# Stability analysis of BAM neural networks with inertial term and time delay

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*Abstract:* - In this paper, the global exponential stability of BAM neural networks with inertial term and time delay are investigated. By chosen proper variable substitution the system is transformed to first order differential equation. Then some new sufficient conditions, which can ensure the globally exponential stability of the system are obtained by constructing suitable Lyapunov functional, using Halanay inequality and the fundamental solution matrix of coefficient matrix. Finally, two examples are given to illustrate the effectiveness of the results.

*Key–Words:* BAM neural networks, inertial term, Halanay inequality, Lyapunov functional, exponential stability.

## 1 Introduction

The bidirectional associative memory (BAM) neural networks were first introduced by Kosko Refs.$[1-3]$. It is a special class of recurrent neural networks that can store bipolar vector pairs. The BAM neural networks are composed of neurons arranged in two layers, the U-layer and V-layer. The neurons in one layer are fully interconnected to the neurons in the other layer, while there are no interconnections among neurons in the same layer. Through iterations of forward and backward information flows between the two layer, it performs a two-way associative search for stored bipolar vector pairs and generalize the single-layer auto-associative Hebbian correlation to a two-layer pattern-matched hetero-associative circuits. Therefore, this class of networks possesses good applications prospects in signal processing, image processing, pattern recognition and associative memories, etc. In Refs.$[4-8]$, the authors have investigated the stability and periodic solutions of BAM neural networks, some sufficient conditions have been given, respectively. The stability of BAM neural networks with reaction-diffusion terms has been considered in Refs.$[9-11]$.

On the other hand, the inertia can be considered a useful tool that is added to help in the generation of chaos in neural systems. Babcock and Westervelt [12] combined inertia and driving to explore chaos in one- and two-neuron systems.Tani et al.[13-16] added inertia and a nonlinear oscillating resistance to neural equations as a way of chaotically searching for memories in neural networks. In [17], the authors considered the bifurcation and chaos in a single inertial neuron model with both time delay and inertial term. Liu et al.[18-20] investigated the Hopf bifurcation and dynamics of an inertial two-neuron system or in a single inertial neuron mode. In [21], the authors investigated the dynamical characteristics of a single inertial neuron model with time delay under periodic external stimuli.

To the best of our knowledge, few authors have considered the globally exponential stability of BAM neural networks with inertial term and time delay, which is very important in theories and applications. In this paper, we will investigate the globally exponential stability of system.

We consider the following BAM neural networks with inertial term and time delay

$$\begin{cases} \frac{d^2u_i(t)}{dt^2} = -\alpha_i \frac{du_i(t)}{dt} - a_i u_i(t) + \sum\limits_{j=1}^{m} c_{ij} f_j(v_j(t)) \\ \qquad + \sum\limits_{j=1}^{m} d_{ij} f_j(v_j(t-\tau_{ji})) + I_i, \\ \frac{d^2v_j(t)}{dt^2} = -\beta_j \frac{dv_j(t)}{dt} - b_j v_j(t) + \sum\limits_{i=1}^{n} p_{ji} g_i(u_i(t)) \\ \qquad + \sum\limits_{i=1}^{n} q_{ji} g_i(u_i(t-\sigma_{ij})) + J_j, \end{cases} \tag{1}$$

for $i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$, where the second derivative are called an inertial term of system (1); $\alpha_i > 0$ and $\beta_j > 0$ are constants; $u_i(t)$ and $v_j(t)$ are the state of the $ith$ neurons from the neural field $F_U$ and the $jth$ neurons from the neural field $F_V$ at time $t$, respectively; $a_i > 0$ and $b_j > 0$ denote the rate which the $ith$ neurons and the $jth$ neurons will reset its potential to the resting state in isolation when disconnected from the networks and external inputs, respectively; $c_{ij}, d_{ij}, p_{ji}$ and $q_{ji}$ are constants, and denote the connection strengths; $f_j$ and $g_i$ denote the activation function of the $jth$ neurons and the $ith$ neurons at time $t$, respectively; $\tau_{ji}$ and $\sigma_{ij}$ denote correspond to the transmission delays and satisfy $0 \leq \tau_{ji} \leq \tau$ and $0 \leq \sigma_{ij} \leq \sigma$; $I_i$ and $J_j$ denote the external inputs on the $ith$ neurons from the neural f eld $F_U$ and the $jth$ neurons from the neural f eld $F_V$ , respectively.

The initial values of system (1) are

$$\begin{cases} u_i(s) = \varphi_{ui}(s), & \frac{du_i(s)}{dt} = \psi_{ui}(s), -\tau \leq s \leq 0, \\ v_j(s) = \varphi_{vj}(s), & \frac{dv_j(s)}{dt} = \psi_{vj}(s), -\sigma \leq s \leq 0, \end{cases} \tag{2}$$

where $\varphi_{ui}(s), \psi_{ui}(s), \varphi_{vj}(s)$ and $\psi_{vj}(s)$ are bounded and continuous function.

This paper is organized as follows. Some preliminaries are given in Section 2. In Section 3, the suff cient conditions are derived for the globally exponential stability of BAM neural networks with inertial term and time delay by constructing a suitable Lyapunov functional and utilizing some analytical techniques. In Section 4, two illustrative examples are given to show the effectiveness of the proposed theory.

## 2 Preliminaries

Throughout this paper, we make the following assumptions.

$(H)$ : The activation functions $f_j, g_i$ satisfy Lipschitz condition, i.e., there exist constant $l_j > 0, k_i > 0$, such that

$|f_j(v_1) - f_j(v_2))| \leq l_j |v_1 - v_2|$,

$|g_i(v_1) - g_i(v_2))| \leq k_i |v_1 - v_2|$,

$i = 1, 2, \cdots, n, j =, 2, \cdots, m$, for any $v_1, v_2 \in R$.

Using variable transformation:

$y_i(t) = \frac{du_i(t)}{dt} + u_i(t), \quad i = 1, 2, \cdots, n,$

$z_j(t) = \frac{dv_j(t)}{dt} + v_j(t), \quad j = 1, 2, \cdots, m,$

then of (1) and (2) can be rewritten as

$$\begin{cases} \frac{du_i(t)}{dt} = -u_i(t) + y_i(t), \\ \frac{dy_i(t)}{dt} = -(a_i - \alpha_i + 1)u_i(t) - (\alpha_i - 1)y_i(t) \\ + \sum\limits_{j=1}^{m} c_{ij} f_j(v_j(t)) + \sum\limits_{j=1}^{m} d_{ij} f_j(v_j(t - \tau_{ji})) + I_i, \\ \frac{dv_j(t)}{dt} = -v_j(t) + z_j(t), \\ \frac{dz_j(t)}{dt} = -(b_j - \beta_j + 1)v_j(t) - (\beta_j - 1)z_j(t) \\ + \sum\limits_{i=1}^{n} p_{ji} g_i(u_i(t)) + \sum\limits_{i=1}^{n} q_{ji} g_i(u_i(t - \sigma_{ij})) + J_j, \end{cases} \tag{3}$$

for $i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$,
and

$$\begin{cases} u_i(s) = \varphi_{ui}(s), & \frac{du_i(s)}{dt} = \psi_{ui}(s), -\tau \leq s \leq 0, \\ y_i(s) = \varphi_{ui}(s) + \psi_{ui}(s) \doteq \bar{\varphi}_{ui}(s), -\tau \leq s \leq 0, \\ v_j(s) = \varphi_{vj}(s), & \frac{dv_j(s)}{dt} = \psi_{vj}(s), -\sigma \leq s \leq 0, \\ z_j(s) = \varphi_{vj}(s) + \psi_{vj}(s) \doteq \bar{\varphi}_{vj}(s), -\sigma \leq s \leq 0, \end{cases} \tag{4}$$

for $i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$.
Let

$$U_i(t) = \begin{pmatrix} u_i(t) \\ y_i(t) \end{pmatrix}, \quad V_j(t) = \begin{pmatrix} v_j(t) \\ z_j(t) \end{pmatrix},$$

system (3) can becomes

$$\frac{dU_i(t)}{dt} = -A_i U_i(t) + P \begin{pmatrix} \sum\limits_{j=1}^{m} c_{ij} f_j(v_j(t)) \\ 0 \end{pmatrix}$$
$$+ P \begin{pmatrix} \sum\limits_{j=1}^{m} d_{ij} f_j(v_j(t - \tau_{ji})) \\ 0 \end{pmatrix} + P \begin{pmatrix} I_i \\ 0 \end{pmatrix}, \tag{5}$$

$$\frac{dV_j(t)}{dt} = -B_j V_j(t) + P \begin{pmatrix} \sum\limits_{i=1}^{n} p_{ji} g_i(u_i(t)) \\ 0 \end{pmatrix}$$
$$+ P \begin{pmatrix} \sum\limits_{i=1}^{n} q_{ji} g_i(u_i(t - \sigma_{ij})) \\ 0 \end{pmatrix} + P \begin{pmatrix} J_j \\ 0 \end{pmatrix}, \tag{6}$$

where

$$P = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad A_i = \begin{pmatrix} 1 & -1 \\ a_i + 1 - \alpha_i & \alpha_i - 1 \end{pmatrix},$$

$$B_j = \begin{pmatrix} 1 & -1 \\ b_j + 1 - \beta_j & \beta_j - 1 \end{pmatrix},$$

for $i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$.
Let

$$u^* = (u_1^*, u_2^*, \cdots, u_n^*)^T, \quad v^* = (v_1^*, v_2^*, \cdots, v_m^*)^T,$$

$$y^* = (y_1^*, y_2^*, \cdots, y_n^*)^T, \quad z^* = (z_1^*, z_2^*, \cdots, z_m^*)^T.$$

**Definition 1**. The point $({u^*}^T, {v^*}^T)$ is called an equilibrium point of system (1), if it satisf es the following equations

$$\begin{cases} -a_i u_i^* + \sum\limits_{j=1}^{m} c_{ij} f_j(v_j^*) + \sum\limits_{j=1}^{m} d_{ij} f_j(v_j^*) + I_i = 0, \\ -b_j v_j^* + \sum\limits_{i=1}^{n} p_{ji} g_i(u_i^*) + \sum\limits_{i=1}^{n} q_{ji} g_i(u_i^*) + J_j = 0, \end{cases} \tag{7}$$

for $i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$.

The point $(u^{*^T}, y^{*^T}, v^{*^T}, z^{*^T})$ is called an equilibrium point of system (3), if it satisfies the following equations

$$\begin{cases} -u_i^* + y_i^* = 0, \\ -a_i u_i^* + \sum\limits_{j=1}^m c_{ij} f_j(v_j^*) + \sum\limits_{j=1}^m d_{ij} f_j(v_j^*) + I_i = 0, \\ -v_j^* + z_j^* = 0, \\ -b_j v_j^* + \sum\limits_{i=1}^n p_{ji} g_i(u_i^*) + \sum\limits_{i=1}^n q_{ji} g_i(u_i^*) + J_j = 0, \end{cases} \quad (8)$$

for $i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$.

