

Fig. 5. The second order FLL stable region

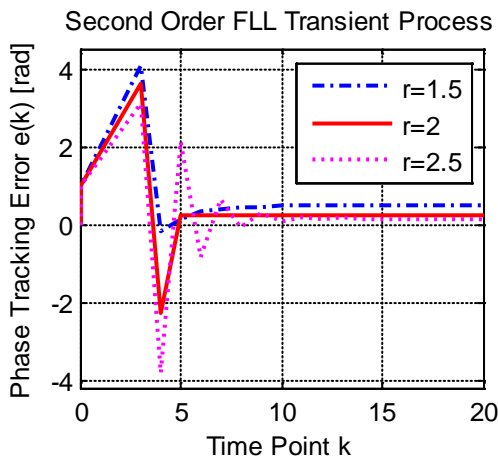


Fig. 6. The second order FLL transient response

3.1.2 Number of Steps for Convergence

In order to calculate the steps for convergence, we need first extend the theory in [12] to higher dimensions as:

Theorem 1[12]: Suppose that X is a normed vector space and K is a subset of X such that the following conditions hold:

- 1) $G: K \rightarrow K$
- 2) $G(x^*)=x^*, x^* \in K$
- 3) There exists a constant $L: 0 \leq L < 1$ such that $\|G(\bar{x}) - G(\bar{x}^*)\| \leq L \|\bar{x} - \bar{x}^*\|, \forall \bar{x} \in K$

Then the following conclusions can be made:

- 1) $\lim_{m \rightarrow \infty} \bar{x}_m = \bar{x}^*, \forall \bar{x}_0 \in K$

and \bar{x}^* is the unique fixed point of G in K.

- 2) $\|\bar{x}_m - \bar{x}^*\| \leq L^m \|\bar{x}_0 - \bar{x}^*\|$

Based on this theory, the steps for convergence to a radius ϵ beside the stationary point can be represented as[11,12]

$$m < \frac{\ln \left(\frac{\epsilon}{\left[(\varphi_0 - \varphi_{ss})^2 + (\varphi_1 - \varphi_{ss})^2 \right]} \right)}{\ln(L)} \quad (36)$$

Where φ_0 and φ_1 are the initial phase errors, φ_{ss} is the stationary phase error, arguments ψ_0 and ψ_1 satisfy $\psi_0 \in [-\varphi_0, \varphi_0], \psi_1 \in [-\varphi_1, \varphi_1]$ and L satisfies the following equation

$$L = \max \left\{ \begin{aligned} & \left[\frac{\left[(2 - r^\omega G_1^\omega) \psi_1 + (G_1^\omega - 1) \psi_0 + 2a_2 - \varphi_{ss} \right]^2}{\left[(\varphi_0 - \varphi_{ss})^2 + (\varphi_1 - \varphi_{ss})^2 \right]} \right. \\ & \left. + \frac{(\psi_0 - \varphi_{ss})^2}{\left[(\varphi_0 - \varphi_{ss})^2 + (\varphi_1 - \varphi_{ss})^2 \right]} \right] \end{aligned} \right\} \quad (37)$$

3.1.3 Oscillatory Behavior

We now examine the behavior of the phase error for loop parameters outside the bounds given in (3.1.13). An oscillatory solution exists[11,12] where

$$\begin{aligned} \varphi_2 &= 2a_2 + (2 - r^\omega G_1^\omega) \varphi_1 + (G_1^\omega - 1) \varphi_2 \\ \varphi_1 &= 2a_2 + (2 - r^\omega G_1^\omega) \varphi_2 + (G_1^\omega - 1) \varphi_1 \end{aligned} \quad (38)$$

These yields

$$\varphi_1 = \varphi_2 = \frac{2a_2}{G_2^\omega}$$

It is easy to see that these two oscillatory points are the same as the steady-state point and this situation also suits for the three oscillatory points. This result illustrates that if the loop oscillates at some particular points then these points are the same to the only one stationary point and loop operates on the stable regions. Otherwise, the loop will oscillate with increasing amplitude like Fig. 7.

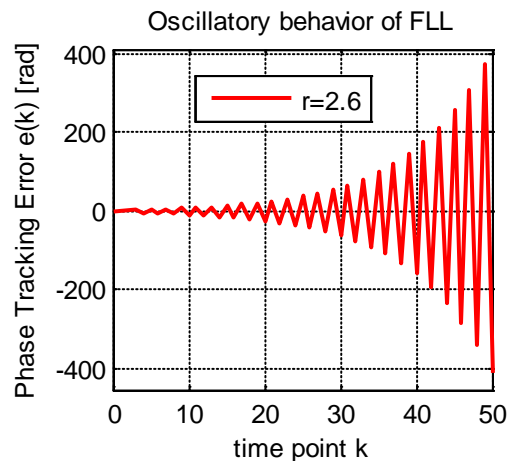


Fig. 7. Oscillatory Behavior of the second order FLL

3.2 The Third-Order PLL

As for the third order PLL, the loop transfer function is[6]

$$\Phi(z) = \frac{\Theta(z)}{1 + K^\theta F^\theta(z) N(z)} \quad (39)$$

By expanding the function we have

$$\begin{aligned} &\Theta(z)(1 - 3z^{-1} + 3z^{-2} - z^{-3}) \\ &= \Phi(z) \left[1 + (G_1^\theta + G_2^\theta + G_3^\theta - 3)z^{-1} \right. \\ &\quad \left. - (2G_1^\theta + G_2^\theta - 3)z^{-2} + (G_1^\theta - 1)z^{-3} \right] \end{aligned} \quad (40)$$

The corresponding difference equation is

$$\begin{aligned} &\theta_k - 3\theta_{k-1} + 3\theta_{k-2} - \theta_{k-3} \\ &= \varphi_k + (p^\theta G_1^\theta - 3)\varphi_{k-1} - [(1+r^\theta)G_1^\theta - 3]\varphi_{k-2} \\ &\quad + (G_1^\theta - 1)\varphi_{k-3} \end{aligned} \quad (41)$$

Here, the loop parameters are

$$p^\theta = 1 + \frac{G_2^\theta}{G_1^\theta} + \frac{G_3^\theta}{G_1^\theta}, \quad r^\theta = 1 + \frac{G_2^\theta}{G_1^\theta} \quad (42)$$

For a quadratic phase input, the difference equation can be expressed as

$$\begin{aligned} &\varphi_{k+3} + (p^\theta G_1^\theta - 3)\varphi_{k+2} \\ &\quad - [(1+r^\theta)G_1^\theta - 3]\varphi_{k+1} + (G_1^\theta - 1)\varphi_k = 0 \end{aligned} \quad (43)$$

It is easy to see that the stationary solution of (43) is

$$\varphi^* = 0 \quad (44)$$

3.2.1 Range of Gain for Stability

The equation (43) can be expressed as a system of three first-order equations, so it is in the form $x_{k+1} = G(x_k)$.

Thus, if $z_k = \varphi_{k+2}$, $y_k = \varphi_{k+1}$, $x_k = \varphi_k$, then equation (43) becomes

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} y_k \\ z_k \\ -(p^\theta G_1^\theta - 3)z_k \\ \quad + [(1+r^\theta)G_1^\theta - 3]y_k - (G_1^\theta - 1)x_k \end{bmatrix} \quad (45)$$

where, $x_k = \varphi_k$, $x_0 = (\varphi_0, \varphi_1, \varphi_2)^T$.

The Jacobian $G'(x) = (\partial g_i / \partial x^j)$ is given by

$$G'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 - G_1^\theta & (1+r^\theta)G_1^\theta - 3 & 3 - p^\theta G_1^\theta \end{bmatrix} \quad (46)$$

At the fixed point $x^* = 0$, we must have $|\lambda_i| < 1$, $i=1, 2, 3$. where the λ 's satisfy the characteristic equation

$|\lambda I - G'(x^*)| = 0$ or

$$\lambda^3 + \lambda^2 (p^\theta G_1^\theta - 3) + \lambda [3 - (1+r^\theta)G_1^\theta] + G_1^\theta - 1 = 0 \quad (47)$$

The Routh-Hurwitz criteria can be used to obtain stability bounds for (46) if the following transformation is used to map the interior of the unit circle to the left half-plan[12]:

$$\lambda = \frac{v+1}{v-1} \quad (48)$$

Substitute this transformation into (47), the resulting equation in v is

$$\begin{aligned} &v^3 (p^\theta G_1^\theta - r^\theta G_1^\theta) + v^2 (p^\theta G_1^\theta + r^\theta G_1^\theta - G_1^\theta) \\ &\quad + v (-p^\theta G_1^\theta + r^\theta G_1^\theta + 4G_1^\theta) \\ &\quad + (8 - p^\theta G_1^\theta - r^\theta G_1^\theta - 2G_1^\theta) = 0 \end{aligned} \quad (49)$$

