A new type of sequence space of non-absolute type and matrix transformation

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Abstract : In this paper, we introduce the space $r^q(\triangle_u^p)$, where

$$r^q(\triangle_u^p) = \{x = (x_k) \in \omega : (\triangle x_k) \in r^q(u,p)\};$$

where $r^q(u,p)$ has recently been introduced and studied by Neyaz and Hamid (Acta Math. Acad. Paeda. Nyreg., 28, 2012, pp. 47-58). We show its completeness property, prove that the space $r^q(\triangle_u^p)$ and $l(p)$ are linearly isomorphic and compute α -, β- and γ-duals of $r^q(\triangle_u^p)$. Moreover, we construct the basis of $r^q(\triangle_u^p)$. Finally, we characterize some matrix class.

Key–Words : Sequence space of non-absolute type; paranormed sequence space; α-, β- and γ-duals ; matrix transformations.

1 Introduction

We denote the set of all sequences with complex terms by ω. It is a routine verification that ω is a linear space with respect to the co-ordinatewise addition and scalar multiplication of sequences which are defined, as usual, by

$$x + y = (x_k) + (y_k) = (x_k + y_k)$$

and

$$\alpha x = \alpha(x_k) = (\alpha x_k),$$

respectively; where $x = (x_k)$, $y = (y_k) \in \omega$ and $\alpha \in \mathbf{C}$. By *sequence space* we understand a linear subspace of ω i.e. the sequence space is the set of scalar sequences (real or complex) which is closed under co-ordinate wise addition and scalar multiplication. Throughout the paper N, R and C denotes the set of non-negative integers, the set of real numbers and the set of complex numbers, respectively. Let l_∞, c and c_0, respectively, denotes the space of all bounded sequences, the space of all convergent sequences and the sequences converging to zero. Also, by l_1 , $l(p)$, cs and bs we denote the spaces of all absolutely convergent, p-absolutely convergent, convergent and bounded series, respectively.

The classical summability theory deals with a generalization of convergence of sequences and series. One original idea was to assign a limit to divergent sequences or series. Toeplitz [1] was the first to study summability methods as a class of transformations of complex sequences by complex infinite matrices. The theory of matrix transformations is a wide field in summability; it deals with the characterizations of classes of matrix mappings between sequence spaces by giving necessary and sufficient conditions on the entries of the infinite matrices.

Let X, Y be two sequence spaces and let $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in N$. Then, the matrix A defines the A-transformation from X *into* Y, if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x exists and is in Y; where $(Ax)_n = \sum_k a_{nk}x_k$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $A \in (X : Y)$ we mean the characterizations of matrices from X to Y *i.e.*, $A : X \to Y$. A sequence x is said to be A-summable to l if Ax converges to l which is called as the A-limit of x.

We denote by (A) the set of all sequences which are summable A. The set (A) is called *summability field* of the matrix A. Thus, if $Ax = \{(Ax)_n\}$, then $(A) = \{x : Ax \in c\}$, where c is the set of convergent sequences. For example, $(I) = c$.

For a sequence space X, the matrix domain X_A

of an infinite matrix A is defined as

$$X_A = \{x = (x_k) : Ax \in X\}. \quad (1)$$

A infinite matrix $A = (a_{nk})$ is said to be regular [2, 3] if and only if the following conditions hold:

$$(i) \quad \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 1,$$

$$(ii) \quad \lim_{n\to\infty} a_{nk} = 0, \ \ (k = 0, 1, 2, ...),$$

$$(iii) \ \sum_{k=0}^{\infty} |a_{nk}| < M, \ \ (M > 0, \ n = 0, 1, 2, ...).$$

Let (q_k) be a sequence of positive numbers and let us write, $Q_n = \sum_{k=0}^{n} q_k$ for $n \in N$. Then the matrix $R^q = (r^q_{nk})$ of the Riesz mean (R, q_n) is given by

$$r^q_{nk} = \begin{cases} \frac{q_k}{Q_n}, & \text{if } 0 \le k \le n, \\ 0, & \text{if } k > n \end{cases}$$

The Riesz mean (R, q_n) is regular if and only if $Q_n \to \infty$ as $n \to \infty$ [3].

Kizmaz [4] defined the difference sequence spaces $Z(\triangle)$ as follows

$$Z(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in Z\}$$

where, $Z \in \{l_\infty, c, c_0\}$ and $\triangle x_k = x_k - x_{k+1}$.

