x})\mathbf{x}(t), \quad (4) \end{aligned}$$

where $\mathbf{f}(\mathbf{x}) = \mathbf{A}(\mathbf{x})\mathbf{x}(t)$, $\mathbf{B}(\mathbf{x}) = \mathbf{g}(\mathbf{x})$, and $\mathbf{h}(\mathbf{x}) = \mathbf{C}(\mathbf{x})\mathbf{x}(t)$.

Let $\mathbf{z}(t)$ be the desired, or reference output.

The goal is to find a state feedback, control law that minimizes a cost function given by

$$\mathbf{J}(\mathbf{x},\mathbf{u}) = \frac{1}{2}\mathbf{e}'(t_f)\mathbf{F}(\mathbf{x})\mathbf{e}(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\left[\mathbf{e}'(t)\mathbf{Q}(\mathbf{x})\mathbf{e}(t) + \mathbf{u}'(\mathbf{x})\mathbf{R}(\mathbf{x})\mathbf{u}(\mathbf{x})\right] dt, \quad (5)$$

where $\mathbf{Q}(\mathbf{x})$ and $\mathbf{F}(\mathbf{x})$ are symmetric positive semi-definite matrices, and $\mathbf{R}(\mathbf{x})$ is a symmetric positive definite matrix. Moreover, $\mathbf{x}'\mathbf{Q}(\mathbf{x})\mathbf{x}$ is a measure of state accuracy and $\mathbf{u}'(\mathbf{x})\mathbf{R}(\mathbf{x})\mathbf{u}(\mathbf{x})$ is a measure of control effort, and the error $\mathbf{e}$ is the difference between the reference output and the actual output, $\mathbf{e}(t) = \mathbf{z}(t) - \mathbf{y}(t)$ .

## 2.2 Solution for Finite-Horizon Optimal Control using Differential SDRE

To minimize the above cost function (5), a feedback control law can be given as

$$\mathbf{u}(\mathbf{x}) = -\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}'(\mathbf{x})[\mathbf{P}(\mathbf{x})\mathbf{x} - \mathbf{g}(\mathbf{x})], \quad (6)$$

where $\mathbf{P}(\mathbf{x})$ is a symmetric, positive-definite solution of the differential SDRE of the form

$$-\dot{\mathbf{P}}(\mathbf{x}) = \mathbf{P}(\mathbf{x})\mathbf{A}(\mathbf{x}) + \mathbf{A}'(\mathbf{x})\mathbf{P}(\mathbf{x}) - \mathbf{P}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}'(\mathbf{x})\mathbf{P}(\mathbf{x}) + \mathbf{C}'(\mathbf{x})\mathbf{Q}(\mathbf{x})\mathbf{C}(\mathbf{x}), \quad (7)$$

with the final condition

$$\mathbf{P}(\mathbf{x}, t_f) = \mathbf{C}'(t_f)\mathbf{F}(\mathbf{x})\mathbf{C}(t_f), \quad (8)$$

and $\mathbf{g}(\mathbf{x})$ is a solution of the state-dependent non-homogeneous vector differential equation

$$\dot{\mathbf{g}}(\mathbf{x}) = -[\mathbf{A}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\mathbf{R}^{-1}\mathbf{B}'(\mathbf{x})\mathbf{P}(\mathbf{x})]'\mathbf{g}(\mathbf{x}) - \mathbf{C}'(\mathbf{x})\mathbf{Q}(\mathbf{x})\mathbf{z}(\mathbf{x}), \quad (9)$$

with the final condition

$$\mathbf{g}(\mathbf{x}, t_f) = \mathbf{C}'(t_f)\mathbf{F}(\mathbf{x})\mathbf{z}(t_f). \quad (10)$$

The resulting differential SDRE-controlled trajectory becomes the solution of the state-dependent closed-loop dynamics

$$\dot{\mathbf{x}}(t) = [\mathbf{A}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}'(\mathbf{x})\mathbf{P}(\mathbf{x})]\mathbf{x}(t) + \mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}'(\mathbf{x})\mathbf{g}(\mathbf{x}). \quad (11)$$