When applying the Routh-Hurwitz criteria[11,12], we have the stable conditions for the coefficients

$$\begin{aligned} &p^\theta - r^\theta > 0 \\ &p^\theta + r^\theta - 2 > 0 \\ &4 - p^\theta + r^\theta > 0 \\ &8 - G_1^\theta (p^\theta + r^\theta + 2) > 0 \\ &G_1^\theta (p^\theta - 1) - p^\theta + r^\theta > 0 \end{aligned} \quad (50)$$

The faster convergence can be achieved when applying $\rho[G(x^*)] = 0$ by examining (46); these are

$$G_1^\theta = 1, \quad r^\theta = 2, \quad p^\theta = 3 \quad (51)$$

It is also required that $|\varphi_{k+3}| < \pi$ when $|\varphi_{k+2}| = \pi$, $|\varphi_{k+1}| = \pi$, $|\varphi_k| = \pi$ [8]. So the overall loop stable conditions independent of the initial phase tracking error can be further derived as the following expressions.

a. when $\varphi(k+2) = \pi$, $\varphi(k+1) = \pi$ and $\varphi(k) = \pi$, we have $|\varphi(k+3)| < \pi$, thus

$$-2 < (r^\theta - p^\theta)G_1^\theta < 0$$

b. when $\varphi(k+2) = \pi$, $\varphi(k+1) = \pi$ and $\varphi(k) = -\pi$, we have $|\varphi(k+3)| < \pi$, thus

$$0 < (2 + r^\theta - p^\theta)G_1^\theta < 2$$

c. when $\varphi(k+2) = \pi$, $\varphi(k+1) = -\pi$ and $\varphi(k) = \pi$, we have $|\varphi(k+3)| < \pi$, thus

$$6 < (p^\theta + r^\theta + 2)G_1^\theta < 8$$

d. when $\varphi(k+2) = \pi$, $\varphi(k+1) = -\pi$ and $\varphi(k) = -\pi$, we have $|\varphi(k+3)| < \pi$, thus

$$4 < (p^\theta + r^\theta)G_1^\theta < 6$$

e. when $\varphi(k+2) = -\pi$, $\varphi(k+1) = \pi$ and $\varphi(k) = \pi$, we have $|\varphi(k+3)| < \pi$, thus

$$4 < (p^\theta + r^\theta)G_1^\theta < 6$$

f. when $\varphi(k+2)=-\pi$, $\varphi(k+1)=\pi$ and $\varphi(k)=-\pi$, we have $|\varphi(k+3)|<\pi$, thus

$$6 < (p^\theta + r^\theta + 2)G_1^\theta < 8$$

g. when $\varphi(k+2)=-\pi$, $\varphi(k+1)=-\pi$ and $\varphi(k)=\pi$, we have $|\varphi(k+3)|<\pi$, thus

$$0 < (2 + r^\theta - p^\theta)G_1^\theta < 2$$

h. when $\varphi(k+2)=-\pi$, $\varphi(k+1)=-\pi$ and $\varphi(k)=-\pi$, we have $|\varphi(k+3)|<\pi$, thus

$$-2 < (r^\theta - p^\theta)G_1^\theta < 0$$

The above analysis can be concluded as

$$\begin{aligned} -2 < (r^\theta - p^\theta)G_1^\theta < 0 \\ 0 < (2 + r^\theta - p^\theta)G_1^\theta < 2 \\ 6 < (p^\theta + r^\theta + 2)G_1^\theta < 8 \\ 4 < (p^\theta + r^\theta)G_1^\theta < 6 \end{aligned} \quad (52)$$

According to (52), the stable region on the r - p plane of loop parameters is show in Fig. 8 with $G_1^\theta=1$. The stable regions are invariant when just consider the quadratic phase input. Figs. 9 and 10 shows the loop transient responses with initial phase error $\varphi(-2)=\varphi(-1)=0$, $\varphi(0)=1$ (rad) and $G_1^\theta=1$, $r^\theta=1.5, 2, 3$ and $p^\theta=2.5, 3, 3.5$. It is noticed that whatever the loop gain parameters change while inside the stable regions, the loop will converge to zero steady-state phase error, and the fast convergence is achieved with $G_1^\theta=1$, $r^\theta=2$ and $p^\theta=3$.

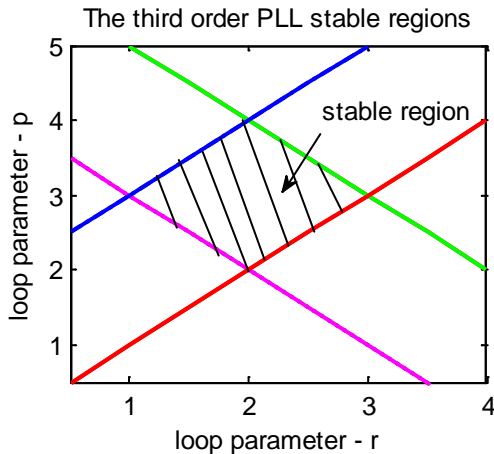


Fig. 8 The third order PLL stable regions

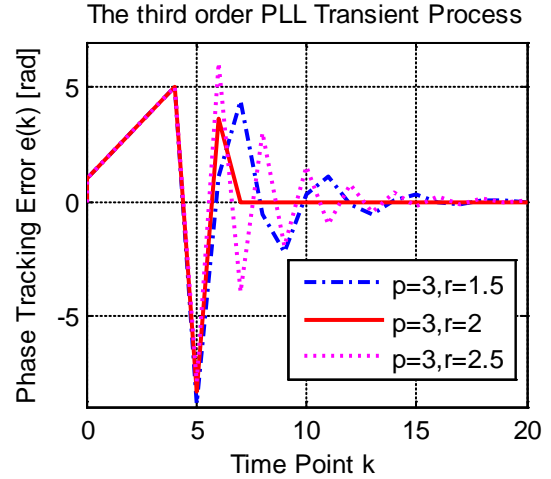


Fig. 9 FLL assisted PLL transient processes; $G_1=1.0$, $\varphi(-2)=\varphi(-1)=0$, $\varphi(0)=1$ (rad) $p=3$ and $r=1.5, 2, 2.5$

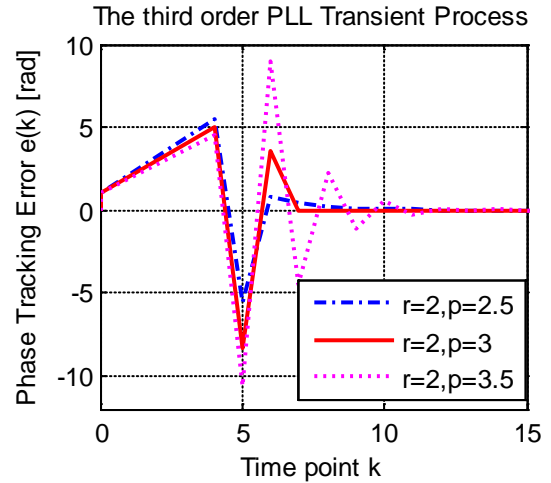


Fig. 10 FLL assisted PLL transient processes; $G_1=1.0$, $\varphi(-2)=\varphi(-1)=0$, $\varphi(0)=1$ (rad) $r=2$ and $p=2.5, 3, 3.5$

3.2.2 Number of Steps for Convergence

Applying the method of Section 3.1.2, the steps to convergences is

$$m < \frac{\ln \left(\frac{\varepsilon}{\left[(\varphi_0 - \varphi_{ss})^2 + (\varphi_1 - \varphi_{ss})^2 + (\varphi_2 - \varphi_{ss})^2 \right]} \right)}{\ln(L)} \quad (53)$$

And L satisfies

$$L = \max \left\{ \begin{aligned} & \left[\frac{\left[(3 - p^\theta G_1^\theta) \psi_2 + ((1 + r^\theta) G_1^\theta - 3) \psi_1 - (G_1^\theta - 1) \psi_0 - \varphi_{ss} \right]^2}{\left[(\varphi_0 - \varphi_{ss})^2 + (\varphi_1 - \varphi_{ss})^2 + (\varphi_2 - \varphi_{ss})^2 \right]} \right. \\ & + \left. \frac{(\psi_1 - \varphi_{ss})^2}{\left[(\varphi_0 - \varphi_{ss})^2 + (\varphi_1 - \varphi_{ss})^2 + (\varphi_2 - \varphi_{ss})^2 \right]} \right. \\ & + \left. \frac{(\psi_0 - \varphi_{ss})^2}{\left[(\varphi_0 - \varphi_{ss})^2 + (\varphi_1 - \varphi_{ss})^2 + (\varphi_2 - \varphi_{ss})^2 \right]} \right\} \end{aligned} \right.$$

Where φ_0, φ_1 and φ_2 are the initial phase errors, φ_{ss} is the stationary phase error, arguments ψ_0, ψ_1 and ψ_2 satisfy $\psi_0 \in [-\varphi_0, \varphi_0], \psi_1 \in [-\varphi_1, \varphi_1], \psi_2 \in [-\varphi_2, \varphi_2]$.