**Definition 2**. Let $u = (u_1, u_2, \cdots, u_n)^T, v = (v_1, v_2, \cdots, v_m)^T$, $(u^{*^T}, y^{*^T}, v^{*^T}, z^{*^T})$ be the equilibrium point of system (3), we define norm

$$\|u\|^2 = \sum_{i=1}^n |u_i|^2, \quad \|v\|^2 = \sum_{j=1}^m |v_i|^2.$$

$$\|\varphi_u - u^*\|^2 = \sup_{-\tau \le t \le 0} \sum_{i=1}^n (\varphi_{ui}(t) - u_i^*)^2,$$

$$\|\varphi_v - v^*\|^2 = \sup_{-\sigma \le t \le 0} \sum_{j=1}^m (\varphi_{vj}(t) - v_j^*)^2,$$

$$\|\bar{\varphi}_u - u^*\|^2 = \sup_{-\tau \le t \le 0} \sum_{i=1}^n (\bar{\varphi}_{ui}(t) - u_i^*)^2,$$

$$\|\bar{\varphi}_v - v^*\|^2 = \sup_{-\sigma \le t \le 0} \sum_{j=1}^m (\bar{\varphi}_{vj}(t) - v_j^*)^2,$$

where $\varphi_u = (\varphi_{u1}, \varphi_{u2}, \cdots, \varphi_{un})^T$,
$\varphi_v = (\varphi_{v1}, \varphi_{v2}, \cdots, \varphi_{vm})^T$,
$\bar{\varphi}_u = (\bar{\varphi}_{u1}, \bar{\varphi}_{u2}, \cdots, \bar{\varphi}_{un})^T$
and $\bar{\varphi}_v = (\bar{\varphi}_{v1}, \bar{\varphi}_{v2}, \cdots, \bar{\varphi}_{vm})^T$ are initial value.

**Definition 3**. The equilibrium point $(u^{*^T}, v^{*^T})$ of system (1) is said to be globally exponentially stable, if there exist constants $\eta > 0$ and $M > 0$ such that

$$\sum_{i=1}^n (u_i(t) - u_i^*)^2 + \sum_{j=1}^m (v_j(t,x) - v_j^*)^2 \le Me^{-\eta t}[\|\varphi_u - u^*\|^2 + \|\varphi_v - v^*\|^2],$$

for all $t \ge 0$, where
$u(t) = (u_1(t), u_2(t), \cdots, u_n(t))^T$,
$v(t) = (v_1(t), v_2(t), \cdots, v_m(t))^T$,
are solution of system (1) with initial value (2).

**Lemma 1** ([22]). If $H(u) \in C^0$, and it satisfies the following conditions
1) $H(u)$ is injective on $R^n$,
2) $\|H(u)\| \to +\infty$, as $\|u\| \to +\infty$,
then $H(u)$ is a homeomorphism of $R^n$.

**Lemma 2.** For matric

$$A_i = \begin{pmatrix} 1 & -1 \\ a_i + 1 - \alpha_i & \alpha_i - 1 \end{pmatrix},$$

$$B_j = \begin{pmatrix} 1 & -1 \\ b_j + 1 - \beta_j & \beta_j - 1 \end{pmatrix},$$

$i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$. If $\alpha_i^2 - 4a_i \ne 0, \beta_j^2 - 4b_j \ne 0 (i = 1, 2, \cdots, n, j = 1, 2, \cdots, m)$, then

$$\|exp(-A_i)t\| \le M_i e^{-\eta t}, i = 1, 2, \cdots, n,$$
$$\|exp(-B_j)t\| \le N_j e^{-\eta t}, j = 1, 2, \cdots, m, t \ge 0,$$

where

$$M_i = \frac{\sqrt{2(\alpha_i - a_i)^2 + 2(\alpha_i - 2)^2}}{\sqrt{|\alpha_i^2 - 4a_i|}},$$

$$N_j = \frac{\sqrt{2(\beta_j - b_j)^2 + 2(\beta_j - 2)^2}}{\sqrt{|\beta_j^2 - 4b_j|}},$$

$$\eta = \min_{1 \le i \le n, 1 \le j \le m} \{\frac{\alpha_i - \sqrt{|\alpha_i^2 - 4a_i|}}{2}, \frac{\beta_j - \sqrt{|\beta_j^2 - 4b_j|}}{2}\}.$$

**Proof.** We consider the following linear differential equation

$$Z_i'(t) = -A_i Z_i(t). \quad (9)$$

By calculation, we can obtain the eigenvalue of matric $-A_i$

$$\lambda_1 = \frac{1}{2}[-\alpha_i + \sqrt{\alpha_i^2 - 4a_i}],$$
$$\lambda_2 = \frac{1}{2}[-\alpha_i - \sqrt{\alpha_i^2 - 4a_i}].$$

Corresponding eigenvector of the $\lambda_1$ and $\lambda_2$, respectively

$$V_1 = (1, \lambda_1 + 1)^T, \quad V_2 = (1, \lambda_2 + 1)^T.$$

Thus, we obtain the fundamental solution matrix of system (9) which is

$$\phi_i(t) = \begin{bmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} \\ (\lambda_1 + 1)e^{\lambda_1 t} & (\lambda_2 + 1)e^{\lambda_2 t} \end{bmatrix}.$$

By calculation, we obtain

$$\phi_i^{-1}(0) = \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} \lambda_2 + 1 & -1 \\ -(\lambda_1 + 1) & 1 \end{bmatrix}.$$

Since $exp(-A_i)t = \phi_i(t)\phi_i^{-1}(0)$, we can obtain

$$exp(-A_i)t = \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} (\lambda_2 + 1)e^{\lambda_1 t} - (\lambda_1 + 1)e^{\lambda_2 t} & e^{\lambda_2 t} - e^{\lambda_1 t} \\ (\lambda_1 + 1)(\lambda_2 + 1)(e^{\lambda_1 t} - e^{\lambda_2 t}) & (\lambda_2 + 1)e^{\lambda_2 t} - (\lambda_1 + 1)e^{\lambda_1 t} \end{bmatrix}.$$

$$\|exp(-A_i)t\| \le \frac{\sqrt{2(\alpha_i - a_i)^2 + 2(\alpha_i - 2)^2}}{\sqrt{|\alpha_i^2 - 4a_i|}} \sqrt{e^{2\lambda_1 t} + e^{2\lambda_2 t}}.$$

If $\alpha_i^2 - 4a_i > 0$, then $\|exp(-A_i)t\| \le M_i e^{\lambda_1 t}$.

If $\alpha_i^2 - 4a_i < 0$, then $\|exp(-A_i)t\| \le M_i e^{-\frac{\alpha_i}{2} t}$.

Thus, we have

$$\|exp(-A_i)t\| \le M_i e^{-\eta t}, i = 1, 2, \cdots, n, t \ge 0.$$

Similarly we can get

$$\|exp(-B_j)t\| \le N_j e^{-\eta t}, j = 1, 2, \cdots, m, t \ge 0,$$

where

$$M_i = \frac{\sqrt{2(\alpha_i - a_i)^2 + 2(\alpha_i - 2)^2}}{\sqrt{|\alpha_i^2 - 4a_i|}},$$

$$N_j = \frac{\sqrt{2(\beta_j - b_j)^2 + 2(\beta_j - 2)^2}}{\sqrt{|\beta_j^2 - 4b_j|}},$$

$\eta = \min_{1\le i\le n, 1\le j\le m}\{\frac{\alpha_i-\sqrt{|\alpha_i^2-4a_i|}}{2}, \frac{\beta_j-\sqrt{|\beta_j^2-4b_j|}}{2}\}$.

**Lemma 3.**([23]) For $x(t)\ge 0, t\in[t_0-\tau, t]$, $\dot{x}(t) = -ax(t)+bx(t-\tau), \tau>0$, if

$$\dot{x}(t) \le -ax(t) + b\bar{x}(t),$$

and $a > b$, $\bar{x}(t) = \sup_{t-\tau\le s\le t} x(s)$. Then, there exists $\gamma > 0$ such that

$$x(t) \le \bar{x}(t_0)e^{-\gamma(t-t_0)}, t > t_0.$$

## 3 Main Results

In this section, we can derive some suff cient conditions which ensure the globally exponential stability of the system (1) by constructing a suitable Lyapunov functional and using some analysis techniques.

**Theorem 1**. Under the hypotheses $(H)$, system (1) has a unique equilibrium point, if there exist constants $0\le\gamma_j,\delta_j\le 1 (j=1,2,\cdots,m)$, $0\le\gamma_i^*,\delta_i^*\le 1 (i=1,2,\cdots,n)$, such that

$$\sum_{j=1}^m \frac{|c_{ij}|}{2}(l_j^{1-\gamma_j})^2 + \sum_{j=1}^m \frac{|d_{ij}|}{2}(l_j^{1-\delta_j})^2 + 1 - \alpha_i + \frac{|a_i-\alpha_i|}{2} < 0,$$

$$\sum_{i=1}^n \frac{|c_{ij}|}{2}(l_j^{\gamma_j})^2 + \sum_{i=1}^n \frac{|d_{ij}|}{2}(l_j^{\delta_j})^2 + \frac{|b_j-\beta_j|}{2} - 1 < 0,$$

$$\sum_{i=1}^n \frac{|p_{ji}|}{2}(k_i^{1-\gamma_i^*})^2 + \sum_{i=1}^n \frac{|q_{ji}|}{2}(k_i^{1-\delta_i^*})^2 + 1 - \beta_j + \frac{|b_j-\beta_j|}{2} < 0,$$

$$\sum_{j=1}^m \frac{|p_{ji}|}{2}(k_i^{\gamma_i^*})^2 + \sum_{j=1}^m \frac{|q_{ji}|}{2}(k_i^{\delta_i^*})^2 + \frac{|a_i-\alpha_i|}{2} - 1 < 0,$$

for $i=1,2,\cdots,n, j=1,2,\cdots,m$.

**Proof.** From def nition 1, we know that equilibrium point $(u^{*T}, v^{*T})$ of system (1) satisf es the following equation:

$$\begin{cases} -a_iu_i^* + \sum_{j=1}^m c_{ij}f_j(v_j^*) + \sum_{j=1}^m d_{ij}f_j(v_j^*) + I_i = 0, \\ -b_jv_j^* + \sum_{i=1}^n p_{ji}g_i(u_i^*) + \sum_{i=1}^n q_{ji}g_i(u_i^*) + J_j = 0, \end{cases} \tag{10}$$

for $i=1,2,\cdots,n, j=1,2,\cdots,m$.

Let $H(u,v) = (H_1(u,v), H_2(u,v),\cdots,\cdots, H_n(u,v), H_{n+1}(u,v), H_{n+2}(u,v),\cdots, H_{n+m}(u,v))^T$, where

$$H_i(u,v) = -a_iu_i + \sum_{j=1}^m c_{ij}f_j(v_j) + \sum_{j=1}^m d_{ij}f_j(v_j) + I_i,$$

$$H_{n+j}(u,v) = -b_jv_i + \sum_{i=1}^n p_{ji}g_i(u_i) + \sum_{i=1}^n q_{ji}g_i(u_i) + J_j,$$

for $i=1,2\cdots,n, j=1,2\cdots,m$.

It is known that the solutions of $H(u,v)=0$ are equilibriums of system (1). If the mapping $H(u,v)$ is a homeomorphism on $R^{n+m}$, then there exists a unique point $(u^*,v^*)$, such that $H(u^*,v^*)=0$, i.e., system (1) has a unique equilibrium point $(u^*,v^*)$. In the following, we shall prove that $H(u,v)$ is a homeomorphism.

First, we prove that $H(u,v)$ is an injective map on $R^{n+m}$.

In fact, if there exist $(u,v) = (u_1,u_2,\cdots,u_n,v_1,v_2,\cdots,v_m)^T\in R^{n+m}$, $(\bar{u},\bar{v}) = (\bar{u}_1,\bar{u}_2,\cdots,\bar{u}_n,\bar{v}_1,\bar{v}_2,\cdots,\bar{v}_m)^T\in R^{n+m}$ and $(u,v)\ne(\bar{u},\bar{v})$ such that $H(u,v) = H(\bar{u},\bar{v})$, then

$$-a_i(u_i-\bar{u}_i) + \sum_{j=1}^m c_{ij}(f_j(v_j)-f_j(\bar{v}_j)) + \sum_{j=1}^m d_{ij}(f_j(v_j)-f_j(\bar{v}_j)) = 0,$$

$$-b_j(v_j-\bar{v}_j) + \sum_{i=1}^n p_{ji}(g_i(u_i)-g_i(\bar{u}_i)) + \sum_{i=1}^n q_{ji}(g_i(u_i)-g_i(\bar{u}_i)) = 0,$$

for $i=1,2\cdots,n, j=1,2\cdots,m$.