As the differential SDRE is a function of $(\mathbf{x}, t)$, we do not know the value of the states ahead of present time step. Consequently, the state dependent coefficients cannot be calculated to solve (7 & 9 ) with the final condition (8 & 10 ) by backward integration from $t_f$ to $t_0$ respectively. To overcome this problem, an approximate analytical approach is used [12], which converts the original nonlinear differential Ricatti equation to a linear DLE, which can be solved in closed form at each time step [7, 6]. In order to solve both (7 & 9 ), one can follow the following steps at each time step [9, 11]:

- Solve the ARE to calculate the steady state value $\mathbf{P_{ss}}(\mathbf{x})$

$$\mathbf{P_{ss}}(\mathbf{x})\mathbf{A}(\mathbf{x}) + \mathbf{A}'(\mathbf{x})\mathbf{P_{ss}}(\mathbf{x}) - \mathbf{P_{ss}}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}'(\mathbf{x})\mathbf{P_{ss}}(\mathbf{x}) + \mathbf{Q}(\mathbf{x}) = 0. \quad (12)$$

- Use change-of-variables technique and assume that

$$\mathbf{K}(\mathbf{x}, \mathbf{t}) = [\mathbf{P}(\mathbf{x}, \mathbf{t}) - \mathbf{P_{ss}}(\mathbf{x})]^{-1}. \quad (13)$$

- Calculate the value of $\mathbf{A_{cl}}(\mathbf{x})$ as

$$\mathbf{A_{cl}}(\mathbf{x}) = \mathbf{A}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}'(\mathbf{x})\mathbf{P_{ss}}(\mathbf{x}). \quad (14)$$

- Calculate the value of $\mathbf{D}$ by solving the algebraic Lyapunov equation [13]

$$\mathbf{A_{cl}}\mathbf{D} + \mathbf{D}\mathbf{A}'_{\mathbf{cl}} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}' = 0. \quad (15)$$

- Solve the DLE

$$\dot{\mathbf{K}}(\mathbf{x}, t) = \mathbf{K}(\mathbf{x}, t)\mathbf{A}'_{\mathbf{cl}}(\mathbf{x}) + \mathbf{A_{cl}}(\mathbf{x})\mathbf{K}(\mathbf{x}, t) - \mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}'(\mathbf{x}). \quad (16)$$

The solution of (38), as shown by [14], is given by

$$\mathbf{K}(\mathbf{x}, t) = \mathbf{e}^{\mathbf{A_{cl}}(\mathbf{t}-\mathbf{t_f})}(\mathbf{K}(\mathbf{x}, t_f) - \mathbf{D})\mathbf{e}^{\mathbf{A_{cl}}'(\mathbf{t}-\mathbf{t_f})} + \mathbf{D}. \quad (17)$$

- Use change-of-variables technique to calculate the value of $\mathbf{P}(\mathbf{x}, \mathbf{t})$ from (35)

$$\mathbf{P}(\mathbf{x}, t) = \mathbf{K}^{-1}(\mathbf{x}, t) + \mathbf{P_{ss}}(t). \quad (18)$$

- Calculate the steady state value $\mathbf{g_{ss}}(\mathbf{x})$ from the equation

$$\mathbf{g_{ss}}(\mathbf{x}) = [\mathbf{A}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\mathbf{R}^{-1}\mathbf{B}'(\mathbf{x})\mathbf{P_{ss}}(\mathbf{x})]'^{-1} \mathbf{C}'(\mathbf{x})\mathbf{Q}(\mathbf{x})\mathbf{z}(\mathbf{x}). \quad (19)$$

- Use change-of-variables technique and assume that

$$\mathbf{K_g}(\mathbf{x}, t) = [\mathbf{g}(\mathbf{x}, t) - \mathbf{g_{ss}}(\mathbf{x})]. \quad (20)$$

- Solve the differential equation

$$\mathbf{K_g}(\mathbf{x}, t) = \mathbf{e}^{-(\mathbf{A}-\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\mathbf{P})'(\mathbf{t}-\mathbf{t_f})}[\mathbf{g}(\mathbf{x}, t_f) - \mathbf{g_{ss}}(\mathbf{x})]. \quad (21)$$