3.2.3 Oscillatory Behavior

We now examine the behavior of the phase error for loop parameters outside the bounds given in (50). An oscillatory solution exists where

$$\begin{aligned} \varphi_2 &= (p^\theta G_1^\theta - 3) \varphi_1 - \left[(1 + r^\theta) G_1^\theta - 3 \right] \varphi_2 + (G_1^\theta - 1) \varphi_1 \\ \varphi_1 &= (p^\theta G_1^\theta - 3) \varphi_2 - \left[(1 + r^\theta) G_1^\theta - 3 \right] \varphi_1 + (G_1^\theta - 1) \varphi_2 \end{aligned} \quad (54)$$

These yields

$$\varphi_1 = \varphi_2 = 0$$

These two oscillatory points are the same as the steady-state point and this situation also suits for three or four oscillatory points. This situation is similar to that of the second FLL. The stable oscillatory point is unique and equals to zero. The other oscillatory situations occur at unstable regions as Fig. 11.

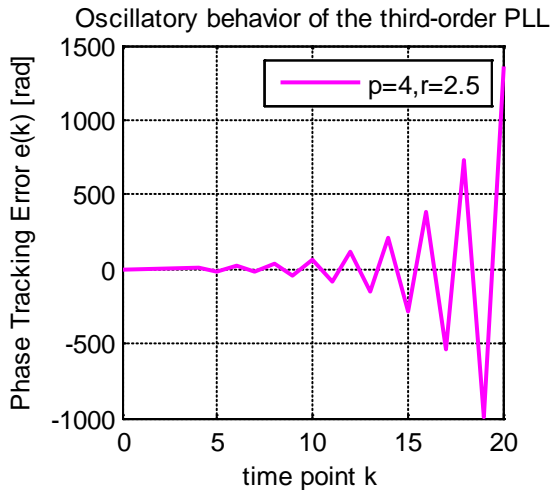


Fig. 11. Oscillatory Behavior of the third order PLL
3.3 The Combined Tracking Loop
 In the assisted carrier phase fine tracking, loop can be represented on z-plane as[6,8]

$$\Phi(z) = \frac{\Theta(z)}{1 + \left[K^\theta F^\theta(z) + D(z) K^\omega F^\omega(z) \right] N(z)} \quad (55)$$

Substitute (9)、(10)、(11) and (12) into (55), therefore, the combined tracking loop equation is

$$\begin{aligned} & \Theta(z) (1 - 3z^{-1} + 3z^{-2} - z^{-3}) \\ & = \Phi(z) \left[1 + (G_1^\theta + G_2^\theta + G_3^\theta + G_1^\omega + G_2^\omega - 3) z^{-1} \right. \\ & \quad \left. - (2G_1^\theta + G_2^\theta + 2G_1^\omega + G_2^\omega - 3) z^{-2} \right. \\ & \quad \left. + (G_1^\theta + G_1^\omega - 1) z^{-3} \right] \end{aligned} \quad (56)$$

Where, the coefficients of the filter are

$$\begin{aligned} G_1^\theta &= K_o K^\theta C_1^\theta, G_2^\theta = K_o K^\theta C_2^\theta \\ G_3^\theta &= K_o K^\theta C_3^\theta, G_1^\omega = K_o K^\omega C_1^\omega \\ G_2^\omega &= K_o K^\omega C_2^\omega \end{aligned} \quad (57)$$

The difference equation characterizing loop performance becomes

$$\begin{aligned} & \theta_k - 3\theta_{k-1} + 3\theta_{k-2} - \theta_{k-3} \\ & = \varphi_k + K_1 \varphi_{k-1} - K_2 \varphi_{k-2} + K_3 \varphi_{k-3} \end{aligned} \quad (58)$$

Where, the coefficients of the equation are

$$\begin{aligned} K_1 &= (G_1^\theta + C_1^\omega) + (G_2^\theta + C_2^\omega) + G_3^\theta - 3 \\ & \triangleq G_1 + G_2 + G_3 - 3 \\ K_2 &= 2(G_1^\theta + G_1^\omega) + (G_2^\theta + G_2^\omega) - 3 \\ & \triangleq 2G_1 + G_2 - 3 \\ K_3 &= (G_1^\theta + G_1^\omega) - 1 \triangleq G_1 - 1 \end{aligned} \quad (59)$$

For the quadratic phase input, we have

$$\varphi_k + K_1 \varphi_{k-1} - K_2 \varphi_{k-2} + K_3 \varphi_{k-3} = 0 \quad (60)$$

Also

$$\varphi_{k+3} + K_1 \varphi_{k+2} - K_2 \varphi_{k+1} + K_3 \varphi_k = 0 \quad (61)$$

It is easy to see that the stationary solution of (3.3.7) is

$$\varphi^* = 0 \quad (62)$$

3.3.1 Range of Gain for Stability

The combined difference equation can be represented in the form of a set of first-order, three-dimensional state equations as

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} y_k \\ z_k \\ -K_1 z_k + K_2 y_k - K_3 x_k \end{bmatrix} \quad (63)$$

where $z_k = \varphi_{k+2}$, $y_k = \varphi_{k+1}$, $x_k = \varphi_k$, and $x_0 = (\varphi_0, \varphi_1, \varphi_2)^T$.

The Jacobian $G'(x) = (\partial g_i / \partial x^j)$ is given by

$$G'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -K_3 & K_2 & -K_1 \end{bmatrix} \quad (64)$$

At the fixed point $x^* = 0$, we must have $|\lambda_i| < 1$, $i=1, 2, 3$. where the λ 's satisfy the characteristic equation $|\lambda I - G'(x^*)| = 0$ or

$$\lambda^3 + \lambda^2(K_1) + \lambda(-K_2) + K_3 = 0 \quad (65)$$

By using the transformation of (3.2.10), we can get the equation in v as

$$v^3(1 + K_1 - K_2 + K_3) + v^2(3 + K_1 + K_2 - 3K_3) + v(3 - K_1 + K_2 + 3K_3) + (1 - K_1 - K_2 - K_3) = 0 \quad (66)$$

When applying the Routh-Hurwitz criteria [12], we have

$$\begin{aligned} 1 + K_1 - K_2 + K_3 &> 0 \\ 3 + K_1 + K_2 - 3K_3 &> 0 \\ 3 - K_1 + K_2 + 3K_3 &> 0 \\ 1 - K_1 - K_2 - K_3 &> 0 \end{aligned} \quad (67)$$

So, we have the stable conditions for the parameters

$$\begin{aligned} p - r &> 0 \\ p + r - 2 &> 0 \\ (p - 1)G_1 - p + r &> 0 \\ 8 - (p + r + 2)G_1 &> 0 \end{aligned} \quad (68)$$

where $r = 1 + G_2/G_1$, $p = 1 + G_2/G_1 + G_3/G_1$.

The conditions for faster convergence, i.e., $\rho[G'(x^*)] = 0$, can be obtained by examining (64); these are

$$K_1 = K_2 = K_3 = 0 \quad (69)$$

This means

$$G_1 = G_2 = G_3 = 1 \quad (70)$$

It is also required that $|\varphi_{k+3}| < \pi$ when $|\varphi_{k+2}| = \pi$, $|\varphi_{k+1}| = \pi$, $|\varphi_k| = \pi$ [8]. So the overall loop stable conditions independent of the initial phase tracking error can be further derived as

$$\begin{aligned} -2 < (r - p)G_1 &< 0 \\ 4 < (p + r)G_1 &< 6 \\ 6 < (p + r + 2)G_1 &< 8 \\ 0 < (2 + r - p)G_1 &< 2 \end{aligned} \quad (71)$$

By comparing the analysis of this part with that of the third order PLL, it is possible to consider the combined loop as an equivalent PLL with filter coefficients $G_1^\theta + G_1^\omega$, $G_2^\theta + G_2^\omega$, G_3^θ , instead of G_1^θ , G_2^θ and G_3^θ . In this way the influence of the FLL can be inserted into the model of the PLL at a design stage. A wide bandwidth FLL allows the loop to

keep stable working while the PLL is unlocked when the frequency error is within the linear range of its discriminator. At this time, the loop response is governed by the FLL and the phase error input acts like a zero-mean perturbation [18]. So from the design point, the second order FLL assisted third PLL can be optimized to a single third order PLL, this will be discussed at Section 5.

