Reliability Analysis of a Deteriorating System with Delayed Vacation of Repairman

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Abstract: In this paper, we discuss a deteriorating system with one repairman. In this system, it is assumed that the component cannot be repaired "as good as new" after failures and the repairman takes a delayed vacation, the repair time is taken into account. Under these assumptions, we derive a model of partial differential equations by using the geometric process and supplementary variable technique. We get some reliability indices by the Laplace transform method. The working probability of the repairman, the delayed rate and the rate of occurrence of failures of the steady state of the system are explicitly given. In particular, the rate of occurrence of failures in the steady state satisfies $m_f \neq 0$.

Key–Words: deteriorating system, delayed vacation, Laplace transform, partial differential equation model, reliability.

1 Introduction

The reliability analysis of the system is an important content at the planning,design and operational stages of various system(see, [1], [2], [3], [4]), these papers study the availability of the system and the maximum reliability of the component. Many authors studied some problems about the reliability of the systems, especially the complex systems. For example: the reliability of repairable system, queuing system, electronic product and network (see,[5], [6], [7], [8]). Much more attentions have been paid to the study of repairable systems. Repairable system is not only a kind of important system discussed in reliability theory but also one of the main objects studied in reliability mathematics. Many authors have worked in this field, including system modeling and model analysis, for example, see [9], [10], [11], [12], [13] and references therein.

In recent years, there have been two hot topics in reliability analysis, one is that the repairman cannot repair in time, or the system after failures cannot be repaired immediately because of the absence of the repairman, the other is that the system cannot be repaired as good as new after failures. This is true particularly for the electronic products(see, [5], [13], [14]). These problems have been considered in the papers mentioned above and studied under the assumption on the deteriorating systems with constant repair rate. Some papers considered the problem of repairman vacation. Obviously, if the repairman is on vacation and the system fails, then the system must be in state of waiting for repair. It is well known that the longer the time is, the more failure times the system will have, even a system can be repaired as good as new. If the repairman goes to vacation at once, then the waiting time of the system to be repaired will be much longer. Based on this reason, we consider a strategy of delayed vacation for a repairable system in the present paper. The delayed vacation means that the repairman waits for certain time after the system finished repairing and then goes to his vacation. If the system fails during waiting vacation, the repairman will begin repairing at once, otherwise the repairman starts his vacation. Here we consider more general case that the repair time satisfies the general distribution but has finite excepted value. This is different from that in considered in [12] and [13]. Firstly we establish mathematical model for such a system, and then we study the reliability indices of the system.

This paper is organized as follows: in section 2, we at first establish the governed equation of dynamic behavior of the system under the geometric process of deteriorating system. Due to the general distribution of repair time, we derive a model of partial differential equations by the supplementary variable and analysis probability; In section 3, we further discuss the state model of the system; In section 4, we use the Laplace transform method to investigate some reliability indices of the system, including the working probability of the repairman, the delayed rate and the rate of occurrence of failures. Finally, in section 5, we give a conclusion remark.

2 Mathematical modeling

In this section we shall model a deteriorating system under the geometric process. Firstly we make the following assumptions:

1). A system consists of one machine and one repairman, the machine is new and starts to work at time $t = 0$;

2). The repairman takes delayed vacation. The repairman will prepare his vacation when the system begins working. If the system fails during this period, the system will be repaired at once. Otherwise the repairman will start his vacation; If the machine fails during vacation of the repairman, the system is in the state of waiting for repair;

3). When the repairman comes back after his vacation, there are two cases: one is that the system is in the state of waiting for repair, in this case he begins his repair immediately; the other is the machine is running, he will continue his delayed vacation immediately;

4). After completion of repair, the system returns to the working state. The time interval between the completion of the $(k-1)$-th repair and the k-th repair of the system is called the k-th cycle of the system, $k = 1, 2, \cdots$. When the system is in the k-th cycle, let