We also have

$$(u_i-\bar{u}_i)[-a_i(u_i-\bar{u}_i) + \sum_{j=1}^m c_{ij}(f_j(v_j)-f_j(\bar{v}_j)) + \sum_{j=1}^m d_{ij}(f_j(v_j)-f_j(\bar{v}_j))] = 0,$$

$$(v_j-\bar{v}_j)[-b_j(v_j-\bar{v}_j) + \sum_{i=1}^n p_{ji}(g_i(u_i)-g_i(\bar{u}_i)) + \sum_{i=1}^n q_{ji}(g_i(u_i)-g_i(\bar{u}_i))] = 0,$$

for $i=1,2\cdots,n, j=1,2\cdots,m$.

From assumptions $(H)$, we can get

$$-a_i|u_i-\bar{u}_i|^2 + \sum_{j=1}^m |c_{ij}|l_j|u_i-\bar{u}_i||v_j-\bar{v}_j| + \sum_{j=1}^m |d_{ij}|l_j|u_i-\bar{u}_i||v_j-\bar{v}_j| \ge 0,$$

$$-b_j|v_j-\bar{v}_j|^2 + \sum_{i=1}^n |p_{ji}|k_i|u_i-\bar{u}_i||v_j-\bar{v}_j| + \sum_{i=1}^n |q_{ji}|k_i|u_i-\bar{u}_i||v_j-\bar{v}_j| \ge 0,$$

for $i=1,2\cdots,n, j=1,2\cdots,m$.

We have

$$\sum_{i=1}^n[-a_i|u_i-\bar{u}_i|^2 + \sum_{j=1}^m |c_{ij}|l_j|u_i-\bar{u}_i||v_j-\bar{v}_j| + \sum_{j=1}^m |d_{ij}|l_j|u_i-\bar{u}_i||v_j-\bar{v}_j|]$$
$$\ge \sum_{i=1}^n\{-a_i|u_i-\bar{u}_i|^2 + \sum_{j=1}^m \frac{|c_{ij}|}{2}[(l_j^{1-\gamma_j})^2|u_i-\bar{u}_i|^2 + (l_j^{\gamma_j})^2|v_j-\bar{v}_j|^2] + \sum_{j=1}^m \frac{|d_{ij}|}{2}[(l_j^{1-\delta_j})^2|u_i-\bar{u}_i|^2 + (l_j^{\delta_j})^2|v_j-\bar{v}_j|^2]\} \ge 0.$$

$$\sum_{j=1}^{m}[-b_j|v_j-\bar{v}_j|^2+\sum_{i=1}^{n}|p_{ji}|k_i|u_i-\bar{u}_i||v_j-\bar{v}_j| + \sum_{i=1}^{n}|q_{ji}|k_i|u_i-\bar{u}_i||v_j-\bar{v}_j|]$$
$$\geq \sum_{j=1}^{m}\{-b_j|v_j-\bar{v}_j|^2+\sum_{i=1}^{n}\frac{|p_{ji}|}{2}[(k_i^{\gamma_i^*})^2|u_i-\bar{u}_i|^2 + (k_i^{1-\gamma_i^*})^2|v_j-\bar{v}_j|^2]+\sum_{i=1}^{n}\frac{|q_{ji}|}{2}[(k_i^{\delta_i^*})^2|u_i-\bar{u}_i|^2 + (k_i^{1-\delta_i^*})^2|v_j-\bar{v}_j|^2]\}\geq 0,$$

where $0\leq\gamma_j,\delta_j\leq 1(j=1,2,\cdots,m)$, $0\leq\gamma_i^*,\delta_i^*\leq 1(i=1,2,\cdots,n)$.
We can obtain

$$\sum_{i=1}^{n}\{-a_i+\sum_{j=1}^{m}[\frac{|c_{ij}|}{2}(l_j^{1-\gamma_j})^2+\frac{|d_{ij}|}{2}(l_j^{1-\delta_j})^2] + \sum_{j=1}^{m}[\frac{|p_{ji}|}{2}(k_i^{\gamma_i^*})^2+\frac{|q_{ji}|}{2}(k_i^{\delta_i^*})^2]\}|u_i-\bar{u}_i|^2 + \sum_{j=1}^{m}\{-b_j+\sum_{i=1}^{n}[\frac{|c_{ij}|}{2}(l_j^{\gamma_j})^2+\frac{|d_{ij}|}{2}(l_j^{\delta_j})^2] + \sum_{i=1}^{n}[\frac{|p_{ji}|}{2}(k_i^{1-\gamma_i^*})^2+\frac{|q_{ji}|}{2}(k_i^{1-\delta_i^*})^2]\}|v_j-\bar{v}_j|^2\geq 0. \quad (11)$$

From the condition of Theorem 1, we have

$$-a_i+\sum_{j=1}^{m}[\frac{|c_{ij}|}{2}(l_j^{1-\gamma_j})^2+\frac{|d_{ij}|}{2}(l_j^{1-\delta_j})^2] + \sum_{j=1}^{m}[\frac{|p_{ji}|}{2}(k_i^{\gamma_i^*})^2+\frac{|q_{ji}|}{2}(k_i^{\delta_i^*})^2] < -a_i+b_i-|b_i-a_i|\leq 0,$$
$$-b_j+\sum_{i=1}^{n}[\frac{|c_{ij}|}{2}(l_j^{\gamma_j})^2+\frac{|d_{ij}|}{2}(l_j^{\delta_j})^2] + \sum_{i=1}^{n}[\frac{|p_{ji}|}{2}(k_i^{1-\gamma_i^*})^2+\frac{|q_{ji}|}{2}(k_i^{1-\delta_i^*})^2] < -b_j+\beta_j-|b_j-\beta_j|\leq 0,$$

for $i=1,2\cdots,n, j=1,2\cdots,m$.

From (11), we can get $u_i=\bar{u}_i, v_j=\bar{v}_j$, for $i=1,2\cdots,n, j=1,2\cdots,m$, which contradict $(u,v)\neq(\bar{u},\bar{v})$. So $H(u,v)$ is an injective on $R^{n+m}$. Second, we prove that $\|H(u,v)\|\to+\infty$ as $\|(u,v)\|\to+\infty$.

Let $\tilde{H}(u,v)=H(u,v)-H(0,0)$
$=(\tilde{H}_1(u,v),\tilde{H}_2(u,v),\cdots,\tilde{H}_n(u,v), \tilde{H}_{n+1}(u,v),\tilde{H}_{n+2}(u,v),\cdots,\tilde{H}_{n+m}(u,v))^T$,
i.e.,

$$\tilde{H}_i(u,v)=-a_iu_i+\sum_{j=1}^{m}c_{ij}(f_j(v_j)-f_j(0)) + \sum_{j=1}^{m}d_{ij}(f_j(v_j)-f_j(0)),$$
$$\tilde{H}_{n+j}(u,v)=-b_jv_j+\sum_{i=1}^{n}p_{ji}(g_i(u_i)-g_i(0)) + \sum_{i=1}^{n}q_{ji}(g_i(u_i)-g_i(0)),$$

for $i=1,2\cdots,n, j=1,2\cdots,n$.

Calculating $(u,v)\tilde{H}(u,v)$, we obtain

$$(u,v)\tilde{H}(u,v)\leq\sum_{i=1}^{n}\{-a_i+\sum_{j=1}^{m}[\frac{|c_{ij}|}{2}(l_j^{1-\gamma_j})^2 + \frac{|d_{ij}|}{2}(l_j^{1-\delta_j})^2]+\sum_{j=1}^{m}[\frac{|p_{ji}|}{2}(k_i^{\gamma_i^*})^2+\frac{|q_{ji}|}{2}(k_i^{\delta_i^*})^2]\}|u_i|^2 + \sum_{j=1}^{m}\{-b_j+\sum_{i=1}^{n}[\frac{|c_{ij}|}{2}(l_j^{\gamma_j})^2+\frac{|d_{ij}|}{2}(l_j^{\delta_j})^2] + \sum_{i=1}^{n}[\frac{|p_{ji}|}{2}(k_i^{1-\gamma_i^*})^2+\frac{|q_{ji}|}{2}(k_i^{1-\delta_i^*})^2]\}|v_j|^2$$
$$\leq -\min_{1\leq i\leq n}\{a_i-\sum_{j=1}^{m}[\frac{|c_{ij}|}{2}(l_j^{1-\gamma_j})^2+\frac{|d_{ij}|}{2}(l_j^{1-\delta_j})^2] - \sum_{j=1}^{m}[\frac{|p_{ji}|}{2}(k_i^{\gamma_i^*})^2+\frac{|q_{ji}|}{2}(k_i^{\delta_i^*})^2]\}||u||^2 - \min_{1\leq j\leq m}\{b_j-\sum_{i=1}^{n}[\frac{|c_{ij}|}{2}(l_j^{\gamma_j})^2+\frac{|d_{ij}|}{2}(l_j^{\delta_j})^2] - \sum_{i=1}^{n}[\frac{|p_{ji}|}{2}(k_i^{1-\gamma_i^*})^2+\frac{|q_{ji}|}{2}(k_i^{1-\delta_i^*})^2]\}||v||^2.$$

Using Schwartz inequality, we get

$$\|(u,v)\|\cdot\|\tilde{H}\|\geq\min_{1\leq i\leq n}\{a_i-\sum_{j=1}^{m}[\frac{|c_{ij}|}{2}(l_j^{1-\gamma_j})^2 + \frac{|d_{ij}|}{2}(l_j^{1-\delta_j})^2]-\sum_{j=1}^{m}[\frac{|p_{ji}|}{2}(k_i^{\gamma_i^*})^2+\frac{|q_{ji}|}{2}(k_i^{\delta_i^*})^2]\}\|u\|^2 + \min_{1\leq j\leq m}\{b_j-\sum_{i=1}^{n}[\frac{|c_{ij}|}{2}(l_j^{\gamma_j})^2+\frac{|d_{ij}|}{2}(l_j^{\delta_j})^2] - \sum_{i=1}^{n}[\frac{|p_{ji}|}{2}(k_i^{1-\gamma_i^*})^2+\frac{|q_{ji}|}{2}(k_i^{1-\delta_i^*})^2]\}\|v\|^2 \geq M(\|u\|^2+\|v\|^2)=M\|(u,v)\|^2,$$

where

$$M=\min\{\min_{1\leq i\leq n}\{a_i-\sum_{j=1}^{m}[\frac{|c_{ij}|}{2}(l_j^{1-\gamma_j})^2 + \frac{|d_{ij}|}{2}(l_j^{1-\delta_j})^2]-\sum_{j=1}^{m}[\frac{|p_{ji}|}{2}(k_i^{\gamma_i^*})^2+\frac{|q_{ji}|}{2}(k_i^{\delta_i^*})^2]\}, \min_{1\leq j\leq m}\{b_j-\sum_{i=1}^{n}[\frac{|c_{ij}|}{2}(l_j^{\gamma_j})^2+\frac{|d_{ij}|}{2}(l_j^{\delta_j})^2] - \sum_{i=1}^{n}[\frac{|p_{ji}|}{2}(k_i^{1-\gamma_i^*})^2+\frac{|q_{ji}|}{2}(k_i^{1-\delta_i^*})^2]\}\}\geq 0.$$

When $\|(u,v)\|\neq 0$, we have $\|\tilde{H}\|\geq M\|(u,v)\|$. Therefore $\|\tilde{H}(u,v)\|\to+\infty$ as $\|(u,v)\|\to+\infty$, which implies that $\|H(u,v)\|\to+\infty$ as $\|(u,v)\|\to+\infty$. From Lemma 1, we know that $H(u,v)$ is a homeomorphism on $R^{n+m}$. Thus, system (1) has a unique equilibrium point.