Due to the fact that the combined loop can be regarded as an equivalent third order PLL, so the convergence steps and the oscillatory behavior are almost the same as the third order PLL except that the loop parameters are different. The steps to convergences is the same as (53),

$$m < \frac{\ln \left(\frac{\varepsilon}{\left[(\varphi_0 - \varphi_{ss})^2 + (\varphi_1 - \varphi_{ss})^2 + (\varphi_2 - \varphi_{ss})^2 \right]} \right)}{\ln(L)} \quad (72)$$

The difference comes up to the value of L which is

$$L = \max \left\{ \begin{aligned} &\frac{\left[-K_1\psi_2 + K_2\psi_1 - K_3\psi_0 - \varphi_{ss} \right]^2}{\left[(\varphi_0 - \varphi_{ss})^2 + (\varphi_1 - \varphi_{ss})^2 + (\varphi_2 - \varphi_{ss})^2 \right]} \\ &+ \frac{(\psi_1 - \varphi_{ss})^2}{\left[(\varphi_0 - \varphi_{ss})^2 + (\varphi_1 - \varphi_{ss})^2 + (\varphi_2 - \varphi_{ss})^2 \right]} \\ &+ \frac{(\psi_0 - \varphi_{ss})^2}{\left[(\varphi_0 - \varphi_{ss})^2 + (\varphi_1 - \varphi_{ss})^2 + (\varphi_2 - \varphi_{ss})^2 \right]} \end{aligned} \right\}$$

The oscillatory behavior is the same as Section 3.3.2 except for the loop parameters.

4. Performance in the Presence of Dynamics and Noise

In this section we consider the behavior for an input with frequency ramp in the presence of noise. Due to the stochastic nature of the input phase noise, the phase tracking error is also a stochastic process. We will apply the C-K equation to study the PDF, mean and variance of the phase error. In the following, we will derive the mathematical expressions of the PDF and variance of phase error due to the input noise and present how the phase error depends on the loop parameters and input noise statistics.

4.1 The Second Order FLL

4.1.1 Steady State PDF

For the quadratic phase input, the loop equation

of the restricted (i.e. $[-\pi, +\pi]$) phase error process $\{\varphi(k)\}$ is

$$\begin{aligned} \varphi_{k+2} - 2\varphi_{k+1} + \varphi_k \\ = 2a_2 + G_1^\omega (\varphi_k + n_k^\omega) - r^\omega G_1^\omega (\varphi_{k+1} + n_{k+1}^\omega) \end{aligned} \quad (73)$$

Note shall be taken that the frequency noise component n^ω are mutually dependent for the consecutive samples due to the difference discriminator. When given $(\varphi_k, \varphi_{k+1})$, the pdf of $(\varphi_{k+1}, \varphi_{k+2})$ can be achieved which is equal to that of n^ω . So the phase error tracking error process $\{\varphi(k)\}$ can be regarded as a first-order, two-dimensional, discrete time, continuously variable Markov process. Therefore we may rewrite it in a set of first-order stochastic difference equations[13]. Before this, we define $s(\cdot)$ by

$$\varphi_k = s(k) - r^\omega s(k+1) + \frac{2a_2}{G_2^\omega} \quad (74)$$

Equation (4.1) then becomes

$$\begin{aligned} [s(k+2) - 2s(k+1) + s(k)] \\ - r^\omega [s(k+3) - 2s(k+2) + s(k+1)] \\ = G_1^\omega [s(k) - r^\omega s(k+1) + n_k^\omega] \\ - r^\omega G_1^\omega [s(k+1) - r^\omega s(k+2) + n_{k+1}^\omega] \end{aligned} \quad (75)$$

Equation (75) may be broken up into two equivalent equations. Letting $x_1(k)=s(k)$ and $x_2(k)=s(k+1)$, one gets the following vector set of equations:

$$\begin{aligned} x_1(k+1) &= x_2(k) \\ x_2(k+1) &= (G_1^\omega - 1)x_1(k) \\ &\quad + (2 - r^\omega G_1^\omega)x_2(k) + G_1^\omega n_k^\omega \end{aligned} \quad (76)$$

Note that $\varphi_k = x_1(k) - r^\omega x_2(k) + 2a_2/G_2^\omega$ according to(74).

The vector process $\{x_1(k), x_2(k)\}$ can be regarded as a second-order, two dimensional Markov process. Consequently, the joint PDF of $x_1(k), x_2(k)$ satisfies the vector Chapman-Kolmogorov equation. The steady-state joint PDF $p(x_1, x_2)$ can be obtained by

$$\begin{aligned} p_{k+1}(x_1, x_2 | x_{10}, x_{20}) \\ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} q_k(x_1, x_2 | z_1, z_2) P_k(z_1, z_2 | x_{10}, x_{20}) dz_1 dz_2 \end{aligned} \quad (77)$$

where

$x_{10}=x_1(0)$	initial value of $x_1(\cdot)$
$x_{20}=x_2(0)$	initial value of $x_2(\cdot)$
$p_k(\cdot, \cdot x_{10}, x_{20})$	joint pdf of $x_1(k)$ and $x_2(k)$ given x_{10} and x_{20}
$q_k(x_1, x_2 z_1, z_2)$	transient joint pdf of $x_1(k+1), x_2(k+1)$ and $x_3(k+1)$ given $x_1(k)=z_1, x_2(k)=z_2$

we can derived from (76) that

$$\begin{aligned} q_{k+1}(x_1, x_2 | z_1, z_2) \\ = \delta(x_1 - z_2) \times \left\{ \frac{\exp(-SNR)}{2\pi G_1^\omega} + \frac{1}{2G_1^\omega} \sqrt{\frac{SNR}{\pi}} \cos(x^*) \right. \\ \left. \times \exp[-SNR \cdot \sin^2(x^*)] \times [1 + erf(\sqrt{SNR} \cos(x^*))] \right\} \end{aligned} \quad (78)$$

which is independent of k and

$$x^* = \frac{1}{G_1^\omega} \{x_2 - (2 - r^\omega G_1^\omega)z_2 - (G_1^\omega - 1)z_1\}$$

When $k \rightarrow \infty$, the steady-state pdf of the phase error φ is independent of initial phase error (φ_0, φ_1) which yields[13,14]

$$p_{k+1}(x_1, x_2) = \int_{-\infty}^{\infty} q_k(x_2 | z_1, z_2) p_k(z_1, x_1) dz_1 \quad (79)$$

Due to the periodicity of the discriminator, we have

$$\begin{aligned} P_k(x_1, x_2) &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} p_k(x_1 + 2m\pi, x_2 + 2n\pi) \\ K(x_1, x_2, z_1) &= \sum_{n=-\infty}^{\infty} q_k\left(\frac{\hat{x} + 2n\pi}{G_1^\omega}\right) \end{aligned} \quad (80)$$

where

$$\hat{x} = x_2 - (2 - r^\omega G_1^\omega)x_1 - (G_1^\omega - 1)z_1 + 2n\pi \quad (81)$$

Consequently, the steady-state PDF $P(x_1, x_2)$ is the unique solution to the following steady-state integral equation

$$P(x_1, x_2) = \int_{-\pi}^{\pi} K(x_1, x_2, z_1) P(z_1, x_1) dz_1 \quad (82)$$

4.1.2 Steady State Mean and Variance

The steady-state mean of the phase tracking error can be obtained by taking the expectations of both sides of (74) and (76) and letting $k \rightarrow \infty$, we have[14]

$$\begin{aligned} E[x_{1,ss}] &= E[x_{2,ss}] \\ E[x_{2,ss}] &= (G_1^\omega - 1)E[x_{1,ss}] \\ &\quad + (2 - r^\omega G_1^\omega)E[x_{2,ss}] + G_1^\omega \lim_{k \rightarrow \infty} E[n_k^\omega] \end{aligned} \quad (83)$$

$$E[\varphi_{ss}] = E[x_{1,ss}] - r^\omega E[x_{2,ss}] + \frac{2a_2}{G_2^\omega}$$

It is easy to see that $E[n^\omega(k)]$ is very small compared with other terms, so we have that the steady-state mean of the phase tracking error is

$$E[\varphi_{ss}] = \frac{2a_2}{G_2^\omega} \quad (84)$$

Modify (73) to yield

$$\text{var}(\varphi_{ss}) = E[x_{1,ss}^2] + r^2 E[x_{2,ss}^2] - 2r E[x_{1,ss} x_{2,ss}] \quad (85)$$

By multiplying x_{1ss} with both sides of (76) and taking expectation of each term while considering the dependency of the consecutive frequency samples, we have

$$E[x_{1ss}x_{2ss}] = (G_1^\omega - 1)E[x_{1ss}^2] + (2 - r^\omega G_1^\omega)E[x_{1ss}x_{2ss}] + (G_1^\omega)^2 R_1^n \quad (86)$$

Squaring both sides of (76) and taking the expectation of each term, we have

$$E[x_{1ss}^2] = E[x_{2ss}^2] \\ E[x_{2ss}^2] = (G_1^\omega - 1)^2 E[x_{1ss}^2] + (2 - r^\omega G_1^\omega)^2 E[x_{2ss}^2] + (G_1^\omega)^2 E[n_k^\omega]^2 + 2(G_1^\omega - 1)(2 - r^\omega G_1^\omega)E[x_{1ss}x_{2ss}] + 2(2 - r^\omega G_1^\omega)(G_1^\omega)^2 R_1^n \quad (87)$$

From (85)-(87) we have

$$\text{var}(\varphi_{ss}) = \frac{[R_1^n - R_0^n + G_1^\omega R_1^n] [(G_1^\omega)^2 + 2G_1^\omega + 3]}{(3 - r^\omega G_1^\omega)(G_1^\omega + 1)} - \frac{2R_0^n}{(G_1^\omega + 1)(G_1^\omega + r^\omega G_1^\omega - 2)} - R_1^n (2G_1^\omega - r^\omega G_1^\omega + 1) \quad (88)$$

According to [10], the correlations of n_k^ω are

$$R_0^n \approx \frac{2}{SNR} \left(1 - e^{-0.05243 \cdot SNR^2}\right) + \frac{\pi^2}{3} e^{-0.50301 \cdot SNR} \quad (89) \\ R_1^n \approx -\frac{1}{2} \left(1 - e^{-0.4864 \cdot SNR^2}\right) R_0^n$$

4.2 The Third Order PLL^[8]

The third order case can be concluded from [8] while just considering the quadratic phase input.