$c^{k-1}\lambda_0$: is the constant failure rate of the system during the repairman preparing vacation, and $c \geq 1$ is deteriorating ratio [12];

$a^{k-1}\lambda_1$: is the constant failure rate of the system during the repairman on vacation, and $a \geq 1$ is deteriorating ratio [12];

$b^{k-1}\gamma$: is the constant waiting repairing rate of the system, and $b \geq 1$ is deteriorating ratio [12];

ε_0: is the constant rate of the repairman preparing vacation;

ε_1: is the constant rate of the repairman returning system;

$\mu(y)$: is the repaired rate, when the failed system has an elapsed repairing time of y, the distribution is

$$G(t) = \int_0^t g(y)dy = 1 - \exp(-\int_0^t \mu(y)dy)$$

with expected value

$$\int_0^\infty tg(t)dt = \int_0^\infty t\mu(t)e^{-\int_0^t \mu(y)dy}dt = \frac{1}{\mu} < \infty.$$

It satisfies $\int_0^y \mu(\tau) < \infty$, $\int_0^\infty \mu(\tau) = \infty$;

5). All above random variables are independent;

6). Each switchover of the states is perfect and each switchover time is instantaneous.

All above are the basic assumptions on the system. Under these hypotheses we will establish a mathematical model. We begin with describing the process and state of the system.

Let $\{I(t), t \geq 0\}$ be a stochastic process taken value in state space $\mathbb{N}$. $I(t) = k$ means that the system is in the k-th cycle at time t.

Let $E = \{0, 1, 2, 3\}$ be the event space. The events are defined as follows:

0: The system is working at time t, the repairman is on preparing holiday;

1: The system is working at time t, the repairman is on holiday;

2: The system fails at time t, the repairman is on holiday;

3: The system fails at time t, the repairman is repairing the failed component.

Clearly, the set of the working states of the system is $W = \{0, 1\} \subset E$ and the set of the failed states is $F = \{2, 3\} \subset E$.

Again, let $\{N(t), t \geq 0\}$ be a stochastic process value in the event space $E = \{0, 1, 2, 3\}$. $N(t) = i \in E$ means that the system is in the event i at time t. Thus $P_i(t) = P\{N(t) = i\}$ denotes the probability of the system at the event i at time t.

Denote by $A_0(t) = P\{N(t) = 0\}$ and $A_1(t) = P\{N(t) = 1\}$. Set $A(t) = A_0(t) + A_1(t)$.Then $A(t)$ is the probability of the system in working state at t, which is called the availability of the system at time t.

$P_2(t) = P\{N(t) = 2\}$ is the probability of the system waiting for repair at time t and $P_W(t) = P\{N(t) = 3\}$ is the probability of the repairman working at time t.

Observe that the process $\{N(t), I(t); t \geq 0\}$ does not constitute a Markov process. To describe the behavior of the system we introduce a new stochastic variable. Let $Y(t)$ be a stochastic process value in $\mathbb{R}_+$. $Y(t) = y$ means that the system has an elapsed repair time at t. Then the equality

$$p(y,t)dy = P\{y \leq Y(t) < y + dy\}$$

denotes the probability density that the failed system has an elapse repair time of y. Obviously we have $P_W(t) = \int_0^\infty p(y,t)dy$. From above we see that the three stochastic process $\{N(t), Y(t), I(t); t \geq 0\}$ constitutes a generalized Markov process. This approach is said to be the supplement variable technique.