Başar and Altay [5] has studied the sequence space as

$$bv_p = \left\{ x = (x_k) \in \omega : \sum_k |x_k - x_{k-1}|^p < \infty \right\},$$

where $1 \le p < \infty$. With the notation of (1), the space bv_p can be redefined as

$$bv_p = (l_p)_\triangle, 1 \le p < \infty$$

where, $\triangle$ denotes the matrix $\triangle = (\triangle_{nk})$ defined as

$$\triangle_{nk} = \begin{cases} (-1)^{n-k}, & \text{if } n-1 \le k \le n, \\ 0, & \text{if } k < n-1 \text{ or } k > n. \end{cases}$$

This space was further studied by Başar, Altay and Mursaleen [6] and have introduced $bv(u, p)$ and $bv_\infty(u, p)$ which are defined as follows:

$$bv(u, p) = \left\{ x = (x_k) \in \omega : \sum_k |u_k \Delta x_k|^{p_k} < \infty \right\},$$

where $0 \le p_k < \infty$ and

$$bv_\infty(u, p) = \left\{ x = (x_k) \in \omega : \sup_k |u_k \Delta x_k|^{p_k} < \infty \right\}.$$

With the notation of (1), the space $bv(u, p)$ and $bv_\infty(u, p)$ can be redefined as

$$bv(u, p) = [l(p)]_{\triangle^u} \ \ and \ \ bv_\infty(u, p) = \ [l_\infty(p)]_{\triangle^u}$$

where, $\triangle^u$ denotes the matrix $\triangle = (\triangle^u_{nk})$ defined as:

$$\triangle^u_{nk} = \begin{cases} (-1)^{n-k} u_k, & \text{if } n-1 \le k \le n, \\ 0, & \text{if } k < n-1 \text{ or } k > n. \end{cases}$$

for all $n, k \in N$.

The approach of constructing a new sequence space by means of matrix domain of a particular limitation method has been studied by several authors [5-30]. They introduced the sequence spaces

$(l_p)_\triangle = bv_p$ [5],

$(l_p)_{E^r} = e^r_p$ and $(l_\infty)_{E^r} = e^r_\infty$ [7],

$(l_\infty)_{N_q}$ and c_{N_q} [8],

$(l_p)_{C_1} = X_p$ and $(l_\infty)_{C_1} = X_\infty$ [9],

$(l_\infty)_{R^t} = r^t_\infty$, $(c)_{R^t} = r^t_c$ and $(c_o)_{R^t} = r^t_0$ [10],

$(c_0)_{A^r} = a^r_0$ and $c_{A^r} = a^r_c$ [11],

$(l_p)_{R^t} = r^t_p$ [12],

$[c_0(u, p)]_{A^r} = a^r_0(u, p)$ and $[c(u, p)]_{A^r} = a^r_c(u, p)$ [13],

$\mu_G = Z(u, v, \mu)$ [14],

$(l_p)_{A^r} = a^r_p$ and $(l_\infty)_{A^r} = a^r_\infty$ [15],

$(c_0)_{C_1} = \widehat{c}_0$, $c_{C_1} = \widehat{c}$ [16],

$c^\lambda_0(\triangle) = \left(c^\lambda_0\right)_\triangle$ and $c^\lambda(\triangle) = \left(c^\lambda\right)_\triangle$ [17],

$r^q(u, p) = \{l(p)\}_{R^q}$ [18],

$c\left(\triangle^\lambda_u\right) = (c)_{\widehat{\wedge}}$ and $c_0\left(\triangle^\lambda_u\right) = (c_0)_{\widehat{\wedge}}$ [19];

where N_q, C_1, R^t and E^r denotes the Nörland, Cesäro, Riesz and Eular means, respectively, A^r and C are respectively defined in [6, 8, 9], $\mu = \{c_0, c, l_p\}$ and $1 \le p < \infty$, $c_0(u, p)$ and $c(u, p)$ also denote the sequence spaces generated from the Maddox's spaces $c_0(p)$ and $c(p)$ by Başarir [20].

2 The Riesz Sequence space $r^q(\triangle^p_u)$ of non-absolute type

In this section, we define the Riesz sequence space $r^q(\triangle^p_u)$, prove that the space $r^q(\triangle^p_u)$ is a complete paranormed linear space and it is shown to be linearly isomorphic to the space $l(p)$.