4.2.1 Steady State PDF

The third order loop difference equation can be expressed as

$$\varphi_{k+3} - 3\varphi_{k+2} + 3\varphi_{k+1} - \varphi_k = -p^\theta G_1^\theta (\varphi_{k+2} + n_{k+2}^\theta) + (1 + r^\theta) G_1^\theta (\varphi_{k+1} + n_{k+1}^\theta) - G_1^\theta (\varphi_k + n_k^\theta) \quad (90)$$

The loop equation can be represented in the form of a set of first-order, three-dimensional state equations as

$$x_1(k+1) = x_2(k) \\ x_2(k+1) = x_3(k) \\ x_3(k+1) = (1 - G_1^\theta) x_1(k) + [(1 + r^\theta) G_1^\theta - 3] x_2(k) + (3 - p^\theta G_1^\theta) x_3(k) + G_1^\theta n^\theta(k) \quad (91)$$

The corresponding output equation is

$$\varphi_k = -x_1(k) + (1 + r^\theta) x_2(k) - p^\theta x_3(k) \quad (92)$$

The steady-state joint pdf $p(x_1, x_2, x_3)$ can be obtained by

$$p_{k+1}(x_1, x_2, x_3 | x_{10}, x_{20}, x_{30}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} q_k(x_1, x_2, x_3 | z_1, z_2, z_3) \times p_k(z_1, z_2, z_3 | x_{10}, x_{20}, x_{30}) dz_1 dz_2 dz_3 \quad (93)$$

where

$$x_{10} = x_1(0) \quad \text{initial value of } x_1(\cdot) \\ x_{20} = x_2(0) \quad \text{initial value of } x_2(\cdot) \\ x_{30} = x_3(0) \quad \text{initial value of } x_3(\cdot)$$

Using the fact that the noise $n^\theta(k)$ is independent of $x_1(k)$, $x_2(k)$ and $x_3(k)$, we can derived from () that

$$q_{k+1}(x_1, x_2, x_3 | z_1, z_2, z_3) = \delta(x_1 - z_2) \delta(x_2 - z_3) \times \frac{1}{G_1^\theta} \frac{1}{2\pi} \exp(-SNR) + \frac{1}{G_1^\theta} \frac{1}{2} \sqrt{\frac{SNR}{\pi}} \cos(x^*) \times \exp\{-SNR \cdot \sin^2(x^*)\} \times [1 + \text{erf}(\sqrt{SNR} \cos(x^*))] \quad (94)$$

where

$$x^* = \frac{1}{G_1^\theta} \left\{ x_3 - (1 - G_1^\theta) z_1 - [(1 + r^\theta) G_1^\theta - 3] z_2 - (3 - p^\theta G_1^\theta) z_3 \right\}$$

As for $k \rightarrow \infty$, the steady-state PDF of the phase error φ exists and is unique, independent of initial phase error φ_0 . Therefore, $\lim_{k \rightarrow \infty} p_k(\varphi | \varphi_0) \triangleq p(\varphi)$ could be obtained by

$$p_{k+1}(x_1, x_2, x_3) = \int_{-\infty}^{\infty} q_k(x_3 | z_1, z_2, z_3) p_k(z_1, x_1, x_2) dz_1 \quad (95)$$

We introduce periodicity into the pdf of the phase tracking error as

$$P_k(x_1, x_2, x_3) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} p_k(x_1 + 2m\pi, x_2 + 2n\pi, x_3 + 2l\pi) \quad (96)$$

As for the conditional PDF of the phase tracking error, by using periodicity we have

$$K(x_1, x_2, x_3, z_1) = \sum_{n=-\infty}^{\infty} q_k \left(\frac{\hat{x} + 2n\pi}{G_1^\theta} \right) \quad (97)$$

where the argument \hat{x} is

$$\hat{x} = x_3 - (3 - p^\theta G_1^\theta) x_2 - [(1 + r^\theta) G_1^\theta - 3] x_1 - (1 - G_1^\theta) z_1 + 2n\pi \quad (98)$$

Consequently, (95) becomes

$$P_{k+1}(x_1, x_2, x_3) = \int_{-\pi}^{\pi} K(x_1, x_2, x_3, z_1) P_k(z_1, x_1, x_2) dz_1 \quad (99)$$

$$\begin{aligned} \text{var}[\varphi_{ss}] &= E[\varphi_{ss}^2] - \{E[\varphi_{ss}]\}^2 \\ &= \frac{[(r^2 + 4p^2 - 8rp) + (2p + p^2 + 2rp + rp^2 - 2r - 3r^2 - r^3)] G_1 E[n_\theta^2]}{16(r - p) + (-12r + 12p - 10r^2 + 2p^2 + 8rp) G_1 + (1 + r)(2r + r^2 - 2p - p^2) G_1^2} \end{aligned} \quad (102)$$

4.3 The Combined Tracking Loop

4.3.1 Steady-State PDF of Phase Error

The phase tracking error in the presence of noise can be represented on z-plane as

$$\Phi(z) = \frac{\Theta(z) - n_k^\omega D(z) F^\omega(z) N(z) - n_k^\theta F^\theta(z) N(z)}{1 + [K^\theta F^\theta(z) + D(z) K^\omega F^\omega(z)] N(z)} \quad (103)$$

For the quadratic input phase, the loop difference equation of the restricted phase error process $\{\varphi(k)\}$ is, from (103) which can be presented as

$$\begin{aligned} \varphi_{k+3} - 3\varphi_{k+2} + 3\varphi_{k+1} - \varphi_k &= -p^\theta G_1^\theta (\varphi_{k+2} + n_{k+2}^\theta) + (1 + r^\theta) G_1^\theta (\varphi_{k+1} + n_{k+1}^\theta) \\ &\quad - G_1^\theta (\varphi_k + n_k^\theta) - r^\omega G_1^\omega (\varphi_{k+2} + n_{k+2}^\omega) \\ &\quad + (1 + r^\omega) G_1^\omega (\varphi_{k+1} + n_{k+1}^\omega) - G_1^\omega (\varphi_k + n_k^\omega) \end{aligned} \quad (104)$$

In order to use the results of the third order PLL, we must do some modification on equation (104) as

As for $k \rightarrow \infty$, the steady-state PDF is independent of the subscript k, which yields

$$P(x_1, x_2, x_3) = \int_{-\pi}^{\pi} K(x_1, x_2, x_3, z_1) P(z_1, x_1, x_2) dz_1 \quad (100)$$

4.2.2 Steady State Mean and Variance

By taking the expectations of (91) the steady-state mean of the phase error is obtained as

$$E[\varphi_{ss}] = 0 \quad (101)$$

the steady-state variance of the phase error is [6]

$$\begin{aligned} G &= G_1^\theta + G_1^\omega \\ P &= \frac{p^\theta G_1^\theta + r^\omega G_1^\omega}{G_1^\theta + G_1^\omega} \\ R &= \frac{(1 + r^\theta) G_1^\theta + (1 + r^\omega) G_1^\omega}{G_1^\theta + G_1^\omega} \\ N_{k+2} &= \frac{G(p^\theta G_1^\theta n_{k+2}^\theta + r^\omega G_1^\omega n_{k+2}^\omega)}{p^\theta G_1^\theta + r^\omega G_1^\omega} \\ N_{k+1} &= \frac{G[(1 + r^\theta) G_1^\theta n_{k+1}^\theta + (1 + r^\omega) G_1^\omega n_{k+1}^\omega]}{(1 + r^\theta) G_1^\theta + (1 + r^\omega) G_1^\omega} \\ N_k &= \frac{G(G_1^\theta n_k^\theta + G_1^\omega n_k^\omega)}{G_1^\theta + G_1^\omega} \end{aligned} \quad (105)$$