**Theorem 2**. Under the hypotheses $(H)$, the unique equilibrium point of system (1) is globally exponentially stably if conditions of Theorem 1 hold.

**Proof.** By using Theorem 1, system (1) has a unique equilibrium point. In the following we will prove the unique equilibrium point $(u^*,v^*)=(u_1^*,u_2^*,\cdots,u_n^*,v_1^*,v_2^*,\cdots,v_m^*)^T$ is globally exponential stable.

Let

$$\bar{U}_i=\begin{pmatrix} u_i-u_i^* \\ y_i-y_i^* \end{pmatrix}, i=1,2\cdots,n,$$

$$\bar{V}_j = \begin{pmatrix} v_j - v_j^* \\ z_j - z_j^* \end{pmatrix} j = 1, 2 \cdots, m.$$

From (5)- (8), we have

$$\frac{d\bar{U}_i(t)}{dt} = -A_i\bar{U}_i(t) + P\begin{pmatrix} \sum_{j=1}^{m} c_{ij}(f_j(v_j(t)) - f_j(v_j^*)) \\ 0 \end{pmatrix} + P\begin{pmatrix} \sum_{j=1}^{m} d_{ij}(f_j(v_j(t-\tau_{ji})) - f_j(v_j^*)) \\ 0 \end{pmatrix}, \quad (12)$$

$$\frac{d\bar{V}_j(t)}{dt} = -B_jV_j(t) + P\begin{pmatrix} \sum_{i=1}^{n} p_{ji}(g_i(u_i(t)) - g_i(u_i^*)) \\ 0 \end{pmatrix} + P\begin{pmatrix} \sum_{i=1}^{n} q_{ji}(g_i(u_i(t-\sigma_{ij})) - g_i(u_i^*)) \\ 0 \end{pmatrix}, \quad (13)$$

for $i = 1, 2 \cdots, n, j = 1, 2, \cdots, m$.

By left multiplying both sides of (12) with $\bar{U}_i^T = (u_i - u_i^*, y_i - y_i^*)$, we get

$$\begin{aligned} \bar{U}_i^T \tfrac{d\bar{U}_i}{dt} &= -[(u_i - u_i^*)^2 + (\alpha_i - 1)(y_i - y_i^*)^2 \\ &+ (a_i - \alpha_i)(u_i - u_i^*)(y_i - y_i^*)] \\ &+ \sum_{j=1}^{m} c_{ij}(y_i - y_i^*)[f_j(v_j(t)) - f_j(v_j^*)] \\ &+ \sum_{j=1}^{m} d_{ij}(y_i - y_i^*)[f_j(v_j(t-\tau_{ji})) - f_j(v_j^*)] \\ &\le -[(1 - \tfrac{|a_i - \alpha_i|}{2})(u_i - u_i^*)^2 + (\alpha_i - 1 - \tfrac{|a_i-\alpha_i|}{2}) \\ &\cdot (y_i - y_i^*)^2] + \sum_{j=1}^{m} |c_{ij}| l_j |y_i - y_i^*||v_j(t) - v_j^*| \\ &+ \sum_{j=1}^{m} |d_{ij}| l_j |y_i - y_i^*||v_j(t-\tau_{ji}) - v_j^*| \\ &\le -[(1 - \tfrac{|a_i - \alpha_i|}{2})(u_i - u_i^*)^2 \\ &+ (\alpha_i - 1 - \tfrac{|a_i-\alpha_i|}{2})(y_i - y_i^*)^2] \\ &+ \sum_{j=1}^{m} \tfrac{|c_{ij}|}{2}[(l_j^{1-\gamma_j})^2|y_i - y_i^*|^2 + (l_j^{\gamma_j})^2|v_j(t) - v_j^*|^2] \\ &+ \sum_{j=1}^{m} \tfrac{|d_{ij}|}{2}[(l_j^{1-\delta_j})^2|y_i - y_i^*|^2 \\ &+ (l_j^{\delta_j})^2|v_j(t-\tau_{ji}) - v_j^*|^2], \quad i = 1, 2, \cdots, n. \quad (14) \end{aligned}$$

By left multiplying both sides of (13) with $\bar{V}_j^T = (v_j - v_j^*, y_j - z_j^*)$, we get

$$\begin{aligned} \bar{V}_j^T \tfrac{d\bar{V}_j}{dt} &= -[(v_j - v_j^*)^2 + (\beta_j - 1)(z_j - z_j^*)^2 \\ &+ (b_j - \beta_j)(v_j - v_j^*)(z_j - z_j^*)] \\ &+ \sum_{i=1}^{n} p_{ji}(z_j - z_j^*)[g_i(u_i(t)) - g_i(u_i^*)] \\ &+ \sum_{i=1}^{n} q_{ji}(z_j - z_j^*)[g_i(u_i(t-\sigma_{ij})) - g_i(u_i^*)] \\ &\le -[(1 - \tfrac{|b_j-\beta_j|}{2})(v_j - v_j^*)^2 \\ &+ (\beta_j - 1 - \tfrac{|b_j-\beta_j|}{2})(z_j - z_j^*)^2] \\ &+ \sum_{i=1}^{n} |p_{ji}| k_i |z_j - z_j^*||u_i(t) - u_i^*| \\ &+ \sum_{i=1}^{n} |q_{ji}| k_i |z_j - z_j^*||u_i(t-\sigma_{ij}) - u_i^*| \\ &\le -[(1 - \tfrac{|b_j-\beta_j|}{2})(v_j - v_j^*)^2 \\ &+ (\beta_j - 1 - \tfrac{|b_j-\beta_j|}{2})(z_j - z_j^*)^2] \\ &+ \sum_{i=1}^{n} \tfrac{|p_{ji}|}{2}[(k_i^{1-\gamma_i^*})^2|z_j - z_j^*|^2 \\ &+ (k_i^{\gamma_i^*})^2|u_i(t) - u_i^*|^2] + \sum_{i=1}^{n} \tfrac{|q_{ji}|}{2}[(k_i^{1-\delta_i^*})^2|z_j - z_j^*|^2 \\ &+ (k_i^{\delta_i^*})^2|u_i(t-\sigma_{ij}) - u_i^*|^2], i = 1, 2, \cdots, n. \quad (15) \end{aligned}$$

We consider the Lyapunov functional:

$$V(t) = V_1(t) + V_2(t),$$

where

$$V_1(t) = \sum_{i=1}^{n}\{\tfrac{\|\bar{U}_i(t)\|^2}{2}e^{2\varepsilon t} + \sum_{j=1}^{m}\frac{|d_{ij}|}{2}(l_j^{\delta_j})^2\int_{t-\tau_{ji}}^{t}(v_j(s) - v_j^*)^2 e^{2\varepsilon(s+\tau_{ji})}ds\} \quad (16)$$

$$V_2(t) = \sum_{j=1}^{m}\{\tfrac{\|\bar{V}_j(t)\|^2}{2}e^{2\varepsilon t} + \sum_{i=1}^{n}\frac{|q_{ji}|}{2}(l_i^{\delta_i^*})^2\int_{t-\sigma_{ij}}^{t}(u_i(s) - u_i^*)^2 e^{2\varepsilon(s+\sigma_{ij})}ds\} \quad (17)$$

$\varepsilon > 0$ is a small number.

Calculating the upper right Dini-derivative $D^+V_1(t)$ of $V_1(t)$ along the solution of (12), by some analysis techniques and (14), we have

$$\begin{aligned} D^+V_1(t) &= \sum_{i=1}^{n}\{\bar{U}_i^T\tfrac{d\bar{U}_i(t)}{dt}e^{2\varepsilon t} + \varepsilon e^{2\varepsilon t}\|\bar{U}_i(t)\|^2 \\ &+ \sum_{j=1}^{m}\tfrac{|d_{ij}|}{2}(l_j^{\delta_j})^2[(v_j(t) - v_j^*)^2 e^{2\varepsilon(t+\tau_{ji})} \\ &- (v_j(t-\tau_{ji}) - v_j^*)^2 e^{2\varepsilon t}]\} \\ &\le \sum_{i=1}^{n} e^{2\varepsilon t} \{\varepsilon\|\bar{U}_i(t)\|^2 \\ &- [(1 - \tfrac{|a_i-\alpha_i|}{2})(u_i - u_i^*)^2 \\ &+ (\alpha_i - 1 - \tfrac{|a_i-\alpha_i|}{2})(y_i - y_i^*)^2] \\ &+ \sum_{j=1}^{m}\tfrac{|c_{ij}|}{2}[(l_j^{1-\gamma_j})^2|y_i - y_i^*|^2 + (l_j^{\gamma_j})^2|v_j(t) - v_j^*|^2] \\ &+ \sum_{j=1}^{m}\tfrac{|d_{ij}|}{2}[(l_j^{1-\delta_j})^2|y_i - y_i^*|^2 + (l_j^{\delta_j})^2|v_j(t-\tau_{ji}) - v_j^*|^2] \end{aligned}$$

$+\sum_{j=1}^{m}\frac{|d_{ij}|}{2}(l_j^{\delta_j})^2[(v_j(t)-v_j^*)^2e^{2\varepsilon\tau_{ji}}$
$-(v_j(t-\tau_{ji})-v_j^*)^2]\}$
$\le\sum_{i=1}^{n}e^{2\varepsilon t}\{\varepsilon\|\bar U_i(t)\|^2-[(1-\frac{|a_i-\alpha_i|}{2})(u_i-u_i^*)^2$
$+(\alpha_i-1-\frac{|a_i-\alpha_i|}{2})(y_i-y_i^*)^2]$
$+\sum_{j=1}^{m}\frac{|c_{ij}|}{2}[(l_j^{1-\gamma_j})^2|y_i-y_i^*|^2+(l_j^{\gamma_j})^2|v_j(t)-v_j^*|^2]\}$
$+\sum_{j=1}^{m}\frac{|d_{ij}|}{2}(l_j^{1-\delta_j})^2|y_i-y_i^*|^2$

$$+\sum_{j=1}^{m}\frac{|d_{ij}|}{2}(l_j^{\delta_j})^2(v_j(t)-v_j^*)^2e^{2\varepsilon\tau_{ji}}. \qquad (18)$$

Calculating the upper right Dini-derivative $D^+V_2(t)$ of $V_2(t)$ along the solution of (13), by some analysis techniques and (15), we have