- Use change-of-variables technique to calculate the value of $\mathbf{g}(\mathbf{x}, t)$

$$\mathbf{g}(\mathbf{x}, t) = \mathbf{K_g}(\mathbf{x}, t) + \mathbf{g_{ss}}(\mathbf{x}). \quad (22)$$

- Calculate the value of the optimal control $\mathbf{u}(\mathbf{x}, t)$ as

$$\mathbf{u}(\mathbf{x}, t) = -\mathbf{R}^{-1}\mathbf{B}'(\mathbf{x})[\mathbf{P}(\mathbf{x}, t)\mathbf{x}(t) - \mathbf{g}(\mathbf{x}, t)]. \quad (23)$$

Fig.1 summarizes the overview of the flow chart of finite-horizon differential SDRE tracking technique.

At least for linear systems, if $t_0 \ll t_f$, the solution of the DRE convergences to that of ARE, and $\mathbf{K}(\mathbf{x}, t) = [\mathbf{P}(\mathbf{x}, t) - \mathbf{P_{ss}}(\mathbf{x})]^{-1}$ becomes singular. To avoid that, the negative definite solution of the ARE can be calculated instead of the positive definite solution, and in this case $[\mathbf{P}(\mathbf{x}, t) - \mathbf{P_{ss}}(\mathbf{x})]$ is guaranteed to be the positive definite, hence, its inverse always exists. This approach works for nonlinear case as well [6]. For calculation of the negative definite solution of the ARE, it suffices to flip the sign of matrix $\mathbf{A}(\mathbf{x})$ and solve the ARE for the positive definite solution, then by using the negative of $\mathbf{P_{ss}}(\mathbf{x})$, the negative definite solution of the original ARE can be obtained [9].

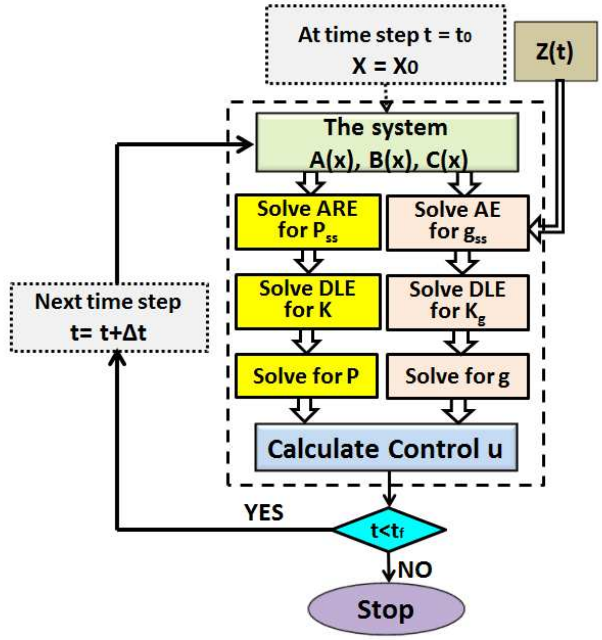


Figure 1: Flow Chart for the Finite-Horizon Differential SDRE Tracking Technique

**Note 1 :** This technique for finite-horizon differential SDRE can be applied for both regulation and tracking problems for nonlinear systems. In finite horizon regulation problems, the reference output $\mathbf{z}(t)$ is assumed to be *zero*.
**Note 2 :** It is easily seen that this technique with finite-horizon differential SDRE can be used for linear systems and the resulting differential SDRE becomes the standard DRE [15].

# 3 Nonlinear, Finite-Horizon Control with Incomplete State Information via SD-DRE

## 3.1 Optimal Estimation

Suppose that the entire state $\mathbf{x}(t)$ is not available, but only the output $\mathbf{y}(t)$ is measurable. Let us reproduce the nonlinear system with noises in state dependent form

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x})\mathbf{x}(t) + \mathbf{B}(\mathbf{x})\mathbf{u}(t) + \mathbf{B}_w(t)\mathbf{w}(t), \quad (24)$$
$$\mathbf{y}(t) = \mathbf{C}(\mathbf{x})\mathbf{x}(t) + \mathbf{v}(t), \quad (25)$$

where, $\mathbf{w}(t)$ and $\mathbf{v}(t)$ are process, and measurement (white, Gaussian, zero mean) random noises, respectively. Note that the system (24,25) is a nonlinear system, but in a linear form. As the nonlinearity will be found in $\mathbf{A}(\mathbf{x})$, $\mathbf{B}(\mathbf{x})$, and $\mathbf{C}(\mathbf{x})$. This form is called linear-like form.