A linear Topological space X over the field of real numbers R is said to be a paranormed space if there is a sub-additive function $h : X \to R$ such that $h(\theta) = 0$, $h(-x) = h(x)$ and scalar multiplication is continuous, that is, $|\alpha_n - \alpha| \to 0$ and $h(x_n - x) \to 0$ imply $h(\alpha_n x_n - \alpha x) \to 0$ for all $\alpha' s$ in R and $x' s$ in X , where θ is a zero vector in the linear space X. Assume here and after that (p_k) be a bounded sequence of strictly positive real numbers with $\sup_k p_k = H$ and $M = \max\{1, H\}$. Then, the linear spaces $l(p)$ and $l_\infty(p)$ were defined by Maddox [2] (see also, [25,26]) as follows :

$$l(p) = \{x = (x_k) : \sum_k |x_k|^{p_k} < \infty\}$$

and

$$l_\infty(p) = \{x = (x_k) : \sup_k |x_k|^{p_k} < \infty\}$$

which are complete spaces paranormed by

$$h_1(x) = \left[\sum_k |x_k|^{p_k}\right]^{1/M}$$

and

$$h_2(x) = \sup_k |x_k|^{p_k/M}$$

iff $\inf p_k > 0$.

We shall assume throughout the text that $p_k^{-1} + \{p_k'\}^{-1} = 1$ provided $1 < \inf p_k \leq H < \infty$ and we denote the collection of all finite subsets of N by F, where N={0, 1, 2, … }.

Neyaz and Hamid [18] have recently introduced $r^q(u, p)$ which is defined as:

$$r^q(u,p) = \left\{x = (x_k) \in \omega : \sum_k \left|\frac{1}{Q_k}\sum_{j=0}^{k} u_k q_j x_j\right|^{p_k} < \infty\right\}$$

where, $0 < p_k \leq H < \infty$.

With the notation of (1) we redefine $r^q(u, p)$ as:

$$r^q(u,p) = \{l(p)\}_{R_u^q}.$$

Following Başar and Altay [5], Mursaleen et al [17, 23], Hamid et al [18, 23, 24], Basarir [27], Choudhary and Mishra [28], Gross Erdmann [30], Tripathy [31], we define the Reisz sequence space $r^q(\triangle_u^p)$ as the set of all sequences such that $R^q\triangle$-transform of it is in the space $l(p)$, that is,

$$r^q(\triangle_u^p) = \left\{x = (x_k) \in \omega : \sum_k \left|\frac{1}{Q_k}\sum_{j=0}^{k} u_k q_j \triangle x_j\right|^{p_k} < \infty\right\}$$

where, $0 < p_k \leq H < \infty$.

Remark 1 *If we take $(u_k) = e = (1, 1, ...)$ in $r^q(\triangle_u^p)$, we get the results obtained in [27].*

With the notation of (1) we redefine $r^q(\triangle_u^p)$ as

$$r^q(\triangle_u^p) = \{l(p)\}_{R^q\triangle}.$$

Define the sequence $y = (y_k)$, which will be used, by the $R^q\triangle$-transform of a sequence $x = (x_k)$, i.e.,

$$y_k = \frac{1}{Q_k}\sum_{j=0}^{k} u_k q_j \triangle x_j. \quad (2)$$

Now, we begin with the following theorem which is essential in the text.

Theorem 2 *$r^q(\triangle_u^p)$ is a complete linear metric space paranormed by $h_\triangle$, defined as*

$$h_\triangle(x) = \left[\sum_k \left|\frac{1}{Q_k}\sum_{j=0}^{k-1} u_k(q_j - q_{j+1})x_j + \frac{q_k u_k}{Q_k} x_k\right|^{p_k}\right]^{\frac{1}{M}}$$

with $0 < p_k \leq H < \infty$.

Proof: The linearity of $r^q(\triangle_u^p)$ with respect to the co-ordinatewise addition and scalar multiplication follows from the inequalities which are satisfied for $z, x \in r^q(\triangle_u^p)$ [2]

$$\begin{aligned} &\left[\sum_k \left|\tfrac{1}{Q_k}\sum_{j=0}^{k-1} u_k(q_j - q_{j+1})(x_j + z_j) + \tfrac{q_k u_k}{Q_k}(x_k + z_k)\right|^{p_k}\right]^{\frac{1}{M}} \\ &\leq \left[\sum_k \left|\tfrac{1}{Q_k}\sum_{j=0}^{k-1} u_k(q_j - q_{j+1})x_j + \tfrac{q_k u_k}{Q_k} x_k\right|^{p_k}\right]^{\frac{1}{M}} \\ &+ \left[\sum_k \left|\tfrac{1}{Q_k}\textstyle\sum_{j=0}^{k-1} u_k(q_j - q_{j+1})z_j + \tfrac{q_k u_k}{Q_k} z_k\right|^{p_k}\right]^{\frac{1}{M}} \end{aligned} \quad (3)$$