At first, let us define the event probability of the system at time t as follows:

For $i = 0, 1, 2$ and $j \in \mathbb{N}$,

$$p_{i,j}(t) = P\{N(t) = i, I(t) = j\}$$

and for $i = 3$ and $j \in \mathbb{N}$,

$$\begin{aligned} &p_{3,j}(t,y)dy \\ &= P\{N(t) = 3, y \le Y(t) < y + dy, I(t) = j\}. \end{aligned}$$

In what follows, we will deduce the differential equations according to change of probability of the system in time interval $(t, t + \Delta t]$ with small Δt:

1). For $i = 0$, this means that the system is working,

$$\begin{aligned} &p_{0,j}(t + \Delta t) \\ &= P\{N(t + \Delta t) = 0, I(t + \Delta t) = j\} \\ &= \sum_{i=0}^{3} P\{N(t + \Delta t) = 0, I(t + \Delta t) = j, N(t) = i\} \\ &= \sum_{i=0}^{3} P\{N(t + \Delta t) = 0, I(t + \Delta t) = j|N(t) = i\} \\ &\times P\{N(t) = i\}. \end{aligned}$$

For Δt small enough, when $N(t+\Delta t) = 0$, there are only two cases: one is $N(t) = 0$ or $N(t) = 1$, which means that the system is working at time t, the system still working at time $t+\Delta t$; the other is $N(t) = 3$, the system is failed at t but the system is working at time $t + \Delta t$. In this case, the system belongs to differential cycle.

Observe that according to assumptions we have the condition probabilities

$$\begin{aligned} &P\{N(t{+}\Delta t) = 0, I(t{+}\Delta t) = j|N(t) = 0, I(t) = j\} \\ &\times P\{N(t) = 0, I(t) = j\} \\ &= (1 - (c^{j-1}\lambda_0 + \varepsilon_0)\Delta t)p_{0,j}(t) + o(\Delta t), \end{aligned}$$

$$\begin{aligned} &P\{N(t{+}\Delta t) = 0, I(t{+}\Delta t) = j|N(t) = 1, I(t) = j\} \\ &\times P\{N(t) = 1, I(t) = j\} \\ &= \varepsilon_1 \Delta t p_{1,j}(t) + o(\Delta t) \end{aligned}$$

and

$$\begin{aligned} &P\{N(t{+}\Delta t) = 0, I(t{+}\Delta t) = j|N(t) = 3, I(t) = j{-}1\} \\ &\times P\{N(t) = 3, I(t) = j - 1\} \\ &= \int_0^\infty \mu(y)p_{3,j-1}(t,y)dy\Delta t + o(\Delta t). \end{aligned}$$

When $j = 1$, it occurs only in the first case. So we have

$$\begin{aligned} &p_{0,1}(t + \Delta t) \\ &= (1 - (\lambda_0 + \varepsilon_0)\Delta t)p_{0,1}(t) + \varepsilon_1 \Delta t p_{1,1}(t) + o(\Delta t). \end{aligned}$$

For $j > 1$, we have

$$\begin{aligned} &p_{0,j}(t + \Delta t) \\ &= (1 - (c^{j-1}\lambda_0 + \varepsilon_0)\Delta t)p_{0,j}(t) + \varepsilon_1 \Delta t p_{1,j}(t) \\ &+ \int_0^\infty \mu(y)p_{3,j-1}(t,y)dy\Delta t + o(\Delta t). \end{aligned}$$

From above we can get a group of ordinary differential equations

$$\frac{dp_{0,1}(t)}{dt} = -(\lambda_0 + \varepsilon_0)p_{0,1}(t) + \varepsilon_1 p_{1,1}(t)$$

and

$$\begin{aligned} \frac{dp_{0,j}(t)}{dt} &= -(c^{j-1}\lambda_0 + \varepsilon_0)p_{0,1}(t) + \varepsilon_1 p_{1,1}(t) \\ &\quad + \int_0^\infty \mu(y)p_{3,j-1}(t,y)dy, \quad \forall j > 1. \end{aligned}$$

2). For $i = 1$ and $j \in \mathbb{N}$,

$$\begin{aligned} &p_{1,j}(t + \Delta t) \\ &= P\{N(t + \Delta t) = 1, I(t + \Delta t) = j\} \\ &= \sum_{i=0}^{3} P\{N(t + \Delta t) = 1, I(t + \Delta t) = j, N(t) = i\} \\ &= \sum_{i=0}^{3} P\{N(t + \Delta t) = 1, I(t + \Delta t) = j|N(t) = i\} \\ &\times P\{N(t) = i\}. \end{aligned}$$

For $N(t + \Delta t) = 1$, the repairman starts vacation and the system is running at time $t + \Delta t$, it must be $N(t) = 0$, the system is working with the repairman waiting vacation at time t or $N(t) = 1$, the system is working when the repairman is vacation at time t.