By substituting (105) into (104) yields

$$\begin{aligned} \varphi_{k+3} - 3\varphi_{k+2} + 3\varphi_{k+1} - \varphi_k &= -PG(\varphi_{k+2} + N_{k+2}) \\ &\quad + RG(\varphi_{k+1} + N_{k+1}) - G(\varphi_k + N_k) \end{aligned} \quad (106)$$

Now the form of (106) is the same as that of (91) except of the loop parameters and noise character. So it can be represented in the form of a set of first-order, three-dimensional state equations as

$$\begin{aligned} x_1(k+1) &= x_2(k) \\ x_2(k+1) &= x_3(k) \\ x_3(k+1) &= (3-PG)x_3(k) + (RG-3)x_2(k) \\ &\quad + (1-G)x_1(k) + GN_k \end{aligned} \quad (107)$$

The corresponding output equation is

$$\varphi_k = -x_1(k) + Rx_2(k) - Px_3(k) \quad (108)$$

Comparing (107) and (108) with (91) and (92), we know that the equation form are the same. So, all the methods studying the steady-state character of the third-order PLL can be used to analyze that of the combined loop. Thus, steady-state PDF suits to the following equation

$$P(x_1, x_2, x_3) = \int_{-\pi}^{\pi} K(x_1, x_2, x_3, z_1) P(z_1, x_1, x_2) dz_1 \quad (109)$$

The expressions for K and P are

$$\begin{aligned} P_k(x_1, x_2, x_3) &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} p_k(x_1 + 2m\pi, x_2 + 2n\pi, x_3 + 2l\pi) \end{aligned} \quad (110)$$

$$K(x_1, x_2, x_3, z_1) = \sum_{n=-\infty}^{\infty} q_k\left(\frac{\hat{x} + 2n\pi}{G}\right) \quad (111)$$

where the argument and transient PDF are

$$\hat{x} = x_3 - (3-PG)x_2 - (RG-3)x_1 - (1-G)z_1 + 2n\pi \quad (112)$$

$$\begin{aligned} q_k(x_1, x_2, x_3 | z_1, z_2, z_3) &= \delta(x_1 - z_2) \delta(x_2 - z_3) \\ &\quad \times \frac{1}{G} \frac{1}{2\pi} \exp(-SNR) + \frac{1}{G} \frac{1}{2} \sqrt{\frac{SNR}{\pi}} \cos(x^*) \\ &\quad \times \exp\{-SNR \cdot \sin^2(x^*)\} \\ &\quad \times \left[1 + \operatorname{erf}\left(\sqrt{SNR} \cos(x^*)\right)\right] \end{aligned} \quad (113)$$

and

$$x^* = \frac{1}{G} [x_3 - (1-G)z_1 - (RG-3)z_2 - (3-PG)z_3]$$

The illustrations for symbols P, K, q and z₁, z₂, z₃ are the same as the explanations for equation (93).

$$\begin{aligned} \operatorname{var}[\varphi_{ss}] &= \frac{\left[-G^2(G + 6R + 6PR + GP^2 + GP^3 - GR^2 - 3P^2 - 3R^2 + GP - GPR^2 - 3) \right] R_0^N + G^2 R_1^N \left[2G^2 P^4 + 2G^2 P^3 \right. \\ &\quad - 2G^2 P^2 R^2 + 2G^2 P^2 - 2G^2 PR^2 + 2G^2 P - 18GP^3 + 16GP^2 R - 12GP^2 + 6GPR^2 + 28GPR - 18GP - 8GR^3 \\ &\quad \left. + 12GR^2 - 4GR - 12G + 42P^2 - 72PR + 42R^2 - 72R + 42 \right]}{G^3(P^3 + P^2 - PR^2 + P - R^2 + 1) + G^2(-9P^2 + 6PR - 8P + 5R^2 + 6R - 5) + G(18P - 30R + 6) + 18} \end{aligned} \quad (119)$$

4.3.2 Steady-State Mean and Variance

By taking the expectations of (91) the steady-state mean of the phase error is obtained as

$$E[\varphi_{ss}] = 0 \quad (114)$$

the steady-state variance of the phase error is

$$\begin{aligned} \operatorname{var}[\varphi_{ss}] &= E[\varphi_{ss}^2] - \{E[\varphi_{ss}]\}^2 \\ &= (1 + R^2 + P^2) E[x_{1ss}^2] - 2R(P+1) E[x_{2ss} x_{3ss}] \end{aligned} \quad (115)$$

Multiply x₂(k) with both sides of (4107) and taking expectation of each term while considering the the dependency of the consecutive two samples, we have

$$\begin{aligned} E[x_{2ss} x_{3ss}] &= (RG-3) E[x_{2ss}^2] \\ &\quad + (4-PG-G) E[x_{2ss} x_{3ss}] + G^2 R_1^N \end{aligned} \quad (116)$$

Squaring both sides of (107) and taking the expectation of each term, we have

$$\begin{aligned} E[x_{3ss}^2] &= \left[(3-PG)^2 + (RG-3)^2 + (1-G)^2 \right] E[x_{3ss}^2] \\ &\quad + \left[2(RG-3)(4-PG-G) \right] E[x_{2ss} x_{3ss}] \\ &\quad + 2G^2(3-PG)R_1^N + G^2R_0^N \end{aligned} \quad (117)$$

In equation (117), we have supposed that E[x_{1ss}x_{2ss}]=E[x_{2ss}x_{3ss}] for simplicity. According to the analysis of section 2, n^θ and n^ω have the same PDF, so the correlations of N_k are

$$\begin{aligned} R_1^N &= \frac{G^2(G_1^\omega)^2 R_1^\omega}{(G_1^\theta + G_1^\omega)^2} \\ R_0^N &= \frac{G^2 \left[(G_1^\theta)^2 R_0^\theta + (G_1^\omega)^2 R_0^\omega \right]}{(G_1^\theta + G_1^\omega)^2} \end{aligned} \quad (118)$$

Where R₀^N, R₁^N are the correlations of noise N_k with correlate spaces are zero and one respectively. R₀^ω, R₁^ω are the correlations of n^ω; R₀^θ, R₁^θ are the correlations of n^θ.

From equations (116), (117) and (118), we have the variance of the combined loop as

5 Loop Optimization

This section will discuss the optimization for the FLL assisted PLL at a design stage. The aim is to find the optimal loop filter structure. The method to design the digital loop filters is based on an optimization process that poses the tradeoff between noise bandwidth and dynamics in a quadratic function which is minimized for a particular dynamic input.

5.1 Optimum Filter for Arbitrary Input

Assuming the performance function to be [15, 16]

$$Q = E[n_0^2(k)] + \lambda \sum_k e^2(k) \quad (120)$$

Where $e(k) = \theta(k) - \hat{\theta}(k)$ is the deterministic component of the phase difference between incoming and generated phase. The parameter λ is determined on the basis of noise bandwidth considerations. The noise part of (120) can be expressed in terms of the closed loop transfer function $H(z)$ as

$$E[n_0^2(k)] = \frac{1}{2\pi j} \oint_{|z|=1} \Phi_{n_0 n_0}(z) \frac{dz}{z} \quad (121)$$

Where $\Phi_{n_0 n_0}$ is the noise spectral density of noise n_0 and is related to the input noise spectral density as

$$\Phi_{n_0 n_0}(z) = H(z)H(z^{-1})\Phi_{n_i n_i}(z) \quad (122)$$

And the closed loop transfer function is

$$H(z) = \frac{F(z)N(z)}{1 + F(z)N(z)} \quad (123)$$

According to Parseval equation, the second part of the performance function can be rewritten as

$$\sum_k e^2(k) = \frac{1}{2\pi j} \oint_{|z|=1} [1 - H(z)][1 - H(z^{-1})]\Phi_{\theta\theta}(z) \frac{dz}{z} \quad (124)$$

Where $\Theta(z)$ is the z transformation of $\theta(k)$ and $\Theta_{\theta\theta}(z) = \Theta(z)\Theta(z^{-1})$.