and for any $\alpha \in \mathbf{R}$ [32]

$$|\alpha|^{p_k} \leq \max(1, |\alpha|^M). \quad (4)$$

It is clear that, $h_\triangle(\theta)$=0 and $h_\triangle(x) = h_\triangle(-x)$ for all $x \in r^q(\triangle_u^p)$. Again the inequality (3) and (4), yield the subadditivity of $h_\triangle$ and

$$h_\triangle(\alpha x) \leq max(1, |\alpha|)h_\triangle(x).$$

Let $\{x^n\}$ be any sequence of points of the space $r^q(\triangle_u^p)$ such that $h_\triangle(x^n - x) \to 0$ and (α_n) is a sequence of scalars such that $\alpha_n \to \alpha$. Then, since the inequality,

$$h_\triangle(x^n) \leq h_\triangle(x) + h_\triangle(x^n - x)$$

holds by subadditivity of $h_\triangle$, $\{h_\triangle(x^n)\}$ is bounded and we thus have

$$h_\triangle(\alpha_n x^n - \alpha x) = \left[\sum_k \left|\frac{1}{Q_k}\sum_{j=0}^{k} u_k(q_j - q_{j+1})(\alpha_n x_j^n - \alpha x_j)\right|^{p_k}\right]^{\frac{1}{M}} \leq |\alpha_n - \alpha|^{\frac{1}{M}} h_\triangle(x^n) + |\alpha|^{\frac{1}{M}} h_\triangle(x^n - x)$$

which tends to zero as $n \to \infty$. That is to say that the scalar multiplication is continuous. Hence, $h_\triangle$ is paranorm on the space $r^q(\triangle_u^p)$.

It remains to prove the completeness of the space $r^q(\triangle_u^p)$. Let $\{x^i\}$ be any Cauchy sequence in the space $r^q(\triangle_u^p)$, where $x^i = \{x_0^i, x_1^i, ...\}$, then for a given $\epsilon > 0$ there exists a positive integer $n_0(\epsilon)$ such that

$$h_\triangle(x^i - x^j) < \epsilon \tag{5}$$

for all $i, j \geq n_0(\epsilon)$. Using definition of $h_\triangle$ and for each fixed $k \in N$ that

$$\left|(R^q\triangle x^i)_k - (R^q\triangle x^j)_k\right| \leq \left[\sum_k \left|(R^q\triangle x^i)_k - (R^q\triangle x^j)_k\right|^{p_k}\right]^{\frac{1}{M}} < \epsilon$$

for $i, j \geq n_0(\epsilon)$, which leads us to the fact that $\{(R^q\triangle x^0)_k, (R^q\triangle x^1)_k, \ldots\}$ is a Cauchy sequence of real numbers for every fixed $k \in N$. Since R is complete, it converges, say, $(R^q\triangle x^i)_k \to ((R^q\triangle x)_k$ as $i \to \infty$. Using these infinitely many limits $(R^q\triangle x)_0, (R^q\triangle x)_1, \ldots$, we define the sequence $\{(R^q\triangle x)_0, (R^q\triangle x)_1, \ldots\}$. From (5) for each $m \in N$ and $i, j \geq n_0(\epsilon)$,

$$\sum_{k=0}^{m} \left|(R^q\triangle x^i)_k - (R^q\triangle x^j)_k\right|^{p_k} \leq h_\triangle(x^i - x^j)^M < \epsilon^M. \tag{6}$$

Take any $i, j \geq n_0(\epsilon)$. First, let $j \to \infty$ in (6) and then $m \to \infty$, we obtain

$$h_\triangle(x^i - x) \leq \epsilon.$$

Finally, taking $\epsilon = 1$ in (6) and letting $i \geq n_0(1)$, we have by Minkowski's inequality for each $m \in N$ that

$$\left[\sum_{k=0}^{m} |(R^q x)_k|^{p_k}\right]^{\frac{1}{M}} \leq h_\triangle(x^i - x) + h_\triangle(x^i) \leq 1 + h_\triangle(x^i)$$

which implies that $x \in r^q(\triangle_u^p)$. Since $h_\triangle(x - x^i) \leq \epsilon$ for all $i \geq n_0(\epsilon)$, it follows that $x^i \to x$ as $i \to \infty$, hence we have shown that $r^q(\triangle_u^p)$ is complete, hence the proof . □

Note that one can easily see the absolute property does not hold on the spaces $r^q(\triangle_u^p)$, that is $h_\triangle(x) \neq h_\triangle(|x|)$ for atleast one sequence in the space $r^q(\triangle_u^p)$ and this says that $r^q(\triangle_u^p)$ is a sequence space of non-absolute type.