Since

$$\begin{aligned} &P\{N(t{+}\Delta t) = 1, I(t{+}\Delta t) = j|N(t) = 0, I(t) = j\} \\ &\times P\{N(t) = 0, I(t) = j\} \\ &= \varepsilon_0 \Delta t p_{0,j}(t) + o(\Delta t) \end{aligned}$$

and

$$\begin{aligned} &P\{N(t{+}\Delta t) = 1, I(t{+}\Delta t) = j|N(t) = 1, I(t) = j\} \\ &\times P\{N(t) = 1, I(t) = j\} \\ &= (1 - (a^{j-1}\lambda_1 + \varepsilon_1)\Delta t)p_{1,j}(t) + o(\Delta t), \end{aligned}$$

so for any $j \in \mathbb{N}$, one has

$$\begin{aligned} &p_{1,j}(t + \Delta t) = (1 - (a^{j-1}\lambda_1 + \varepsilon_1)\Delta t)p_{1,j}(t) \\ &+\varepsilon_0 \Delta t p_{0,j}(t) + o(\Delta t). \end{aligned}$$

Hence we have ordinary differential equations

$$\frac{dp_{1,j}(t)}{dt} = \varepsilon_0 p_{0,j}(t) - (a^{j-1}\lambda_1 + \varepsilon_1)p_{1,j}(t) \quad \forall j \ge 1.$$

3). For $i = 2$ and $j \in \mathbb{N}$,

$$\begin{aligned} &p_{2,j}(t + \Delta t) \\ &= P\{N(t + \Delta t) = 2, I(t + \Delta t) = j\} \\ &= \sum_{i=0}^{3} P\{N(t + \Delta t) = 2, I(t + \Delta t) = j, N(t) = i\} \\ &= \sum_{i=0}^{3} P\{N(t + \Delta t) = 2, I(t + \Delta t) = j|N(t) = i\} \\ &\times P\{N(t) = i\}. \end{aligned}$$

For $N(t+\Delta t) = 2$, the system fails during the vacation of the repairman at time $t+\Delta t$, it must be $N(t) = 1$, the system is working and the repairman starts vacation at time t or $N(t) = 2$, the system fails during the vacation of the repairman at time t. Therefore, for any $j \in \mathbb{N}$,

$$\begin{aligned}&P\{N(t{+}\Delta t) = 2, I(t{+}\Delta t) = j|N(t) = 1, I(t) = j\}\\ &\times P\{N(t) = 1, I(t) = j\}\\ &= a^{j-1}\lambda_1 \Delta t p_{1,j}(t) + o(\Delta t)\end{aligned}$$

and

$$\begin{aligned}&P\{N(t{+}\Delta t) = 2, I(t{+}\Delta t) = j|N(t) = 2, I(t) = j\}\\ &\times P\{N(t) = 2, I(t) = j\}\\ &= (1 - b^{j-1}\gamma\Delta t)p_{2,j}(t) + o(\Delta t),\end{aligned}$$

it holds that

$$\begin{aligned}&p_{2,j}(t+\Delta t)\\ &= (1 - b^{j-1}\gamma\Delta t)p_{2,j}(t) + a^{j-1}\lambda_1\Delta t p_{1,j}(t) + o(\Delta t).\end{aligned}$$

This leads to differential equations

$$\frac{dp_{2,j}(t)}{dt} = a^{j-1}\lambda_1 p_{1,j}(t) - b^{j-1}\gamma p_{2,j}(t) \quad \forall j \geq 1.$$