$D^+V_2(t)=\sum_{i=1}^{n}\{\bar V_j^T\frac{d\bar V_j(t)}{dt}e^{2\varepsilon t}+\varepsilon e^{2\varepsilon t}\|\bar V_j(t)\|^2$
$+\sum_{i=1}^{n}\frac{|d_{ji}^*|}{2}(l_i^{\delta_i})^2[(u_i(t)-u_i^*)^2e^{2\varepsilon(t+\sigma_{ij})}$
$-(u_i(t-\sigma_{ij})-u_i^*)^2e^{2\varepsilon t}]\}$
$\le\sum_{j=1}^{m}e^{2\varepsilon t}\{\varepsilon\|\bar V_j(t)\|^2$
$-[(1-\frac{|b_j-\beta_j|}{2})(v_j-v_j^*)^2+(\beta_j-1-\frac{|b_j-b_j^*|}{2})(z_j-z_j^*)^2]+\sum_{i=1}^{n}\frac{|p_{ji}|}{2}[(k_i^{\gamma_i^*})^2|z_j-z_j^*|^2+(k_i^{1-\gamma_i^*})^2|u_i(t)-u_i^*|^2]+\sum_{i=1}^{n}\frac{|q_{ji}|}{2}[(k_i^{1-\delta_i^*})^2|z_j-z_j^*|^2+(k_i^{\delta_i^*})^2|u_i(t-\sigma_{ij})-u_i^*|^2]+\sum_{i=1}^{n}\frac{|q_{ji}|}{2}(k_i^{\delta_i^*})^2[(u_i(t)-u_i^*)^2e^{2\varepsilon\sigma_{ij}}-(u_i(t-\sigma_{ij})-u_i^*)^2]\}$
$\le\sum_{j=1}^{m}e^{2\varepsilon t}\{\varepsilon\|\bar V_j(t)\|^2-[(1-\frac{|b_j-\beta_j|}{2})(v_j-v_j^*)^2+(\beta_j-1-\frac{|b_j-b_j^*|}{2})(z_j-z_j^*)^2]+\sum_{i=1}^{n}\frac{|p_{ji}|}{2}[(k_i^{1-\gamma_i^*})^2|z_j-z_j^*|^2+(k_i^{\gamma_i^*})^2|u_i(t)-u_i^*|^2]$
$+\sum_{i=1}^{n}\frac{|q_{ji}|}{2}(k_i^{1-\delta_i^*})^2|z_j-z_j^*|^2$

$$+\sum_{i=1}^{n}\frac{|q_{ji}|}{2}(k_i^{\delta_i^*})^2(u_i(t)-u_i^*)^2e^{2\varepsilon\sigma_{ij}}\}. \qquad (19)$$

It follows from (18),(19) that

$D^+V(t)=D^+V_1(t)+D^+V_2(t)$
$\le\sum_{i=1}^{n}e^{2\varepsilon t}\{\varepsilon\|\bar U_i(t)\|^2-[(1-\frac{|a_i-\alpha_i|}{2})(u_i-u_i^*)^2+(\alpha_i-1-\frac{|a_i-\alpha_i|}{2})(y_i-y_i^*)^2]+\sum_{j=1}^{m}\frac{|c_{ij}|}{2}[(l_j^{1-\gamma_j})^2|y_i-y_i^*|^2+(l_j^{\gamma_j})^2|v_j(t)-v_j^*|^2]+\sum_{j=1}^{m}\frac{|d_{ij}|}{2}(l_j^{1-\delta_j})^2|y_i-y_i^*|^2$
$+\sum_{j=1}^{m}\frac{|d_{ij}|}{2}(l_j^{\delta_j})^2(v_j(t)-v_j^*)^2e^{2\varepsilon\tau_{ji}}\}$
$+\sum_{j=1}^{m}e^{2\varepsilon t}\{\varepsilon\|\bar V_j(t)\|^2-[(1-\frac{|b_j-\beta_j|}{2})(v_j-v_j^*)^2+(\beta_j-1-\frac{|b_j-\beta_j|}{2})(z_j-z_j^*)^2]+\sum_{i=1}^{n}\frac{|p_{ji}|}{2}[(k_i^{1-\gamma_i^*})^2|z_j-z_j^*|^2+(k_i^{\gamma_i^*})^2|u_i(t)-u_i^*|^2]+\sum_{i=1}^{n}\frac{|q_{ji}|}{2}(k_i^{1-\delta_i^*})^2|z_j-z_j^*|^2$
$+\sum_{i=1}^{n}\frac{|q_{ji}|}{2}(k_i^{\delta_i^*})^2(u_i(t)-u_i^*)^2e^{2\varepsilon\sigma_{ij}}\}$
$=\sum_{i=1}^{n}e^{2\varepsilon t}[\varepsilon-1+\frac{|a_i-\alpha_i|}{2}+\sum_{j=1}^{m}\frac{|p_{ji}|}{2}(k_i^{\gamma^*})^2$
$+\sum_{j=1}^{m}\frac{|q_{ji}|}{2}(k_i^{\delta^*})^2e^{2\varepsilon\sigma_{ij}}](u_i(t)-u_i^*)^2$
$+\sum_{i=1}^{n}e^{2\varepsilon t}[\varepsilon-\alpha_i+1+\frac{|a_i-\alpha_i|}{2}+\sum_{j=1}^{m}\frac{|c_{ij}|}{2}(l_j^{1-\gamma_j})^2$
$+\sum_{j=1}^{m}\frac{|d_{ij}|}{2}(l_j^{1-\delta_j})^2](y_i-y_i^*)^2$
$+\sum_{j=1}^{m}e^{2\varepsilon t}[\varepsilon-1+\frac{|b_j-\beta_j|}{2}$
$+\sum_{i=1}^{n}\frac{|c_{ij}|}{2}(l_j^{\gamma_j})^2+\sum_{i=1}^{n}\frac{|d_{ij}|}{2}(l_j^{\delta_j})^2e^{2\varepsilon\tau_{ji}}](v_j(t)-v_j^*)^2$
$+\sum_{j=1}^{m}e^{2\varepsilon t}[\varepsilon-\beta_j+1+\frac{|b_j-\beta_j|}{2}+\sum_{i=1}^{n}\frac{|p_{ji}|}{2}(k_i^{1-\gamma_i})^2$

$$+\sum_{i=1}^{n}\frac{|q_{ji}|}{2}(k_i^{1-\delta_i})^2](z_j-z_j^*)^2. \qquad (20)$$

Form condition of Theorem 1, we can choose a small $\varepsilon>0$ such that

$\varepsilon-1+\frac{|a_i-\alpha_i|}{2}+\sum_{j=1}^{m}\frac{|p_{ji}|}{2}(k_i^{\gamma^*})^2+$
$\sum_{j=1}^{m}\frac{|q_{ji}|}{2}(k_i^{\delta^*})^2e^{2\varepsilon\sigma_{ij}}\le 0$
$\varepsilon-\alpha_i+1+\frac{|a_i-\alpha_i|}{2}+\sum_{j=1}^{m}\frac{|c_{ij}|}{2}(l_j^{1-\gamma_j})^2$
$+\sum_{j=1}^{m}\frac{|d_{ij}|}{2}(l_j^{1-\delta_j})^2\le 0,$
$\varepsilon-1+\frac{|b_j-\beta_j|}{2}+\sum_{i=1}^{n}\frac{|c_{ij}|}{2}(l_j^{\gamma_j})^2$
$+\sum_{i=1}^{n}\frac{|d_{ij}|}{2}(l_j^{\delta_j})^2e^{2\varepsilon\tau_{ji}}\le 0,$
$\varepsilon-\beta_j+1+\frac{|b_j-\beta_j|}{2}+\sum_{i=1}^{n}\frac{|p_{ji}|}{2}(k_i^{1-\gamma_i})^2$
$+\sum_{i=1}^{n}\frac{|q_{ji}|}{2}(k_i^{1-\delta_i})^2\le 0,$

for $i=1,2\cdots,n, j=1,2\cdots,m$.
From (20), we get $D^+V(t)\le 0$, and so $V(t)\le V(0)$, for all $t\ge 0$.
From (16) and (17), we have

$V(t)\ge\sum_{i=1}^{n}\frac{\|\bar U_i(t)\|^2}{2}e^{2\varepsilon t}+\sum_{j=1}^{m}\frac{\|\bar V_j(t)\|^2}{2}e^{2\varepsilon t}$

$$= \sum_{i=1}^{n} \frac{e^{2\varepsilon t}}{2}[(u_i - u_i^*)^2 + (y_i - y_i^*)^2] + \sum_{j=1}^{m} \frac{e^{2\varepsilon t}}{2}[(v_j - v_j^*)^2 + (z_j - z_j^*)^2]. \quad (21)$$

$$V(0) = \sum_{i=1}^{n}\{\frac{\|\bar{U}_i(0)\|^2}{2} + \sum_{j=1}^{m} \frac{|d_{ij}|}{2}(l_j^{\delta_j})^2 \int_{-\tau_{ji}}^{0}(v_j(s) - v_j^*)^2 e^{2\varepsilon(s+\tau_{ji})}ds\} + \sum_{j=1}^{m}\{\frac{\|\bar{V}_j(0)\|^2}{2} + \sum_{i=1}^{n} \frac{|q_{ji}|}{2}(k_i^{\delta_i^*})^2 \int_{-\sigma_{ij}}^{0}(u_i(s) - u_j^*)^2 e^{2\varepsilon(s+\sigma_{ij})}ds\}$$
$$= \sum_{i=1}^{n}\{\frac{(\varphi_{ui}(0)-u_i^*)^2}{2} + \frac{(\bar{\varphi}_{ui}(0)-u_i^*)^2}{2} + \sum_{j=1}^{m} \frac{|d_{ij}|}{2}(l_j^{\delta_j})^2 \int_{-\tau_{ji}}^{0}(\varphi_{vj}(s) - v_j^*)^2 e^{2\varepsilon(s+\tau_{ji})}ds\} + \sum_{j=1}^{m}\{\frac{(\varphi_{vj}(0)-v_j^*)^2}{2} + \frac{(\bar{\varphi}_{vj}(0)-v_j^*)^2}{2} + \sum_{i=1}^{n} \frac{|q_{ji}|}{2}(k_i^{\delta_i^*})^2 \int_{-\sigma_{ij}}^{0}(\varphi_{ui}(s) - u_i^*)^2 e^{2\varepsilon(s+\sigma_{ij})}ds\}$$
$$\le \frac{\|\varphi_u-u^*\|^2}{2} + \frac{\|\bar{\varphi}_u-u^*\|^2}{2} + \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{|d_{ij}|}{2}(l_j^{\delta_j})^2 \int_{-\tau_{ji}}^{0}(\varphi_{vj}(s) - v_j^*)^2 e^{2\varepsilon(s+\tau_{ji})}ds + \frac{\|\varphi_v-v^*\|^2}{2} + \frac{\|\bar{\varphi}_v-v^*\|^2}{2} + \sum_{j=1}^{m}\sum_{i=1}^{n} \frac{|q_{ji}|}{2}(k_i^{\delta_i^*})^2 \int_{-\sigma_{ij}}^{0}(\varphi_{ui}(s) - u_i^*)^2 e^{2\varepsilon(s+\sigma_{ij})}ds$$
$$\le \frac{\|\varphi_u-u^*\|^2}{2} + \frac{\|\bar{\varphi}_u-u^*\|^2}{2} + \tau\sum_{i=1}^{n}\max_{1\le j\le m}\{|d_{ij}|(l_j^{\delta_j})^2\}e^{2\varepsilon\tau}\|\varphi_v - v^*\|^2 + \frac{\|\varphi_v-v^*\|^2}{2} + \frac{\|\bar{\varphi}_v-v^*\|^2}{2} + \sigma\sum_{j=1}^{m}\max_{1\le i\le n}\{|q_{ji}|(k_i^{\delta_i^*})^2\}e^{2\varepsilon\sigma}\|\varphi_u - u^*\|^2$$
$$= [\frac{1}{2} + \sigma\sum_{j=1}^{m}\max_{1\le i\le n}\{|q_{ji}|(k_i^{\delta_i^*})^2\}e^{2\varepsilon\sigma}]\|\varphi_u - u^*\|^2 + [\frac{1}{2} + \tau\sum_{i=1}^{n}\max_{1\le j\le m}\{|d_{ij}|(l_j^{\delta_j})^2\}e^{2\varepsilon\tau}]\|\varphi_v - v^*\|^2 + \frac{\|\bar{\varphi}_u - u^*\|^2}{2} + \frac{\|\bar{\varphi}_v - v^*\|^2}{2}. \quad (22)$$