Theorem 3 *The Riesz sequence space $r^q(\triangle_u^p)$ of non-absolute type is linearly isomorphic to the space $l(p)$, where $0 < p_k \leq H < \infty$.*

Proof: To prove the theorem, we will show the existence of a linear bijection between the spaces $r^q(\triangle_u^p)$ and $l(p)$, where $0 < p_k \leq H < \infty$. With the notation of (3), define the transformation T from $r^q(\triangle_u^p)$ to $l(p)$ by $x \to y = Tx$. The linearity of T is trivial. Further, it is obvious that $x = \theta$ whenever $Tx = \theta$ and hence T is injective.

Let $y \in l(p)$ and define the sequence $x = (x_k)$ by

$$x_k = \sum_{n=0}^{k-1}\left(\frac{1}{q_n} - \frac{1}{q_{n+1}}\right) u_k^{-1} Q_k y_k + u_k^{-1}\frac{Q_k}{q_k} y_k,$$

for $k \in N$. Then,

$$h_\triangle(x) = \left[\sum_k \left|\frac{1}{Q_k}\sum_{j=0}^{k-1} u_k(q_j - q_{j+1})x_j + \frac{q_k u_k}{Q_k} x_k\right|^{p_k}\right]^{\frac{1}{M}}$$

$$= \left[\sum_k \left|\sum_{j=0}^{k} \delta_{kj} y_j\right|^{p_k}\right]^{\frac{1}{M}}$$

$$= \left[\sum_k |y_k|^{p_k}\right]^{\frac{1}{M}}$$

$$= h_1(y) < \infty,$$

where,

$$\delta_{kj} = \begin{cases} 1, & \text{if } k = j, \\ 0, & \text{if } k \neq j. \end{cases}$$

Thus, we have $x \in r^q(\triangle_u^p)$. Consequently, T is surjective and is paranorm preserving. Hence, T is a linear bijection and this proves that the spaces $r^q(\triangle_u^p)$ and $l(p)$ are linearly isomorphic, hence the proof.

3 Basis and α-, β- and γ-duals of the space $r^q(\triangle_u^p)$

In this section, we compute α-, β- and γ- duals of the space $r^q(\triangle_u^p)$ and finally in this section we give the basis for the space $r^q(\triangle_u^p)$.

For the sequence space X and Y, define the set

$$S(X:Y) = \{z = (z_k) : xz = (x_k z_k) \in Y\}. \quad (7)$$

With the notation of (7), the α-, β- and γ- duals of a sequence space X, which are respectively denoted by X^α, X^β and X^γ and are defined by

$X^\alpha = S(X : l_1)$, $X^\beta = S(X : cs)$ and $X^\gamma = S(X : bs)$.

If a sequence space X paranormed by h contains a sequence (b_n) with the property that for every $x \in X$ there is a unique sequence of scalars (α_n) such that

$$\lim_n h(x - \sum_{k=0}^{n} \alpha_k b_k) = 0,$$

then (b_n) is called a Schauder basis (or briefly basis) for X. The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and written as $x = \sum \alpha_k b_k$.

First we first state some lemmas which are needed in proving our theorems.

Lemma 4 *[33]*
(i) Let $1 < p_k \leq H < \infty$. Then $A \in (l(p) : l_1)$ if and only if there exists an integer $B > 1$ such that

$$\sup_{K \in F} \sum_k \left| \sum_{n \in K} a_{nk} B^{-1} \right|^{p_k'} < \infty.$$

(ii) Let $0 < p_k \leq 1$.Then $A \in (l(p) : l_1)$ if and only if

$$\sup_{K \in F} \sup_k \left| \sum_{n \in K} a_{nk} B^{-1} \right|^{p_k} < \infty.$$

Lemma 5 *[33]*
(i) Let $1 < p_k \leq H < \infty$. Then $A \in (l(p) : l_\infty)$ if and only if there exists an integer $B > 1$ such that

$$\sup_n \sum_k |a_{nk} B^{-1}|^{p_k'} < \infty. \quad (8)$$

(ii) Let $0 < p_k \leq 1$ for every $k \in N$. Then $A \in (l(p) : l_\infty)$ if and only if

$$\sup_{n,k} |a_{nk}|^{p_k} < \infty. \quad (9)$$

Lemma 6 *[33] Let $0 < p_k \leq H < \infty$ for every $k \in N$. Then $A \in (l(p) : c)$ if and only if (8) and (9) hold along with*

$$\lim_n a_{nk} = \beta_k \; for \; k \in N \quad (10)$$

also holds.