4). If $N(t+\Delta t) = 3$, the system fails. There are two cases in this event: one is that the system is in failure but the repair has not been conducted, the other is that the system being under repair. Here we consider only the first case. In this case, the system at time t is probable in states $N(t) = 0$ or $N(t) = 2$, so $p_{3,j}(t, 0)$ implies that the system has just failed and has not started repairs in j-th cycles of the system at time t. It has the expression

$$p_{3,j}(t, 0) = c^{j-1}\lambda_0 p_{0,j}(t) + b^{j-1}\gamma p_{2,j}(t), j \in \mathbb{N}.$$

5). Here we consider $N(t+\Delta t) = 3$ and the system is under repair. For the repair time $y > 0$, let $\Delta t < y$

$$\begin{aligned}&p_{3,j}(t+\Delta t, y+\Delta t)dy\\ &= P\{N(t+\Delta t) = 3,\\ &y+\Delta t \leq Y(t{+}\Delta t) \leq y{+}\Delta t + dy, I(t{+}\Delta t) = j\}\\ &= \sum_{i=0}^{3} P\{N(t+\Delta t) = 3, y+\Delta t \leq Y(t+\Delta t)\\ &\qquad \leq y+\Delta t+dy, I(t+\Delta t) = j, N(t) = i\}\end{aligned}$$

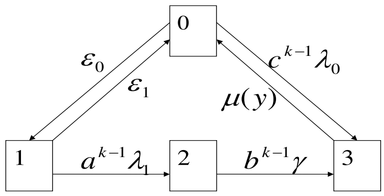
Since the repair time $y > 0$ and $\Delta t < y$, in event of $i = 0, 1, 2$, the repair time of the system is always less than $y+\Delta t$, so the event probabilities of $N(t) = 0, N(t) = 1$ and $N(t) = 2$ are zero. Therefore, at time t, the system is only in state $N(t) = 3$. The term $P\{N(t+\Delta t) = 3, y+\Delta t \leq Y(t+\Delta t) \leq y+\Delta t+dy, I(t+\Delta t) = j, N(t) = 3\}$ shows that the system is in maintenance and elapse time in $y+\Delta t \leq Y(t+\Delta t) \leq y+\Delta t+dy$ at time $t+\Delta t$. Obviously, it holds only if the system is in j-th cycle and has elapsed time in $y \leq Y(t) < y+\Delta t$, the probability density of the repairing is

$$\begin{aligned}&p_{3,j}(t+\Delta t, y+\Delta t)\\ &= (1-\mu(y)\Delta t)p_{3,j}(t, y) + o(\Delta t), j \in \mathbb{N}.\end{aligned}$$

This leads to partial differential equations

$$\frac{\partial p_{3,j}(t,y)}{\partial t} + \frac{\partial p_{3,j}(t,y)}{\partial y} = -\mu(y)p_{3,j}(t,y), j \in \mathbb{N}.$$

Secondly, given the following system state transition diagram:



By the analysis above, we see that the dynamic behavior of the system is governed by the partial differential equations:
When $j = 1$

$$\begin{cases} \{\frac{d}{dt} + \varepsilon_0 + \lambda_0\}p_{0,1}(t) = \varepsilon_1 p_{1,1}(t), \\ \{\frac{d}{dt} + \varepsilon_1 + \lambda_1\}p_{1,1}(t) = \varepsilon_0 p_{0,1}(t), \\ \{\frac{d}{dt} + \gamma\}p_{2,1}(t) = \lambda_1 p_{1,1}(t), \\ \{\frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \mu(y)\}p_{3,1}(t,y) = 0, \\ p_{3,1}(t,0) = \lambda_0 p_{0,1}(t) + \gamma p_{2,1}(t). \end{cases} \tag{1}$$