Since $V(0) \ge V(t)$, from (21) and (22), we obtain

$$\sum_{i=1}^{n} \frac{e^{2\varepsilon t}}{2}[(u_i - u_i^*)^2 + (y_i - y_i^*)^2] + \sum_{j=1}^{m} \frac{e^{2\varepsilon t}}{2}[(v_j - v_j^*)^2 + (z_j - z_j^*)^2])$$
$$\le [\frac{1}{2} + \sigma\sum_{j=1}^{m}\max_{1\le i\le n}\{|q_{ji}|(k_i^{\delta_i^*})^2\}e^{2\varepsilon\sigma}]\|\varphi_u - u^*\|^2 + [\frac{1}{2} + \tau\sum_{i=1}^{n}\max_{1\le j\le m}\{|d_{ij}|(l_j^{\delta_j})^2\}e^{2\varepsilon\tau}]\|\varphi_v - v^*\|^2 + \frac{\|\bar{\varphi}_u - u^*\|^2}{2} + \frac{\|\bar{\varphi}_v - v^*\|}{2}. \quad (23)$$

By multiplying both sides of (23) with $2e^{-2\varepsilon t}$ , we get

$$\sum_{i=1}^{n}[(u_i - u_i^*)^2 + (y_i - y_i^*)^2] + \sum_{j=1}^{m}[(v_j - v_j^*)^2 + (z_j - z_j^*)^2]$$
$$\le e^{-2\varepsilon t}\{[1 + 2\sigma\sum_{j=1}^{m}\max_{1\le i\le n}\{|q_{ji}|(k_i^{\delta_i^*})^2\}e^{2\varepsilon\sigma}]\|\varphi_u - u^*\|^2 + [1 + 2\tau\sum_{i=1}^{n}\max_{1\le j\le m}\{|d_{ij}|(l_j^{\delta_j})^2\}e^{2\varepsilon\tau}]\|\varphi_v - v^*\|^2 + \|\bar{\varphi}_u - u^*\|^2 + \|\bar{\varphi}_v - v^*\|^2\}$$
$$\le e^{-2\varepsilon t}\{M^*[\|\varphi_u - u^*\|^2 + \|\varphi_v - v^*\|^2] + \|\bar{\varphi}_u - u^*\|^2 + \|\bar{\varphi}_v - v^*\|^2\}, \quad (24)$$

for all $t \ge 0$ , where

$$M^* = \max\{[1 + 2\sigma\sum_{j=1}^{m}\max_{1\le i\le n}\{|q_{ji}|(k_i^{\delta_i^*})^2\}e^{2\varepsilon\sigma}, 1 + 2\tau\sum_{i=1}^{n}\max_{1\le j\le m}\{|d_{ij}|(l_j^{\delta_j})^2\}e^{2\varepsilon\tau}\}.$$

Let $M = M^* + \frac{\|\bar{\varphi}_u-u^*\|^2+\|\bar{\varphi}_v-v^*\|^2}{\|\varphi_u-u^*\|^2+\|\varphi_v-v^*\|^2} > 1$, from (24), we obtain

$$\sum_{i=1}^{n}\|u_i - u_i^*\|^2 + \sum_{j=1}^{m}\|v_j - v_j^*\|^2 \le Me^{-2\varepsilon t}(\|\varphi_u - u^*\|^2 + \|\varphi_v - v^*\|^2),$$

for all $t \ge 0$. This implies that the equilibrium $(u^*, v^*)$ of system (1) is globally exponentially stable.

Furthermore, as consequence of Theorem 2 we have the following corollaries.

**Corollary 1**. Under the hypotheses $(H)$, the unique equilibrium point of system (1) is globally exponentially stably if one of the following conditions holds

$$\begin{cases} \sum_{j=1}^{m}\frac{|c_{ij}|}{2} + \sum_{j=1}^{m}\frac{|d_{ij}|}{2} + 1 - \alpha_i + \frac{|a_i-\alpha_i|}{2} < 0, \\ \sum_{i=1}^{n}\frac{|c_{ij}|}{2}(l_j)^2 + \sum_{i=1}^{n}\frac{|d_{ij}|}{2}(l_j)^2 + \frac{|b_j-\beta_j|}{2} - 1 < 0, \\ \sum_{i=1}^{n}\frac{|p_{ji}|}{2} + \sum_{i=1}^{n}\frac{|q_{ji}|}{2} + 1 - \beta_j + \frac{|b_j-\beta_j|}{2} < 0, \\ \sum_{j=1}^{m}\frac{|p_{ji}|}{2}(k_i)^2 + \sum_{j=1}^{m}\frac{|q_{ji}|}{2}(k_i)^2 + \frac{|a_i-\alpha_i|}{2} - 1 < 0, \end{cases} \quad (25)$$

for $i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$,

$$\begin{cases} \sum_{j=1}^{m}\frac{|c_{ij}|}{2}(l_j)^2 + \sum_{j=1}^{m}\frac{|d_{ij}|}{2}(l_j)^2 + 1 - \alpha_i + \frac{|a_i-\alpha_i|}{2} < 0, \\ \sum_{i=1}^{n}\frac{|c_{ij}|}{2} + \sum_{i=1}^{n}\frac{|d_{ij}|}{2} + \frac{|b_j-\beta_j|}{2} - 1 < 0, \\ \sum_{i=1}^{n}\frac{|p_{ji}|}{2}(k_i)^2 + \sum_{i=1}^{n}\frac{|q_{ji}|}{2}(k_i)^2 + 1 - \beta_j + \frac{|b_j-\beta_j|}{2} < 0, \\ \sum_{j=1}^{m}\frac{|p_{ji}|}{2} + \sum_{j=1}^{m}\frac{|q_{ji}|}{2} + \frac{|a_i-\alpha_i|}{2} - 1 < 0, \end{cases} \quad (26)$$

for $i=1,2,\cdots,n, j=1,2,\cdots,m$,

$$\left\{\begin{array}{l}\sum\limits_{j=1}^{m}\frac{|c_{ij}|}{2}+\sum\limits_{j=1}^{m}\frac{|d_{ij}|}{2}+1-\alpha_i+\frac{|a_i-\alpha_i|}{2}<0,\\ \sum\limits_{i=1}^{n}\frac{|c_{ij}|}{2}(l_j)^2+\sum\limits_{i=1}^{n}\frac{|d_{ij}|}{2}(l_j)^2+\frac{|b_j-\beta_j|}{2}-1<0,\\ \sum\limits_{i=1}^{n}\frac{|p_{ji}|}{2}(k_i)^2+\sum\limits_{i=1}^{n}\frac{|q_{ji}|}{2}(k_i)^2\\ \qquad +1-\beta_j+\frac{|b_j-\beta_j|}{2}<0,\\ \sum\limits_{j=1}^{m}\frac{|p_{ji}|}{2}+\sum\limits_{j=1}^{m}\frac{|q_{ji}|}{2}+\frac{|a_i-\alpha_i|}{2}-1<0,\end{array}\right. \tag{27}$$

for $i=1,2,\cdots,n, j=1,2,\cdots,m$, and

$$\left\{\begin{array}{l}\sum\limits_{j=1}^{m}\frac{|c_{ij}|}{2}(l_j)^2+\sum\limits_{j=1}^{m}\frac{|d_{ij}|}{2}(l_j)^2\\ \qquad +1-\alpha_i+\frac{|a_i-\alpha_i|}{2}<0,\\ \sum\limits_{i=1}^{n}\frac{|c_{ij}|}{2}+\sum\limits_{i=1}^{n}\frac{|d_{ij}|}{2}+\frac{|b_j-\beta_j|}{2}-1<0,\\ \sum\limits_{i=1}^{n}\frac{|p_{ji}|}{2}+\sum\limits_{i=1}^{n}\frac{|q_{ji}|}{2}\\ \qquad +1-\beta_j+\frac{|b_j-\beta_j|}{2}<0,\\ \sum\limits_{j=1}^{m}\frac{|p_{ji}|}{2}(k_i)^2+\sum\limits_{j=1}^{m}\frac{|q_{ji}|}{2}(k_i)^2+\frac{|a_i-\alpha_i|}{2}-1<0,\end{array}\right. \tag{28}$$

for $i=1,2,\cdots,n, j=1,2,\cdots,m$.

In fact, conditions (25)-(28) are the special cases of the Theorem 2 as $\gamma_j=\delta_j=1, \gamma_{i^*}=\delta_{i^*}=1$; $\gamma_j=\delta_j=0, \gamma_{i^*}=\delta_{i^*}=0$; $\gamma_j=\delta_j=1, \gamma_{i^*}=\delta_{i^*}=0$; $\gamma_j=\delta_j=0, \gamma_{i^*}=\delta_{i^*}=1$, respectively. Therefore, by Theorem 2 we see that Corollary 1 is true.

**Theorem 3**. Under the hypotheses $(H)$, the equilibrium point of system (1) is globally exponentially stably if the following conditions hold

$\alpha_i^2-4a_i\neq 0, \beta_j^2-4b_j\neq 0 (i=1,2,\cdots,n, j=1,2,\cdots,m)$,

$$-\eta+\max\{\sum_{i=1}^{n}M_i\max_{1\le j\le m}(|c_{ij}|l_j),\sum_{j=1}^{m}N_j\max_{1\le i\le n}(|p_{ji}|k_i)\}$$
$$+\max\{\sum_{j=1}^{m}N_j\max_{1\le i\le n}(|q_{ji}|k_i),\sum_{i=1}^{n}M_i\max_{1\le j\le m}(|d_{ij}|l_j)\}$$
$$<0.$$

**Proof.** Let the equilibrium point $(u^*,v^*)=(u_1^*,u_2^*,\cdots,u_n^*,v_1^*,v_2^*,\cdots,v_m^*)^T$ of system (1),

$$\bar{U}_i=\begin{pmatrix}u_i-u_i^*\\ y_i-y_i^*\end{pmatrix},\quad \bar{V}_j=\begin{pmatrix}v_j-v_j^*\\ z_j-z_j^*\end{pmatrix}.$$

From (12),(13) and Lemma 2, we have

$$\begin{aligned}\|\bar{U}_i(t)\|&\le\|e^{-A_it}\|\|\bar{U}_i(0)\|\\ &+\int_0^t\|e^{-A_i(t-s)}\|[|\sum_{j=1}^{m}c_{ij}(f_j(v_j(s))-f_j(v_j^*))|\\ &+|\sum_{j=1}^{m}d_{ij}(f_j(v_j(s-\tau_{ji}))-f_j(v_j^*))|]ds\\ &\le M_ie^{-\eta t}\|\bar{U}_i(0)\|\\ &+M_i\int_0^t e^{-\eta(t-s)}[\sum_{j=1}^{m}|c_{ij}|l_j|v_j(s)-v_j^*|\\ &+\sum_{j=1}^{m}|d_{ij}|l_j|v_j(s-\tau_{ji})-v_j^*|]ds.\end{aligned}\tag{29}$$

$$\begin{aligned}\|\bar{V}_j(t)\|&\le\|e^{-B_jt}\|\|\bar{V}_j(0)\|\\ &+\int_0^t\|e^{-B_j(t-s)}\|[|\sum_{i=1}^{n}p_{ji}(g_j(u_i(s))-g_i(u_i^*))|\\ &+|\sum_{i=1}^{n}q_{ji}(g_i(u_i(s-\sigma_{ij}))-g_i(u_i^*))|]ds\\ &\le N_je^{-\eta t}\|\bar{V}_j(0)\|\\ &+N_j\int_0^t e^{-\eta(t-s)}[\sum_{i=1}^{n}|p_{ji}|k_i|u_i(s)-u_i^*|\\ &+\sum_{i=1}^{n}|q_{ji}|k_i|u_i(s-\sigma_{ij})-u_i^*|]ds.\end{aligned}\tag{30}$$