At each time step, in order to find the best estimate $\hat{\mathbf{x}}(t)$, the state estimate equations are

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t) &= \mathbf{A}(\hat{\mathbf{x}})\hat{\mathbf{x}}(t) + \mathbf{B}(\hat{\mathbf{x}})\mathbf{u}(t) \\ &+\mathbf{K}_e(\hat{\mathbf{x}}, t)[\mathbf{y}(t) - \mathbf{C}(\hat{\mathbf{x}})\hat{\mathbf{x}}(t)]; \quad (26) \\ \hat{\mathbf{x}}(t_0) &= \bar{\mathbf{x}}(t_0),\end{aligned}$$

where, $\mathbf{K}_e(\hat{\mathbf{x}}, t)$, the optimal Kalman estimator gain, is obtained as

$$\mathbf{K}_e(\hat{\mathbf{x}}, t) = \mathbf{P}_e(\hat{\mathbf{x}}, t)\mathbf{C}'(\hat{\mathbf{x}})\mathbf{R}_v^{-1}(t), \quad (27)$$

$\mathbf{P}_e(\hat{\mathbf{x}}, t)$ is the solution of the matrix differential Riccati equation

$$\begin{aligned}\dot{\mathbf{P}}_e(\hat{\mathbf{x}}, t) &= \mathbf{A}(\hat{\mathbf{x}})\mathbf{P}_e(\hat{\mathbf{x}}, t) + \mathbf{P}_e(\hat{\mathbf{x}}, t)\mathbf{A}'(\hat{\mathbf{x}}) + \\ \mathbf{B}_w(t)\mathbf{Q}_w(t)\mathbf{B}_w'(t) &- \mathbf{P}_e(\hat{\mathbf{x}}, t)\mathbf{C}'(\hat{\mathbf{x}})\mathbf{R}_v^{-1}(t)\mathbf{C}(\hat{\mathbf{x}})\mathbf{P}_e(\hat{\mathbf{x}}, t), \quad (28)\end{aligned}$$

is to be solved in *forward* direction with initial condition $\mathbf{P}_e(\hat{\mathbf{x}}, t_0) = \mathbf{P}_{e0}$ for any real-time implementation, whereas the standard differential Riccati equation, arising in the control problem, is to be solved in *backward* direction with a given final condition. The minimization of the optimal cost function is equivalent to minimization of

$$\begin{aligned}J_a(\mathbf{x}, u) &= \mathcal{E}[\frac{1}{2}\hat{\mathbf{x}}'(t_f)\mathbf{F}\hat{\mathbf{x}}(t_f) + \\ \frac{1}{2}\int_{t_0}^{t_f} &[\hat{\mathbf{x}}'(t)\mathbf{Q}(\hat{\mathbf{x}})\hat{\mathbf{x}}(t) + \mathbf{u}'(\hat{\mathbf{x}}, t)\mathbf{R}(\hat{\mathbf{x}})\mathbf{u}(\hat{\mathbf{x}}, t)dt]. \quad (29)\end{aligned}$$

where $\mathbf{Q}(\mathbf{x})$ and $\mathbf{F}$ are symmetric *positive semi-definite* matrices, and $\mathbf{R}(\mathbf{x})$ is a symmetric *positive definite* matrix. Moreover, $\mathbf{x}'\mathbf{Q}(\mathbf{x})\mathbf{x}$ is a measure of state accuracy and $\mathbf{u}'(\mathbf{x})\mathbf{R}(\mathbf{x})\mathbf{u}(\mathbf{x})$ is a measure of control effort.

## 3.2 Optimal Control

Finite-horizon optimal control of nonlinear systems is a challenging problem in the control field due to the complexity of time-dependency of the Hamilton-Jacobi-Bellman (HJB) differential equation [9]. In finite-horizon optimal nonlinear control problem, the differential Riccati equation (DRE) cannot be solved in real time because the DRE arising in the optimal control can only be solved backward in time from its known final value. To overcome this problem, an approximate analytical approach is used [12] to convert the original nonlinear Ricatti equation to a linear DLE that can be solved in closed format at each time step [11].