And when $j \geq 2$,

$$\begin{cases} \{\frac{d}{dt} + \varepsilon_0 + c^{j-1}\lambda_0\}p_{0,j}(t) = \\ \qquad \varepsilon_1 p_{1,1}(t) + \int_0^\infty \mu(y)p_{3,j-1}(t,y)dy, \\ \{\frac{d}{dt} + \varepsilon_1 + a^{j-1}\lambda_1\}p_{1,j}(t) = \varepsilon_0 p_{0,j}(t), \\ \{\frac{d}{dt} + b^{j-1}\gamma\}p_{2,j}(t) = a^{j-1}\lambda_1 p_{1,j}(t), \\ \{\frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \mu(y)\}p_{3,j}(t,y) = 0, \\ p_{3,j}(t,0) = c^{j-1}\lambda_0 p_{0,j}(t) + b^{j-1}\gamma p_{2,j}(t). \end{cases} \tag{2}$$

By the assumption 1) on the system, the initial conditions are given by

$$\begin{cases} p_{0,1}(0) = 1, p_{0,j}(0) = 0 \ \ (j \geq 2), \\ p_{1,j}(0) = p_{2,j}(0) = 0 \\ p_{3,j}(0,y) = 0, \ y \in (0,\infty) \ \ (j \geq 1). \end{cases} \tag{3}$$

Since all functions are the probability (or probability density) of the system in some state, from practice point of view, they satisfy the normal condition

$$\sum_{j=1}^{\infty} p_{0,j}(t) + \sum_{j=1}^{\infty} p_{1,j}(t) + \sum_{j=1}^{\infty} p_{2,j}(t) + \sum_{j=1}^{\infty} \int_0^{\infty} p_{3,j}(t,y)dy = 1, \quad \forall t \geq 0. \quad (4)$$

3 State model of the system

The model (1) and (2) mainly address the system at t being in the j-th cycle. In this section we shall discuss the state equation of the system.

Clearly, we have $P_i(t) = P\{N(t) = i\}$, $i = 0,1,2,3$. $P_0(t)$ is the probability of the machine working at time t after the repairman repairs failures; $P_1(t)$ is the probability of the machine working at time t when the repairman is on vacation; $P_2(t)$ is the probability of the system waiting for repair; they have the following expressions, respectively,

$$P_0(t) = \sum_{j=1}^{\infty} p_{0,j}(t), \quad P_1(t) = \sum_{j=1}^{\infty} p_{1,j}(t)$$

and

$$P_2(t) = \sum_{j=1}^{\infty} p_{2,j}(t).$$

Set

$$\begin{array}{l} m_{f_0}(t) = \sum\limits_{j=1}^{\infty} c^{j-1}\lambda_0 p_{0,j}(t), \\ m_{f_1}(t) = \sum\limits_{j=1}^{\infty} a^{j-1}\lambda_1 p_{1,j}(t), \\ m_f(t) = m_{f_0}(t) + m_{f_1}(t) \end{array}$$

where $m_{f_0}(t)$ is called the rate of occurrence of failures at time t when the repairman is in system, the system is failures; $m_{f_1}(t)$ is called the rate of occurrence of failures at time t when the repairman is on vacation, the system is failures; $m_f(t)$ is the rate of occurrence of failures at time t.

Denote

$$m_d(t) = \sum_{j=1}^{\infty} b^{j-1}\gamma p_{2,j}(t).$$

The $m_d(t)$ is said to be the delayed rate of occurrence at time t;

Set

$$p_3(t,y) = \sum_{j=1}^{\infty} p_{3,j}(t,y),$$

$p_3(t,y)$ is said to be the repair density of the system. Then we have

$$P_3(t) = \int_0^{\infty} p_3(t,y)dy$$

$P_3(t)$ is probability of the system under repair at t.