From (29) and (30), we obtain

$$\begin{aligned}&\sum_{i=1}^{n}\|\bar{U}_i(t)\|+\sum_{j=1}^{m}\|\bar{V}_j(t)\|\le\sum_{i=1}^{n}M_ie^{-\eta t}\|\bar{U}_i(0)\|\\ &+\sum_{i=1}^{n}M_i\int_0^t e^{-\eta(t-s)}[\sum_{j=1}^{m}|c_{ij}|l_j|v_j(s)-v_j^*|\\ &+\sum_{j=1}^{m}|d_{ij}|l_j|v_j(s-\tau_{ji})-v_j^*|]ds+\sum_{j=1}^{m}N_je^{-\eta t}\|\bar{V}_j(0)\|\\ &+\sum_{j=1}^{m}N_j\int_0^t e^{-\eta(t-s)}[\sum_{i=1}^{n}|p_{ji}|k_i|u_i(s)-u_i^*|\\ &+\sum_{i=1}^{n}|q_{ji}|k_i|u_i(s-\sigma_{ij})-u_i^*|]ds\doteq W(t).\end{aligned}$$

We have

$$\begin{aligned}&W'(t)=-\eta W(t)+\sum_{i=1}^{n}M_i[\sum_{j=1}^{m}|c_{ij}|l_j|v_j(t)-v_j^*|+\\ &\sum_{j=1}^{m}|d_{ij}|l_j|v_j(t-\tau_{ji})-v_j^*|]+\sum_{j=1}^{m}N_j[\sum_{i=1}^{n}|p_{ji}|k_i|u_i(t)-\\ &u_i^*|+\sum_{i=1}^{n}|q_{ji}|k_i|u_i(t-\sigma_{ij})-u_i^*|]\\ &\le-\eta W(t)+\sum_{i=1}^{n}M_i[\max_{1\le j\le m}(|c_{ij}|l_j)\sum_{j=1}^{m}|v_j(t)-v_j^*|\\ &+\max_{1\le j\le m}(|d_{ij}|l_j)\sum_{j=1}^{m}|v_j(t-\tau_{ji})-v_j^*|]\\ &+\sum_{j=1}^{m}N_j[\max_{1\le i\le n}(|p_{ji}|k_i)\sum_{i=1}^{n}|u_i(t)-u_i^*|\\ &+\max_{1\le i\le n}(|q_{ji}|k_i)\sum_{i=1}^{n}|u_i(t-\sigma_{ij})-u_i^*|]\\ &\le-\eta W(t)\\ &+\max\{\sum_{i=1}^{n}M_i\max_{1\le j\le m}(|c_{ij}|l_j),\sum_{j=1}^{m}N_j\max_{1\le i\le n}(|p_{ji}|k_i)\}\\ &\cdot[\sum_{i=1}^{n}|u_i(t)-u_i^*|+\sum_{j=1}^{m}|v_j(t)-v_j^*|]\end{aligned}$$

$$+\max\{\sum_{j=1}^{m} N_j \max_{1\le i\le n}(|q_{ji}|k_i), \sum_{i=1}^{n} M_i \max_{1\le j\le m}(|d_{ij}|l_j)\}$$

$$\cdot[\sum_{i=1}^{n}|u_i(t-\sigma_{ij})-u_i^*|+\sum_{j=1}^{m}|v_j(t-\tau_{ji})-v_j^*|]. \quad (31)$$

Since $W(t) \ge \sum_{i=1}^{n}\|\bar{U}_i(t)\| + \sum_{j=1}^{m}\|\bar{V}_j(t)\|$, $\bar{W}(t) = \sup_{t-\xi\le s\le t} W(t) \ge \sup_{t-\xi\le s\le t}\{\sum_{i=1}^{n}\|\bar{U}_i(s)\|+\sum_{j=1}^{m}\|\bar{V}_j(s)\|\}$, where $\xi = \max\{\tau,\sigma\}$.

Thus, from (31) we can obtain

$$W'(t) \le -\eta W(t)$$
$$+\max\{\sum_{i=1}^{n} M_i \max_{1\le j\le m}(|c_{ij}|l_j), \sum_{j=1}^{m} N_j \max_{1\le i\le n}(|p_{ji}|k_i)\}W(t)$$
$$+\max\{\sum_{j=1}^{m} N_j \max_{1\le i\le n}(|q_{ji}|k_i), \sum_{i=1}^{n} M_i \max_{1\le j\le m}(|d_{ij}|l_j)\}\bar{W}(t).$$

When

$$\max\{\sum_{i=1}^{n} M_i \max_{1\le j\le m}(|c_{ij}|l_j), \sum_{j=1}^{m} N_j \max_{1\le i\le n}(|p_{ji}|k_i)\}$$
$$+\max\{\sum_{j=1}^{m} N_j \max_{1\le i\le n}(|q_{ji}|k_i), \sum_{i=1}^{n} M_i \max_{1\le j\le m}(|d_{ij}|l_j)\} < \eta,$$

by Lemma 3, there exists $\lambda > 0$ such that $\sum_{i=1}^{n}\|\bar{U}_i(t)\| + \sum_{j=1}^{m}\|\bar{V}_j(t)\| \le W(t) \le \bar{W}(0)e^{-\lambda t}$, we obtain $\sum_{i=1}^{n}\|u_i-u_i^*\|^2 + \sum_{j=1}^{m}\|v_j-v_j^*\|^2$

$$\le Me^{-\lambda t}(\|\varphi_u - u^*\|^2 + \|\varphi_v - v^*\|^2), t > 0,$$

where $M > 0$. This implies that the equilibrium $(u^*, v^*)$ of system (1) is globally exponentially stable.

## 4 Numerical Examples

In this section, we give two examples for showing our results.

**Example 4.1** Consider the following BAM neural networks with an inertial term and time delay

$$\begin{cases} \frac{d^2u_i(t)}{dt^2} = -\alpha_i\frac{du_i(t)}{dt} - a_iu_i(t) + \sum_{j=1}^{2} c_{ij}f_j(v_j(t)) \\ \qquad + \sum_{j=1}^{2} d_{ij}f_j(v_j(t-\tau_{ji})) + I_i, \\ \frac{d^2v_j(t)}{dt^2} = -\beta_j\frac{dv_j(t)}{dt} - b_jv_j(t) + \sum_{i=1}^{2} p_{ji}g_i(u_i(t)) \\ \qquad + \sum_{i=1}^{2} q_{ji}g_i(u_i(t-\sigma_{ij})) + J_j, \end{cases} \quad (32)$$

for $i = 1,2, j = 1,2$, where $a_1 = 3$, $a_2 = 2$, $\alpha_1 = 2.8$, $\alpha_2 = 1.96$, $b_1 = 4$, $b_2 = 2.5$, $\beta_1 = 3.5$, $\beta_2 = 3$, $c_{11} = 0.5$, $c_{12} = 0.3$, $c_{21} = 0.1$, $c_{22} = 0.2$, $p_{11} = 0.4$, $p_{12} = 0.2$, $p_{21} = -0.3$, $p_{22} = 0.5$, $d_{11} = 0.4$, $d_{12} = 0.6$, $d_{21} = -0.2$, $d_{22} = -0.3$, $q_{11} = 0.7$, $q_{12} = 0.4$, $q_{21} = 0.3$, $q_{22} = 0.8$, $I_1 = 2, I_2 = 3, J_1 = 2.5, J_2 = 4$,
and $f_i(x) = g_i(x) = \frac{1}{2}(|x+1| - |x-1|)(i = 1,2)$. Obviously, $f_i(x), g_i(x)(i = 1,2)$ satisfy the condition of hypotheses $(H)$ and $l_i = k_i = 1(i = 1,2)$.
From (32), we can get the equation of the equilibrium

$$\begin{cases} -3u_1 + 0.9f_1(v_1) + 0.9f_2(v_2) + 2 = 0, \\ -2u_2 - 0.1f_1(v_1) - 0.1f_2(v_2) + 3 = 0, \\ -4v_1 + 1.1g_1(u_1) + 0.6g_2(u_2) + 2.5 = 0, \\ -2.5v_2 + 1.3g_2(u_2) + 4 = 0, \end{cases} \quad (33)$$

For any $0 \le \gamma_j, \delta_j \le 1(j = 1,2)$, $0 \le \gamma_i^*, \delta_i^* \le 1(i = 1,2)$, we have the following results by simple calculation

$$\sum_{j=1}^{2}\frac{|c_{ij}|}{2} + \sum_{j=1}^{2}\frac{|d_{ij}|}{2} + 1 - \alpha_i + \frac{|a_i-\alpha_i|}{2} < 0,$$

$$\sum_{i=1}^{2}\frac{|c_{ij}|}{2} + \sum_{i=1}^{2}\frac{|d_{ij}|}{2} + \frac{|b_j-\beta_j|}{2} - 1 < 0,$$

$$\sum_{i=1}^{2}\frac{|p_{ji}|}{2} + \sum_{i=1}^{2}\frac{|q_{ji}|}{2} + 1 - \beta_j + \frac{|b_j-\beta_j|}{2} < 0,$$

$$\sum_{j=1}^{2}\frac{|p_{ji}|}{2} + \sum_{j=1}^{2}\frac{|q_{ji}|}{2} + \frac{|a_i-\alpha_i|}{2} - 1 < 0,$$

for $i = 1,2, j = 1,2$.

Then, the conditions of Theorem 1 and Theorem 2 hold. By calculation , there exists a unique equilibrium point $(u_1^*, u_2^*, v_1^*, v_2^*) = (\frac{3.8}{3}, 1.4, \frac{4.2}{4}, \frac{10.6}{5})$ of (33). Figs.1 - Figs.4 depict the time responses of state variables of $u_1(t), u_2(t), v_1(t), v_2(t)$ of system in example 4.1, respectively. Evidently, this consequence is coincident with the results of Theorem 1 and Theorem 2.

**Example 4.2** For system (32), let $a_1 = 6$, $a_2 = 8$, $\alpha_1 = 5$, $\alpha_2 = 6$, $b_1 = 6$, $b_2 = 8$, $\beta_1 = 5$, $\beta_2 = 6$, $c_{11} = 0.1$, $c_{12} = 0.05$, $c_{21} = 0.125$, $c_{22} = 0.1$, $p_{11} = 0.05$, $p_{12} = 0.1$, $p_{21} = 0.125$, $p_{22} = 0.125$, $d_{11} = 0.05$, $d_{12} = 0.1$, $d_{21} = 0.1$, $d_{22} = 0.125$, $q_{11} = 0.1$, $q_{12} = 0.1$, $q_{21} = 0.125$, $q_{22} = 0.125$, $I_1 = 6, I_2 = 8, J_1 = 7, J_2 = 9$,
and $f_i(x) = g_i(x) = \frac{1}{2}(|x+1| - |x-1|)(i = 1,2)$. Obviously, $f_i(x), g_i(x)(i = 1,2)$ satisf es the condition of hypotheses $(H)$ and $l_i = k_i = 1(i = 1,2)$.