At each time step, using the results of finite-horizon nonlinear regulator obtained in [16] except that the state is now the optimal estimate $\hat{\mathbf{x}}(t)$

$$\begin{aligned}\mathbf{u}(\hat{\mathbf{x}}, t) &= -\mathbf{R}^{-1}(\hat{\mathbf{x}})\mathbf{B}'(\hat{\mathbf{x}})\mathbf{P}_c(\hat{\mathbf{x}}, t)\hat{\mathbf{x}}(t) \\ &= -\mathbf{K}_c(\hat{\mathbf{x}}, t)\hat{\mathbf{x}}(t), \quad (30)\end{aligned}$$

where, $\mathbf{K}_c(\hat{\mathbf{x}}, t) = \mathbf{R}^{-1}(\hat{\mathbf{x}})\mathbf{B}'(\hat{\mathbf{x}})\mathbf{P}_c(\hat{\mathbf{x}}, t)$, is the Kalman *controller* gain and $\mathbf{P}_c(\hat{\mathbf{x}}, t)$ is a symmetric, positive-definite solution of the SD-DRE.
The entire algorithm of combined *estimation* and *control* leading to nonlinear regulator problem is shown in the following steps [11, 12]:

### 3.2.1 Optimal Estimator

- At each time step, solve the matrix differential Riccati equation

$$\begin{aligned}\dot{\mathbf{P}}_e(\hat{\mathbf{x}}, t) &= \mathbf{A}(\hat{\mathbf{x}})\mathbf{P}_e(\hat{\mathbf{x}}, t) + \mathbf{P}_e(\hat{\mathbf{x}}, t)\mathbf{A}'(\hat{\mathbf{x}}) + \\ \mathbf{B}_w(t)\mathbf{Q}_w(t)\mathbf{B}_w'(t) &- \mathbf{P}_e(\hat{\mathbf{x}}, t)\mathbf{C}'(\hat{\mathbf{x}})\mathbf{R}_v^{-1}(t)\mathbf{C}(\hat{\mathbf{x}})\mathbf{P}_e(\hat{\mathbf{x}}, t), \quad (31)\end{aligned}$$

  in the forward direction with the initial condition $\mathbf{P}_e(\hat{\mathbf{x}}, t_0) = \mathbf{P}_{e0}$.

- Obtain the optimal estimator (filter) gain as

$$\mathbf{K}_e(\hat{\mathbf{x}}, t) = \mathbf{P}_e(\hat{\mathbf{x}}, t)\mathbf{C}'(\hat{\mathbf{x}})\mathbf{R}_v^{-1}(t). \quad (32)$$

- Solve for the optimal state estimate $\hat{\mathbf{x}}(t)$ from the equation

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t) &= \mathbf{A}(\hat{\mathbf{x}})\hat{\mathbf{x}}(t) + \mathbf{B}(\hat{\mathbf{x}})\mathbf{u}(t) + \\ &\mathbf{K}_e(\hat{\mathbf{x}}, t)[\mathbf{y}(t) - \mathbf{C}(\hat{\mathbf{x}})\hat{\mathbf{x}}(t)], \quad (33)\end{aligned}$$

  with the initial condition $\hat{\mathbf{x}}(t_0) = \bar{\mathbf{x}}_0$.

### 3.2.2 Optimal Controller

- Solve the algebraic Riccati equation (ARE) to calculate the steady state value $\mathbf{P_{ss}}(\hat{\mathbf{x}})$

$$\begin{aligned}&\mathbf{P_{ss}}(\hat{\mathbf{x}})\mathbf{A}(\hat{\mathbf{x}}) + \mathbf{A}'(\hat{\mathbf{x}})\mathbf{P_{ss}}(\hat{\mathbf{x}}) - \\ &\mathbf{P_{ss}}(\hat{\mathbf{x}})\mathbf{B}(\hat{\mathbf{x}})\mathbf{R}^{-1}(\hat{\mathbf{x}})\mathbf{B}'(\hat{\mathbf{x}})\mathbf{P_{ss}}(\hat{\mathbf{x}}) + \mathbf{Q}(\hat{\mathbf{x}}) = 0. \quad (34)\end{aligned}$$