Theorem 7 *Let $1 < p_k \leq H < \infty$ for every $k \in N$. Define the sets $D_1(u,p)$ and $D_2(u,p)$ as follows*

$$D_1(u,p) = \bigcup_{B>1} \{a = (a_k) \in \omega : \sup_{K \in F} \sum_k | \sum_{n \in K} \left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) u_k^{-1} a_n Q_k + \frac{a_n}{q_n} u_k^{-1} Q_n B^{-1} |^{p_k'} < \infty\}$$

and

$$D_2(u,p) = \bigcup_{B>1} \{a = (a_k) \in \omega : \sum_k \left| [\left(\frac{a_k}{q_k} + \left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \sum_{i=k+1}^{n} a_i \right) u_k^{-1} Q_k] B^{-1} \right|^{p_k'} < \infty\}.$$

Then,

$$[r^q(\triangle_u^p)]^\alpha = D_1(u,p)$$

and

$$[r^q(\triangle_u^p)]^\beta = [r^q(\triangle_u^p)]^\gamma = D_2(u,p) \cap \; cs.$$

Proof: Let us take any $a = (a_k) \in \omega$. We can easily derive with (2) that

$$\begin{aligned} & a_n x_n \\ & = \sum_{k=0}^{n-1} \left(\tfrac{1}{q_k} - \tfrac{1}{q_{k+1}} \right) u_k^{-1} a_n Q_k y_k + \tfrac{a_n}{q_n} u_k^{-1} Q_n y_n \\ & = (Cy)_n \end{aligned} \quad (11)$$

where, $C = (c_{nk})$ is defined as

$$c_{nk} = \begin{cases} \left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) u_k^{-1} a_n Q_k, & \text{if } 0 \leq k \leq n-1, \\ \frac{a_n}{q_n} u_k^{-1} Q_n, & \text{if } k = n, \\ 0, & \text{if } k > n, \end{cases}$$

for all $n, k \in N$. Thus we observe by combining (11) with (i) of Lemma 4 that $ax = (a_n x_n) \in l_1$ whenever $x = (x_n) \in r^q(\triangle_u^p)$ if and only if $Cy \in l_1$ whenever $y \in l(p)$. This shows that $[r^q(\triangle_u^p)]^\alpha = D_1(u,p)$.

Further, consider the equation,

$$\sum_{k=0}^{n} a_k x_k$$
$$= \sum_{k=0}^{n} \left[\left(\frac{a_k}{q_k} + \left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \sum_{i=k+1}^{n} a_i \right) u_k^{-1} Q_k \right] y_k$$
$$= (Dy)_n \tag{12}$$

where, $D = (d_{nk})$ is defined as

$$d_{nk} = \begin{cases} \left(\frac{a_k}{q_k} + (\frac{1}{q_k} - \frac{1}{q_{k+1}}) \sum_{i=k+1}^{n} a_i \right) u_k^{-1} Q_k, \\ \qquad \text{if } 0 \le k \le n, \\ 0, \qquad \text{if } k > n. \end{cases}$$

Thus we deduce from Lemma 6 with (12) that $ax = (a_n x_n) \in cs$ whenever $x = (x_n) \in r^q(\triangle_u^p)$ if and only if $Dy \in c$ whenever $y \in l(p)$. Therefore, we derive from (8) that

$$\sum_k \left| \left[\left(\frac{a_k}{q_k} + \left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \sum_{i=k+1}^{n} a_i \right) u_k^{-1} Q_k \right] B^{-1} \right|^{p_k'} < \infty, \tag{13}$$

and $\lim_n d_{nk}$ exists and hence shows that $[r^q(\triangle_u^p)]^\beta = D_2(u,p) \cap cs$. As proved above, from Lemma 5 together with (12) that $ax = (a_k x_k) \in bs$ whenever $x = (x_n) \in r^q(\triangle_u^p)$ if and only if $Dy \in l_\infty$ whenever $y = (y_k) \in l(p)$. Therefore, we again obtain the condition (13) which means that $[r^q(\triangle_u^p)]^\gamma = D_2(u,p) \cap cs$ and the proof of the theorem is complete. □