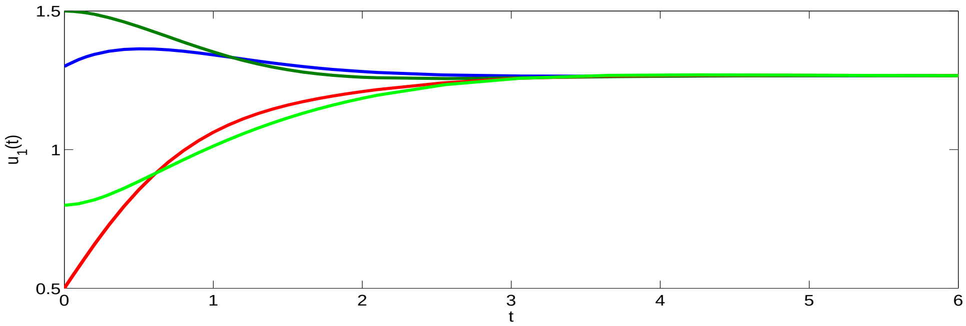


Fig.1. Transient response of state variables $u_1(t)$ of Example 4.1

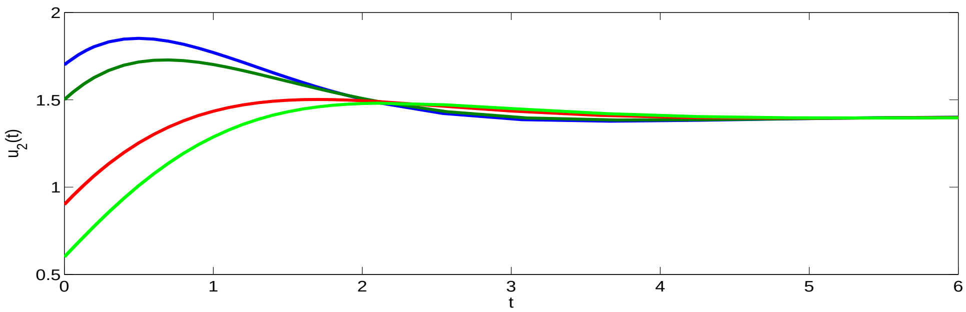


Fig.2. Transient response of state variables $u_2(t)$ of Example 4.1

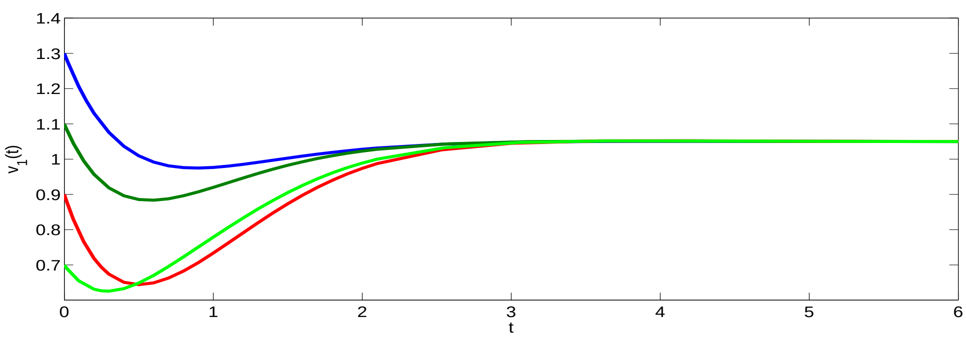


Fig.3. Transient response of state variables $v_1(t)$ of Example 4.1

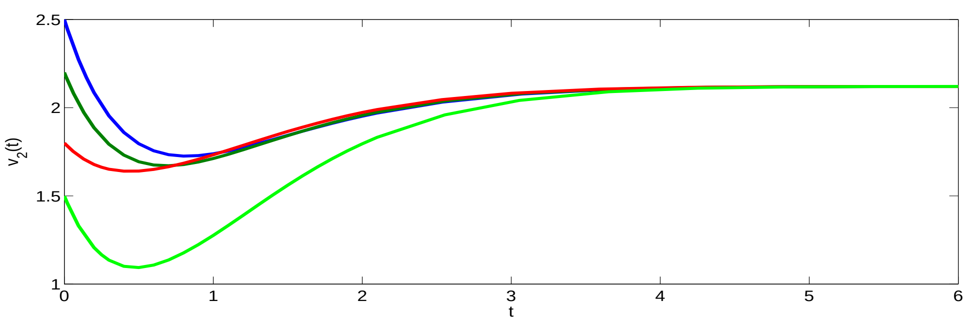


Fig.4. Transient response of state variables $v_2(t)$ of Example 4.1

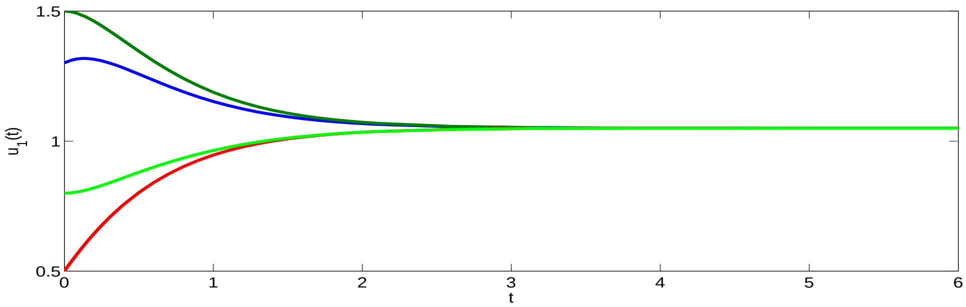


Fig.5. Transient response of state variables $u_1(t)$ of Example 4.2

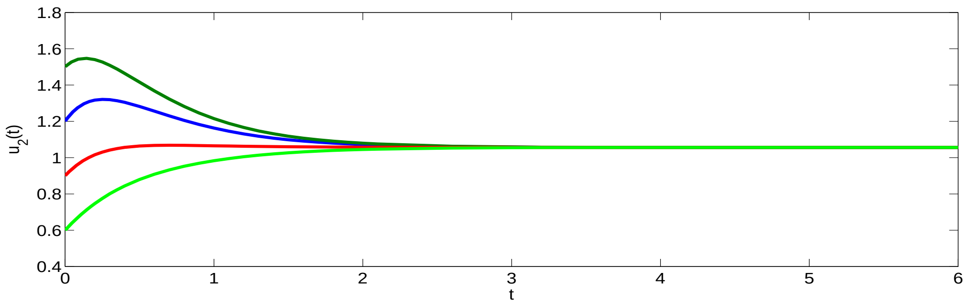


Fig.6. Transient response of state variables $u_2(t)$ of Example 4.2

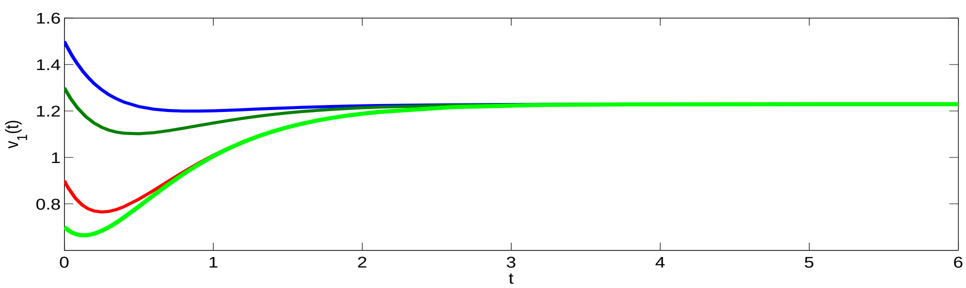


Fig.7. Transient response of state variables $v_1(t)$ of Example 4.2

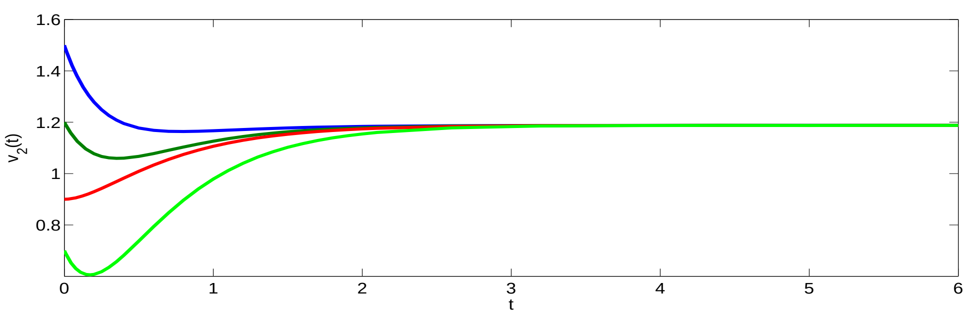


Fig.8. Transient response of state variables $v_2(t)$ of Example 4.2

By simple calculation, we can obtain
$M_1 = \sqrt{20}, M_2 = \sqrt{10}, N_1 = \sqrt{20}, N_2 = \sqrt{10},$
$\eta = \min_{1\le i\le n, 1\le j\le m}\{\frac{\alpha_i - \sqrt{|\alpha_i^2 - 4a_i|}}{2}, \frac{\beta_j - \sqrt{|\beta_j^2 - 4b_j|}}{2}\}$
$= 2.$
From (32), we can get the equation of the equilibrium

$$\begin{cases} -6u_1 + \frac{3}{20}f_1(v_1) + \frac{3}{20}f_2(v_2) + 6 = 0, \\ -8u_2 + \frac{9}{40}f_1(v_1) + \frac{9}{40}f_2(v_2) + 8 = 0, \\ -6v_1 + \frac{3}{20}g_1(u_1) + \frac{1}{5}g_2(u_2) + 7 = 0, \\ -8v_2 + \frac{1}{4}g_1(u_1) + \frac{1}{4}g_2(u_2) + 9 = 0, \end{cases} \quad (34)$$

We have the following results by simple calculation
$\alpha_i^2 - 4a_i \neq 0, \beta_j^2 - 4b_i \neq 0 (i, j = 1, 2),$
$-\eta + \max\{\sum_{i=1}^{n} M_i \max_{1\le j\le m}(|c_{ij}|l_j), \sum_{j=1}^{m} N_j \max_{1\le i\le n}(|p_{ji}|k_i)\}$
$+ \max\{\sum_{j=1}^{m} N_j \max_{1\le i\le n}(|q_{ji}|k_i), \sum_{i=1}^{n} M_i \max_{1\le j\le m}(|d_{ij}|l_j)\}$
$= -2 + \frac{2}{5}\sqrt{5} + \frac{1}{4}\sqrt{10} < 0.$

Then, the conditions of Theorem 3 hold. By calculation , there exists a unique equilibrium point $(u_1^*, u_2^*, v_1^*, v_2^*) = (\frac{63}{60}, \frac{169}{160}, \frac{147}{120}, \frac{19}{16})$ of (34). Figs.5-Figs.8 depict the time responses of state variables of $u_1(t), u_2(t), v_1(t), v_2(t)$ of system in example 4.2, respectively. Evidently, this consequence is coincident with the results of Theorem 3.

**Remark 1** By simple calculation in Example 4.1 there is
$\eta = \min_{1\le i\le n, 1\le j\le m}\{\frac{\alpha_i - \sqrt{|\alpha_i^2 - 4a_i|}}{2}, \frac{\beta_j - \sqrt{|\beta_j^2 - 4b_j|}}{2}\}$
$= \frac{1.96 - \sqrt{4.1584}}{2} < 0,$
$-\eta + \max\{\sum_{i=1}^{n} M_i \max_{1\le j\le m}(|c_{ij}|l_j), \sum_{j=1}^{m} N_j \max_{1\le i\le n}(|p_{ji}|k_i)\}$
$+ \max\{\sum_{j=1}^{m} N_j \max_{1\le i\le n}(|q_{ji}|k_i), \sum_{i=1}^{n} M_i \max_{1\le j\le m}(|d_{ij}|l_j)\}$
$> 0.$
It showed that the conditions of Theorem 3 aren't satisf ed . While in Example 4.2 there is

$$\sum_{j=1}^{2} \frac{|p_{j2}|}{2} + \sum_{j=1}^{2} \frac{|q_{j2}|}{2} + \frac{|a_2 - \alpha_2|}{2} - 1 > 0.$$

It showed that the conditions of Theorem 2 aren't satisf ed . Therefore, the above two examples show that all the Theorems 2-3 in this paper have advantages in different problems and applications.

# 5 Conclusions

In this paper, we give three theorems to ensure the existence and the exponential stability of the equilibrium point for BAM neural networks with inertial term and time delay. Especially, we give different conditions in Theorems 2 and Theorems 3 to ensure the exponential stability of the equilibrium point, which have different advantages in different problems and applications. Finally two examples illustrate the effectiveness in different conditions.

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