- Use change-of-variables procedure and assume that

$$\mathbf{K}(\hat{\mathbf{x}}, t) = [\mathbf{P}(\hat{\mathbf{x}}, t) - \mathbf{P_{ss}}(\hat{\mathbf{x}})]^{-1}. \quad (35)$$

- Calculate the value of $\mathbf{A_{cl}}(\hat{\mathbf{x}})$ as

$$\mathbf{A_{cl}}(\hat{\mathbf{x}}) = \mathbf{A}(\hat{\mathbf{x}}) - \mathbf{B}(\hat{\mathbf{x}})\mathbf{R^{-1}}(\hat{\mathbf{x}})\mathbf{B}'(\hat{\mathbf{x}})\mathbf{P_{ss}}(\hat{\mathbf{x}}). \quad (36)$$

- Calculate the value of $\mathbf{D}$ by solving the algebraic Lyapunov equation [13]

$$\mathbf{A_{cl}}\mathbf{D} + \mathbf{D}\mathbf{A}'_{\mathbf{cl}} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}' = 0. \quad (37)$$

- Solve the differential Lyapunov equation

$$\dot{\mathbf{K}}(\hat{\mathbf{x}}, t) = \mathbf{K}(\hat{\mathbf{x}}, t)\mathbf{A}'_{\mathbf{cl}}(\hat{\mathbf{x}}) + \mathbf{A_{cl}}(\hat{\mathbf{x}})\mathbf{K}(\hat{\mathbf{x}}, t) - \mathbf{B}(\hat{\mathbf{x}})\mathbf{R}^{-1}(\hat{\mathbf{x}})\mathbf{B}'(\hat{\mathbf{x}}). \quad (38)$$

The solution of (38), as shown by [14], is given by

$$\mathbf{K}(\hat{\mathbf{x}}, t) = e^{\mathbf{A_{cl}}(t-t_f)}(\mathbf{K}(\hat{\mathbf{x}}, t_f) - \mathbf{D})e^{\mathbf{A}'_{\mathbf{cl}}(t-t_f)} + \mathbf{D}. \quad (39)$$

- Use change-of-variables procedure to calculate the value of $\mathbf{P_c}(\hat{\mathbf{x}}, t)$

$$\mathbf{P_c}(\hat{\mathbf{x}}, t) = \mathbf{K}^{-1}(\hat{\mathbf{x}}, t) + \mathbf{P_{ss}}(\hat{\mathbf{x}}). \quad (40)$$

- Finally, calculate the value of the optimal control $\mathbf{u}(\hat{\mathbf{x}}, t)$ as

$$\mathbf{u}(\hat{\mathbf{x}}, t) = -\mathbf{R}^{-1}(\hat{\mathbf{x}})\mathbf{B}'(\hat{\mathbf{x}})\mathbf{P_c}(\hat{\mathbf{x}}, t)\hat{\mathbf{x}}(t). \quad (41)$$

## 4 Simulation Results

Time On Target (TOT) is the military co-ordination of artillery fire by many weapons so that all the munitions arrive at the target at precisely the same time. This is useful because more attacks can land on the target at the same time with no warning, and that will improve the overall attack accuracy.

The developed nonlinear finite-horizon tracking technique with incomplete state information is implemented for a motor attached to the tracker of two powered vehicles with the requirement to hit a stationary target at the same time, and these vehicles are initially at different ranges and aspect angles from the target, as illustrated in Fig. 2.

As shown in Fig. 2, the line of sight between the stationary target and vehicle A is more than the line of sight between vehicle B and the same target. To achieve a successful TOT, it's required that the munition from the further vehicle, vehicle B, to make a certain maneuver whereas the munition from vehicle A to make a direct path with no maneuver. This scenario can grantee that the munitions from both vehicles A and B hit the target at the same time, is shown in Fig. 3.