Theorem 8 *Let $0 < p_k \le 1$ for every $k \in N$. Define the sets $D_3(u,p)$ and $D_4(u,p)$ as follows*

$$D_3(u,p) = \{a = (a_k) \in \omega :$$
$$\sup_{K \in F} \sup_k | \sum_{n \in K} [(\frac{1}{q_k} - \frac{1}{q_{k+1}}) u_k^{-1} a_n Q_k$$
$$+ \frac{a_n}{q_n} u_k^{-1} Q_n] B^{-1} |^{p_k} < \infty\}$$

and

$$D_4(u,p) = \{a = (a_k) \in \omega :$$
$$\sup_k \left| \left[\left(\frac{a_k}{q_k} + \left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \sum_{i=k+1}^{n} a_i \right) u_k^{-1} Q_k \right] B^{-1} \right|^{p_k} < \infty\}.$$

Then, $[r^q(\triangle_u^p)]^\alpha = D_3(u,p)$ *and*

$$[r^q(\triangle_u^p)]^\beta = [r^q(\triangle_u^p)]^\gamma = D_4(u,p) \cap cs.$$

Proof: The proof follows easily from Theorem 7 (above) by using second parts of Lemmas 4, 5 and 6 instead of the first parts. □

Theorem 9 *Define the sequence $b^{(k)}(q) = \{b_n^{(k)}(q)\}$ of the elements of the space $r^q(\triangle_u^p)$ for every fixed $k \in N$ by*

$$b_n^{(k)}(q) = \begin{cases} (\frac{1}{q_n} - \frac{1}{q_{n+1}}) u_k^{-1} Q_n + u_k^{-1} \frac{Q_k}{q_k}, \\ \qquad \text{if } 0 \le n \le k-1, \\ 0, \qquad \text{if } n > k-1. \end{cases}$$

Then, the sequence $\{b^{(k)}(q)\}$ is a basis for the space $r^q(\triangle_u^p)$ and any $x \in r^q(\triangle_u^p)$ has a unique representation of

$$x = \sum_k \lambda_k(q) b^{(k)}(q) \tag{14}$$

where, $\lambda_k(q) = (R^q \triangle x)_k$ for all $k \in N$ and $0 < p_k \le H < \infty$.

Proof: It is clear that $b^{(k)}(q) \subset r^q(\triangle_u^p)$, since

$$R^q \triangle b^{(k)}(q) = e^{(k)} \in l(p) \; for \; k \in N \tag{15}$$

and $0 < p_k \le H < \infty$, where $e^{(k)}$ is the sequence whose only non-zero term is 1 in k^{th} place for each $k \in N$.

Let $x \in r^q(\triangle_u^p)$ be given. For every non-negative integer m, we put

$$x^{[m]} = \sum_{k=0}^{m} \lambda_k(q) b^{(k)}(q). \tag{16}$$

Then, we obtain by applying $R^q \triangle$ to (16) with (15) that

$$R^q \triangle x^{[m]} = \sum_{k=0}^{m} \lambda_k(q) R^q \triangle b^{(k)}(q) = \sum_{k=0}^{m} (R^q \triangle x)_k e^{(k)}$$

and

$$\left(R^q \triangle \left(x - x^{[m]} \right) \right)_i = \begin{cases} 0, & \text{if } 0 \le i \le m \\ (R^q \triangle x)_i, & \text{if } i > m \end{cases}$$

where $i, m \in N$. Given $\varepsilon > 0$, there exists an integer m_0 such that

$$\left(\sum_{i=m}^{\infty} |(R^q \triangle x)_i|^{p_k} \right)^{\frac{1}{M}} < \frac{\varepsilon}{2},$$

for all $m \geq m_0$. Hence,

$$h_{\triangle}\left(x - x^{[m]}\right) = \left(\sum_{i=m}^{\infty} |(R^q \triangle x)_i|^{p_k}\right)^{\frac{1}{M}} \leq \left(\sum_{i=m_0}^{\infty} |(R^q \triangle x)_i|^{p_k}\right)^{\frac{1}{M}} < \frac{\varepsilon}{2} < \varepsilon$$

for all $m \geq m_0$, which proves that $x \in r^q(\triangle_u^p)$ is represented as (14).