From (121) and (124), equation (120) can be expressed as

$$Q = \frac{1}{2\pi j} \oint_{|z|=1} \left[\lambda \Phi_{\theta\theta}(z) + P(z)W(z)W(z^{-1}) - \lambda W(z)N(z)\Phi_{\theta\theta}(z) - \lambda W(z^{-1})N(z^{-1})\Phi_{\theta\theta}(z) \right] \frac{dz}{z} \quad (125)$$

Where

$$H(z) = W(z)N(z) \quad (126)$$

And

$$P(z) = [\Phi_{n_i n_i}(z) + \lambda \Phi_{\theta\theta}(z)]N(z)N(z^{-1}) \quad (127)$$

Let $W(z) = W_0(z) + \varepsilon\eta(z)$, where $W_0(z)$ is the optimal loop transfer function and $\varepsilon\eta(z)$ is the variation to be minimized on the true optimum $W_0(z)$. Substitute this equation into (125) and setting the variation of Q to zero at ε equals zero completes the standard vibrational procedure [16].

$$\left. \frac{\partial Q[W_0(z) + \varepsilon\eta(z)]}{\partial \varepsilon} \right|_{\varepsilon=0} = \frac{1}{2\pi j} \oint_{|z|=1} \left[P(z)W(z)\eta(z^{-1}) + P(z)W(z^{-1})\eta(z) - \lambda\eta(z)N(z)\Phi_{\theta\theta}(z) - \lambda\eta(z^{-1})N(z^{-1})\Phi_{\theta\theta}(z) \right] \frac{dz}{z} \quad (128)$$

In order to keep all the terms in (128) split into factors having poles in either, but not both, the inside or outside the unit circle, it is convenient to define

$$P(z) = P^+(z)P^-(z) \quad (129)$$

Where all the poles in $P^+(z)$ are inside the unit circle and $P^-(z)$ has all the poles outside the unit circle. So, we have from (128) that

$$\left. \frac{\partial Q[W_0(z) + \varepsilon\eta(z)]}{\partial \varepsilon} \right|_{\varepsilon=0} = \frac{1}{2\pi j} \oint_{|z|=1} \left\{ \eta(z^{-1})P^-(z) \left[P^+(z)W_0(z) - \lambda \frac{N(z^{-1})\Phi_{\theta\theta}(z)}{P^-(z)} \right] + \eta(z)P^+(z) \left[P^-(z)W_0(z^{-1}) - \lambda \frac{N(z)\Phi_{\theta\theta}(z)}{P^+(z)} \right] \right\} \frac{dz}{z} \quad (130)$$

From (130), we can find that the poles in the first part are z_i and those in the second part are z_i^{-1} , and we have [17]

$$\oint f(z^{-1}) \frac{dz}{z} = \oint f(z) \frac{dz}{z}$$

So, (130) can be reduced to be

$$\left. \frac{\partial Q[W_0(z) + \varepsilon\eta(z)]}{\partial \varepsilon} \right|_{\varepsilon=0} = \frac{1}{\pi j} \oint_{|z|=1} \left\{ \eta(z^{-1})P^-(z) \left[z^{-1}P^+(z)W_0(z) - \lambda \frac{N(z^{-1})\Phi_{\theta\theta}(z)}{zP^-(z)} \right] \right\} \quad (131)$$

Taking (131) equal to zero yields

$$W_0(z) = \frac{z \cdot \left[\frac{\lambda N(z^{-1}) \Phi_{\theta\theta}(z)}{z \cdot P^-(z)} \right]_+}{P^+(z)} \quad (132)$$

are

$$F(z) = \frac{W_0(z)}{1 - W_0(z)N(z)} \quad (133)$$

Where $[\]_+$ illustrates that all the poles are outside the unit circle. The corresponding optimal loop filters

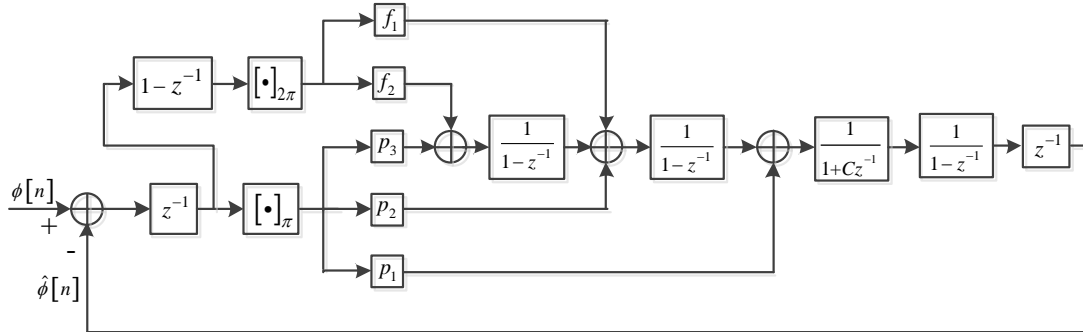


Fig. 12. The structure of the optimum PLL for tracking Doppler rates

5.2 Optimal Filter for Doppler Rates

Utilizing the analysis method of Section 5.1, the optimum filter for a Doppler rate signal can be derived. According to [17,18], the optimal filter for a frequency ramp signal can be expressed as

$$F(z) = \frac{A - Bz^{-1} + cz^{-2}}{(1 - z^{-1})^3 (1 + cz^{-1})} \quad (134)$$

For the purpose of implementation, it is desirable to use a cascade of accumulators. Then (134) can be rewritten as

$$F(z) = \frac{p_3 + (f_2 + p_2)(1 - z^{-1}) + (f_1 + p_1)(1 - z^{-1})^2}{(1 - z^{-1})^3 (1 + p_1 z^{-1})} \quad (135)$$

Where $f_1 + p_1 = C$, $f_2 + p_2 = B - 2C$ and $p_3 = A - B + C$. The corresponding structure of the optimum loop is illustrated in Figure 12. And A, B and C satisfy the

following equations

$$A = 6 - 3z_s + z_d, B = 8 - 3z_s + z_p, C = 3 - z_s \quad (136)$$

$$z_s = z_1 + z_2 + z_3, z_p = z_1 z_2 z_3, z_d = z_1 z_2 + z_1 z_3 + z_2 z_3$$

Where z_1, z_2 and z_3 are the solutions of the following equations

$$\begin{aligned} z_{1,2} + z_{1,2}^{-1} &= 2 - \frac{1 \pm j\sqrt{3}}{2} \sqrt[3]{v} \\ z_3 + z_3^{-1} &= 2 + \sqrt[3]{v} \\ v &= \dot{\Delta} \omega^2 T^4 \gamma^2 \end{aligned} \quad (137)$$

And the corresponding frequency ramp is

$$\Phi(z) = \frac{\dot{\Delta} \omega T^2}{(1 - z^{-1})^3} \quad (138)$$

The relationship of loop noise equivalent bandwidth versus loop parameter v is shown in Figure 13. When designing a loop, we shall first choose the appropriate noise bandwidth so as to get the v value and substitute it into (137), z_1, z_2 and z_3 will be calculated, then A, B and C will be derived by using equation (136). Thus, we get f_1 and f_2, p_1, p_2 and p_3 .

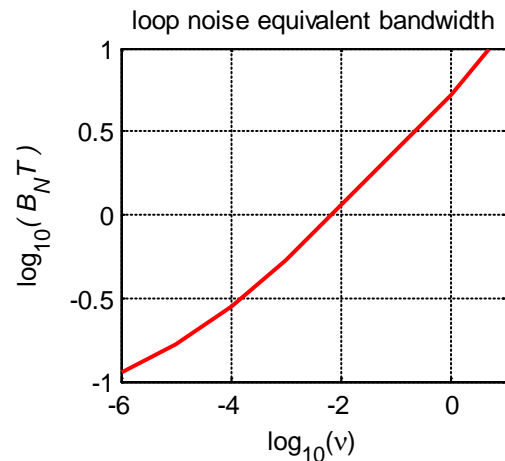


Fig. 13. Loop noise equivalent bandwidth

5.3 Design Example: Loops for Responders

Taking the trajectory in Section 1 into account, we model the input dynamic as an acceleration step, which becomes quadratic ramp in terms of phase and a linear one in term of frequency. For these inputs, the optimal loop filter was obtained in Section 5.2.

A typical rule of thumb for keeping a reasonable distance from the pull-out values of the

loop is that the peak of the error transient has a maximum value given by half the linear range of the phase discriminator, an eighth of cycle [17]. Always we take $B_N T < 1$ to keep loop stable. So, we have $B_N T = 0.125$ (i.e. -0.9 for $\log(\cdot)$) which means $v = 3 \times 10^{-4}$. Substitute this values into (137), (136) and (135) yields $A = 0.6173$, $B = 1.105$ and $C = 0.5$.