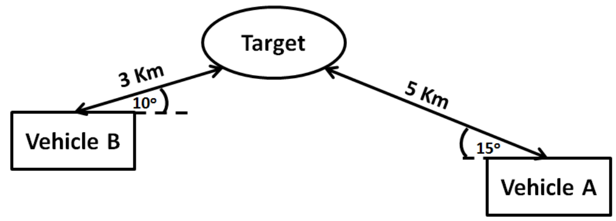


Figure 2: Time On Target Illustrative Diagram

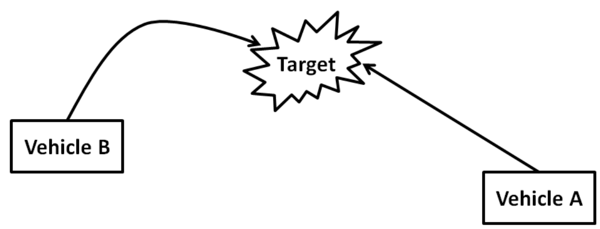


Figure 3: Time On Target Attack Diagram

The dynamic equations for the tracker system

$$V(t) = L\frac{di(t)}{dt} + Ri(t) + k_b\frac{d\theta(t)}{dt}, \quad (42)$$

$$ml^2\frac{d^2\theta(t)}{dt^2} = -mglsin(\theta(t)) - k_m i(t), \quad (43)$$

where, $V$ is the control voltage, $L$ is the motor inductance, $i$ is the current through the motor winding, $R$ the motor winding resistance, $k_b$ the motor's back electro magnetic force constant, $\theta$ the error angle, $m$ the mass of tracker, $g$ the gravitational constant, and $k_m$ the damping (friction) constant.

The nonlinear state equations for the system are written in the state dependent form

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{\frac{g}{l}sin(x_1)}{x_1} & 0 & \frac{k_m}{ml^2} \\ 0 & -\frac{k_b}{L} & \frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} u, \quad (44)$$

where: $\theta = x_1$ , $\dot{\theta} = x_2$ , $i = x_3$, and $V = u$.

### 4.1 Vehicle A

Consider a direct path (shot) to the target, in this case the desired tracker angle will be $z(t) = 0^o$, i.e *the problem is now a regulator problem.*

The simulations were performed for final time of 6 seconds, and the engagement scenario is shown in Fig. 4. The resulting trajectories for the demanded and achieved tracker angles are presented in Fig. 5, and the error is shown in Fig. 6.

In Fig. 5, the solid line denotes the *estimated* (with noise) angle trajectory of the finite-horizon tracking controller, the dashed line denotes the *desired* tracker angle.

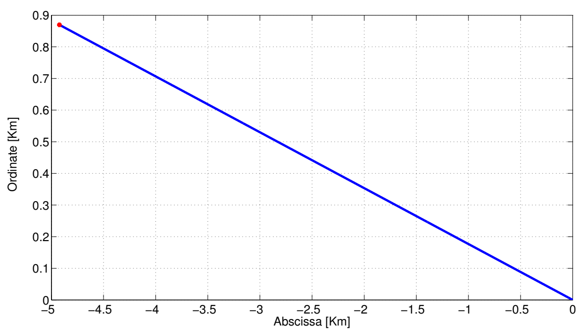


Figure 4: Vehicle A -Target Engagement Scenario

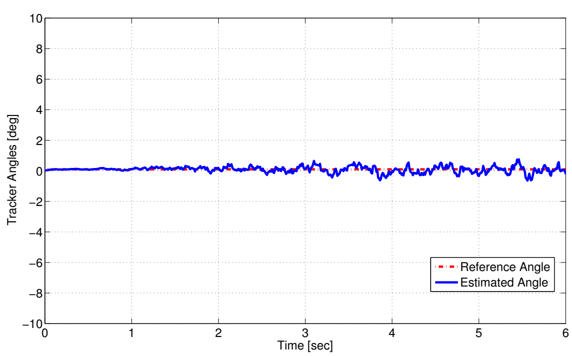


Figure 5: Angle Trajectories for Vehicle A Tracker

As shown in Fig. 6, the finite-horizon differential SDRE nonlinear regulating algorithm with incomplete state information gives excellent results, and is able to solve the nonlinear regulator problem with a zero average angle error.