Let us show the uniqueness of the representation for $x \in r^q(\triangle_u^p)$ given by (13). Suppose, on the contrary; that there exists a representation $x = \sum_k \mu_k(q) b^k(q)$. Since the linear transformation T from $r^q(\triangle_u^p)$ to $l(p)$ used in the Theorem 3 is continuous we have

$$\begin{aligned}(R^q \triangle x)_n &= \sum_k \mu_k(q) \left(R^q \triangle b^k(q)\right)_n \\ &= \sum_k \mu_k(q) e_n^{(k)} = \mu_n(q)\end{aligned}$$

for $n \in N$, which contradicts the fact that $(R^q \triangle x)_n = \lambda_n(q)$ for all $n \in N$. Hence, the representation (14) is unique. This completes the proof. □

4 Matrix Mappings on the Space $r^q(\triangle_u^p)$

In this section, we characterize the matrix mappings from the space $r^q(\triangle_u^p)$ to the space l_∞.

Theorem 10 *(i) : Let $1 < p_k \leq H < \infty$ for every $k \in N$. Then $A \in (r^q(\triangle_u^p) : l_\infty)$ if and only if there exists an integer $B > 1$ such that*

$$C(B) = \sup_n \sum_k \left|\left[\frac{a_{nk}}{q_k} + \left(\frac{1}{q_k} - \frac{1}{q_{k+1}}\right) \sum_{i=k+1}^{n} a_{ni}\right] u_k^{-1} B^{-1} Q_k\right|^{p_k'} < \infty \quad (17)$$

and $\{a_{nk}\}_{k \in N} \in cs$ for each $n \in N$.

(ii) : Let $0 < p_k \leq 1$ for every $k \in N$. Then $A \in (r^q(\triangle_u^p) : l_\infty)$ if and only if

$$\sup_{n,k} \left|\left[\frac{a_{nk}}{q_k} + \left(\frac{1}{q_k} - \frac{1}{q_{k+1}}\right) \sum_{i=k+1}^{n} a_{ni}\right] u_k^{-1} Q_k\right|^{p_k} < \infty, \quad (18)$$

and $\{a_{nk}\}_{k \in N} \in cs$ for each $n \in N$.

Proof: We only prove the part (i) and (ii) follows in a similar fashion. So, let $A \in (r^q(\triangle_u^p) : l_\infty)$ and $1 < p_k \leq H < \infty$ for every $k \in N$. Then Ax exists for $x \in r^q(\triangle_u^p)$ and implies that $\{a_{nk}\}_{k \in N} \in \{r^q(\triangle_u^p)\}^\beta$ for each $n \in N$. Hence necessity of (17) holds.

Conversely, suppose that the necessities (17) hold and $x \in r^q(\triangle_u^p)$, since $\{a_{nk}\}_{k \in N} \in \{r^q(\triangle_u^p)\}^\beta$ for every fixed $n \in N$, so the A-transform of x exists. Consider the following equality obtained by using the relation (11) that

$$\sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m} \left[\frac{a_{nk}}{q_k} + \left(\frac{1}{q_k} - \frac{1}{q_{k+1}}\right) \sum_{i=k+1}^{m} a_{ni}\right] u_k^{-1} Q_k y_k. \quad (19)$$

Taking into account the assumptions we derive from (19) as $m \to \infty$ that

$$\sum_k a_{nk} x_k = \sum_k \left[\frac{a_{nk}}{q_k} + \left(\frac{1}{q_k} - \frac{1}{q_{k+1}}\right) \sum_{i=k+1}^{\infty} a_{ni}\right] u_k^{-1} Q_k y_k \quad (20)$$

Now, by combining (20) and the inequality which holds for any $B > 0$ and any complex numbers a, b

$$|ab| \leq B\left(\left|aB^{-1}\right|^{p'} + |b|^p\right)$$

with $p^{-1} + p'^{-1} = 1$(see [10]), one can easily see that

$$\begin{aligned}&\sup_{n \in N} \left|\sum_k a_{nk} x_k\right| \\ &\leq \sup_{n \in N} \sum_k \left|\left[\frac{a_{nk}}{q_k} + \left(\frac{1}{q_k} - \frac{1}{q_{k+1}}\right) \sum_{i=k+1}^{\infty} a_{ni}\right] u_k^{-1} Q_k\right| |y_k| \\ &\leq B\left[C(B) + h_1^B(y)\right] < \infty.\end{aligned}$$

This shows that $Ax \in l_\infty$ whenever $x \in r^q(\triangle_u^p)$. This completes the proof. □

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