5.4 Stable Range for the Optimum Loop

The optimal loop has an extra pole at $z = -C$ compared with the traditional one. So, the loop transfer function is

$$\begin{aligned} \Theta(z) &= [1 + (C-3)z^{-1} + 3(1-C)z^{-2} + (3C-1)z^{-3} - Cz^{-4}] \\ &= \Phi [1 + (C-3)z^{-1} + (p_1 + p_2 + p_3 - 3C + 3)z^{-2} \\ &\quad + (3C - 2p_1 - p_2 - 1)z^{-3} + (p_1 - C)z^{-4}] \end{aligned} \quad (139)$$

So, we have corresponding difference equation as

$$\begin{aligned} \theta_{k+4} + (C-3)\theta_{k+3} + 3(1-C)\theta_{k+2} + (3C-1)\theta_{k+1} - C\theta_k \\ = \varphi_{k+4} + (C-3)\varphi_{k+3} + (p_1 + p_2 + p_3 - 3C + 3)\varphi_{k+2} \\ + (3C - 2p_1 - p_2 - 1)\varphi_{k+1} + (p_1 - C)\varphi_k \end{aligned} \quad (140)$$

As for the quadratic phase input like (26), we have

$$\begin{aligned} \varphi_{k+4} = (3-C)\varphi_{k+3} - (p_1 + p_2 + p_3 - 3C + 3)\varphi_{k+2} \\ + (2p_1 + p_2 + 1 - 3C)\varphi_{k+1} + (C - p_1)\varphi_k \end{aligned} \quad (141)$$

It's easy to see that the fixed point is

$$\varphi^* = 0 \quad (142)$$

This equation can be rewritten in a set of first-order difference equations as

$$\begin{bmatrix} w_{k+1} \\ x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \\ z_k \\ \kappa_1 z_k - \kappa_2 y_k + \kappa_3 x_k + \kappa_4 w_k \end{bmatrix} \quad (143)$$

where $w_k = \varphi_k$, $w_0 = (\varphi_0, \varphi_1, \varphi_2, \varphi_3)^T$ and the loop parameters are

$$\begin{aligned} \kappa_1 &= 3 - C \\ \kappa_2 &= p_1 + p_2 + p_3 - 3C + 3 \\ &\triangleq pp_1 - 3C + 3 \\ \kappa_3 &= 2p_1 + p_2 + 1 - 3C \\ &\triangleq (1+r)p_1 + 1 - 3C \\ \kappa_4 &= C - p_1 \end{aligned} \quad (144)$$

where $r = 1 + p_2/p_1$, $p = r + p_3/p_1$.

The Jacobian $G'(x) = (\partial g_i / \partial x^j)$ is given by

$$G'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \kappa_4 & \kappa_3 & -\kappa_2 & \kappa_1 \end{bmatrix} \quad (145)$$

At the fixed point $x^* = 0$, we must have $|\lambda_i| < 1$, $i = 1, 2, 3$. where the λ 's satisfy the characteristic equation $|\lambda I - G'(x^*)| = 0$ or

$$\lambda^4 + \lambda^3(-\kappa_1) + \lambda^2(\kappa_2) + \lambda(-\kappa_3) + (-\kappa_4) = 0 \quad (146)$$

By utilizing (48) in (146), we have

$$\begin{aligned} v^4 [\kappa_2 - \kappa_1 - \kappa_3 - \kappa_4 + 1] + v^3 [2\kappa_3 - 2\kappa_1 + 4\kappa_4 + 4] \\ + v^2 [6 - 6\kappa_4 - 2\kappa_2] + v [2\kappa_1 - 2\kappa_3 + 4\kappa_4 + 4] \\ + [\kappa_1 + \kappa_2 + \kappa_3 - \kappa_4 + 1] = 0 \end{aligned} \quad (147)$$

When the Routh-Hurwitz criteria is applied, we have

$$\begin{aligned} \kappa_2 - \kappa_1 - \kappa_3 - \kappa_4 + 1 &> 0 \\ 2\kappa_3 - 2\kappa_1 + 4\kappa_4 + 4 &> 0 \\ 6 - 6\kappa_4 - 2\kappa_2 &> 0 \\ 2\kappa_1 - 2\kappa_3 + 4\kappa_4 + 4 &> 0 \\ \kappa_1 + \kappa_2 + \kappa_3 - \kappa_4 + 1 &> 0 \end{aligned} \quad (148)$$

Also we have from (144) and (148) that

$$\begin{aligned} p - r &> 0 \\ r - 1 &> 0 \\ p &< 3 \\ 4C - (r + 3)p_1 + 4 &> 0 \\ -8C + (p + r + 2)p_1 + 8 &> 0 \end{aligned} \quad (149)$$

From (149), we can derive the region for C as

$$\begin{aligned} -1 < C < 1, \quad \text{for } r \text{ and } p \\ \frac{(r+3)p_1 - 4}{4} < C < \frac{(p+r+2)p_1 + 8}{8}, \quad \text{given } r \text{ and } p \end{aligned} \quad (150)$$

6 Conclusion

The performance of a second order FLL assisted third order PLL for the spread spectrum responder carrier phase tracking has been investigated. Stable regions, oscillatory behavior and steps for convergence for the FLL, PLL and the combined loop have been derived by taking them as a fixed point problem in the absence of noise. The mean and variance of these three loops have been studied in the presence of noise. At last, we have discussed the optimizations for the assisted tracking loop. This work provides an important reference in the design

for high dynamic carrier tracking in digital domain.

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References:

- [1] Shen Rongjun, Zhao Jun. Development Tendency and Strategy of Space TT&C Technology of Our Country. *Journal of Astronautics*, 3(22), 2001.
- [2] GUO Qi, GAO Fei and LI Guo-min. Analysis for performance and measures of anti-interference in a Spectrum TT&C System. *Journal of Spacecraft TT&C Technology*, 2005, 24(1): 1591-1595.
- [3] HAO Xue-kun, MA Wen-feng, FANG Hua and etc. The Study of Simulation for Tracking Statellite Doppler Shift Using Third-order Phase-locked loop. *Journal of System Simulation*, 2004, 4(16): 625-627.
- [4] Vilnrotter, V.A.; Hinedi, S.; Kumar, R., "Frequency estimation techniques for high dynamic trajectories," *Aerospace and Electronic Systems, IEEE Transactions on* , vol.25, no.4, pp.559,577, Jul 1989.
- [5] Ward, Phillip W., "Performance Comparisons Between FLL, PLL and a Novel FLL-Assisted-PLL Carrier Tracking Loop Under RF Interference Conditions," *Proceedings of the 11th International Technical Meeting of the Satellite Division of The Institute of Navigation (ION GPS 1998)*, Nashville, TN, September 1998, pp. 783-795.
- [6] Curran, J.T.; Lachapelle, G.; Murphy, C.C., "Improving the Design of Frequency Lock Loops for GNSS Receivers," *Aerospace and Electronic Systems, IEEE Transactions on* , vol.48, no.1, pp.850,868, Jan. 2012.
- [7] Kaplan E D. *Understanding GPS:Principles and Applications*, Artech House, Chapter 5, pp.153-241.
- [8] Weihua Zhuang, "Performance analysis of GPS carrier phase observable," *IEEE Transactions on Aerospace and Electronic Systems*, vol.32, no.2, pp.754,767, April 1996.
- [9] Yu, W., Lachapelle, G., Skone, S., "PLL Performance for Signals in the Presence of Thermal Noise, Phase Noise, and Ionospheric Scintillation," *Proceedings of the 19th International Technical Meeting of the Satellite Division of The Institute of Navigation (ION GNSS 2006)*, Fort Worth, TX, September 2006, pp. 1341-1357.
- [10] P. Crosta, A Novel Approach to the Performance Evaluation of an Arctangent Discriminator for Phase Locked Loop and application to the carrier tracking of the Ionospheric Scintillation, (ENC-GNSS), Naples, Italy, May 2009.
- [11] Osborne, H., "Stability Analysis of an Nth Power Digital Phase-Locked Loop--Part I: First-Order DPLL," *Communications, IEEE Transactions on* , vol.28, no.8, pp.1343,1354, Aug 1980.
- [12] Osborne, H., "Stability Analysis of an Nth Power Digital Phase-Locked Loop--Part II: Second- and Third-Order DPLL's," *Communications, IEEE Transactions on* , vol.28, no.8, pp.1355,1364, Aug 1980.
- [13] Weinberg, Aaron; Bede Liu, "Discrete Time Analyses of Nonuniform Sampling First- and Second-Order Digital Phase Lock Loops," *Communications, IEEE Transactions on* , vol.22, no.2, pp.123,137, Feb 1974.
- [14] Jae Lee; Chong Un, "Performance Analysis of Digital Tanlock Loop," *Communications, IEEE Transactions on* , vol.30, no.10, pp.2398,2411, Oct 1982.
- [15] Kazemi, Pejman L., "Optimum Digital Filters for GNSS Tracking Loops," *Proceedings of the 21st International Technical Meeting of the Satellite Division of The Institute of Navigation (ION GNSS 2008)*, Savannah, GA, September 2008, pp. 2304-2313.
- [16] Jaffe, R.; Rechten, E., "Design and performance of phase-lock circuits capable of near-optimum performance over a wide range of input signal and noise levels," *Information Theory, IRE Transactions on* , vol.1, no.1, pp.66,76, March 1955.
- [17] Pedro A. Roncagliolo, Javier G. García, and Carlos H. Muravchik, "Optimized Carrier Tracking Loop Design for Real-Time High-Dynamics GNSS Receivers," *International Journal of Navigation and Observation*, vol. 2012, Article ID 651039, 18 pages, 2012.
- [18] P. A. Roncagliolo, C. E. De Blasis and C.H. Muravchik, "GPS Digital Tracking Loops Design for High Dynamic Launching Vehicles," *in Proceedings of The 9th International Symposium on Spread Spectrum Techniques and Applications*, ISSSTA 2006, Manaus, Brazil, August, 2006.