### 4.2 Vehicle B

Consider a maneuvering path (shot) to the target, such that the final time is to be 6 seconds, i.e *the problem is now a tracking problem.*

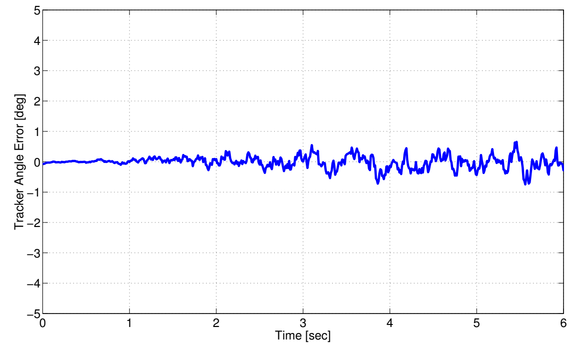


Figure 6: Error for Vehicle A Tracker

The engagement scenario is shown in Fig. 7. The resulting trajectories for the demanded and achieved tracker angles are presented in Fig. 8, and the error is shown in Fig. 9.

In Fig. 8, the solid line denotes the *estimated* (with noise) angle trajectory of the finite-horizon tracking controller, the dashed line denotes the *desired* tracker angle.

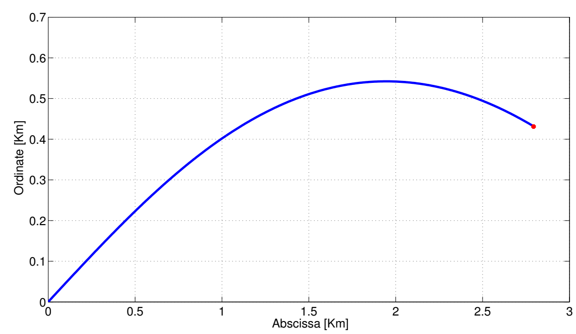


Figure 7: Vehicle B -Target Engagement Scenario

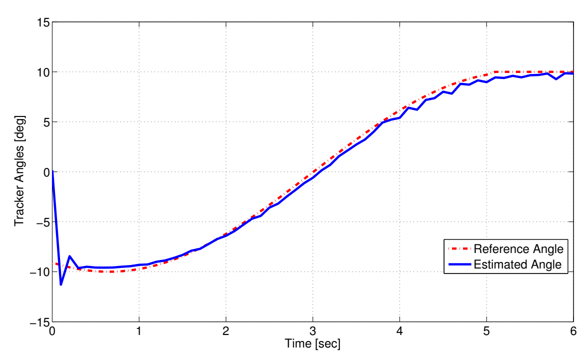


Figure 8: Angle Trajectories for Vehicle B Tracker

Fig. 7 show that a successful hit is observed. Comparing these trajectories in Fig. 8, it's clear that

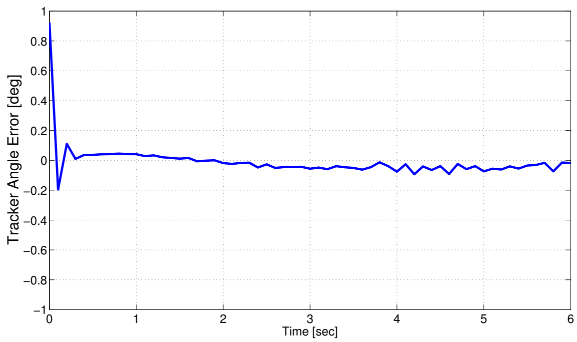


Figure 9: Error for Vehicle B Tracker

the system is achieving a very good tracking even when the vehicle shot is executing a maneuver. The tracker controlled by the developed technique is able to hit the target in exactly 6 seconds with standard deviation error of $0.02^o$.

## 5 Conclusions

In this paper, we proposed a finite-horizon regulation and tracking techniques used for nonlinear systems. The main idea of the proposed technique is to integrate the differential SDRE filter algorithm and the finite-horizon SDRE technique. The finite-horizon SDRE is based on change of variables that converts the nonlinear differential Riccati equation (DRE) to a linear differential Lyapunov equation. The proposed technique is effective for wide range of operating points. Simulation results for Time On Target engagement scenario are included to demonstrate the effectiveness of the presented technique.

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