According to (1) and (2) we have the following state equations

$$\left\{ \begin{array}{l} \frac{dP_0(t)}{dt} + \varepsilon_0 P_0(t) + m_{f_0}(t) = \\ \qquad \varepsilon_1 P_1(t) + \int_0^{\infty} \mu(y)p_3(t,y)dy, \\ \frac{dP_1(t)}{dt} + \varepsilon_1 P_1(t) + m_{f_1}(t) = \varepsilon_0 P_0(t), \\ \frac{dP_2(t)}{dt} + m_d(t) = m_{f_1}(t), \\ \{\frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \mu(y)\}p_3(t,y) = 0, \\ p_3(t,0) = m_d(t) + m_{f_0}(t) \\ (P_0(0), P_1(0), P_2(0), p_3(0,y)) = (1,0,0,0). \end{array} \right. \quad (5)$$

In the study of reliability of a system, the discussion of the existence of the steady state of the system is an important content. This is because some indices of reliability will be deduced from it. If there exists a steady state $(P_0, P_1, P_2, p_3(y))$ of the system, then it must satisfy the following equations

$$\left\{ \begin{array}{l} \varepsilon_0 P_0 + m_{f_0} = \varepsilon_1 P_1 + \int_0^{\infty} \mu(y)p_3(y)dy, \\ \varepsilon_1 P_1 + m_{f_1}(t) = \varepsilon_0 P_0, \\ m_d = m_{f_1}, \\ \{\frac{d}{dy} + \mu(y)\}p_3(y) = 0, \\ p_3(0) = m_d + m_{f_0} \end{array} \right. \quad (6)$$

Solving above equations we get

$$p_3(y) = (m_d + m_{f_0})e^{-\int_0^y \mu(\tau)d\tau}$$

and $m_d = m_{f_1}$. So the steady state probability of the system in repair (or the probability of repairman working) is

$$P_3 = \int_0^{\infty} (m_d + m_{f_0})e^{-\int_0^y \mu(\tau)d\tau}dy = \frac{m_d + m_{f_0}}{\mu}$$

where $\frac{1}{\mu} = \int_0^{\infty} e^{-\int_0^y \mu(\tau)d\tau}dy$ is the expected value. Observe that if there is a steady state of the system, it must satisfy the normal condition

$$P_0 + P_1 + P_2 + P_3 = 1$$

Although we have given P_3, we cannot determine from (6) the steady state probabilities P_0, P_1 and P_2.

The state equations (5) might be a steady state, but the system (1) and (2) cannot have a steady state. In fact, if there exists a steady state $(p_{0,j}, p_{1,j}, p_{2,j}, p_{3,j}(y))$ of the system, it must satisfy

the following equations:
When $j = 1$

$$\begin{cases} \{\varepsilon_0 + \lambda_0\}p_{0,1} = \varepsilon_1 p_{1,1}, \\ \{\varepsilon_1 + \lambda_1\}p_{1,1} = \varepsilon_0 p_{0,1}, \\ \gamma p_{2,1} = \lambda_1 p_{1,1}, \\ \{\frac{d}{dy} + \mu(y)\}p_{3,1}(y) = 0, \\ p_{3,1}(0) = \lambda_0 p_{0,1} + \gamma p_{2,1}. \end{cases} \tag{7}$$

When $j \geq 2$

$$\begin{cases} \{\varepsilon_0 + c^{j-1}\lambda_0\}p_{0,j} = \varepsilon_1 p_{1,j} + \int_0^\infty \mu(y)p_{3,j-1}(y)dy, \\ \{\varepsilon_1 + a^{j-1}\lambda_1\}p_{1,j} = \varepsilon_0 p_{0,j}, \\ b^{j-1}\gamma p_{2,j} = a^{j-1}\lambda_1 p_{1,j}, \\ \{\frac{d}{dy} + \mu(y)\}p_{3,j}(y) = 0, \\ p_{3,j}(0) = c^{j-1}\lambda_0 p_{0,j} + b^{j-1}\gamma p_{2,j}. \end{cases} \tag{8}$$

Clearly, the equations (7) have zero solution only, and hence the equations (8) have also zero solution. Obviously it does not satisfy the normal condition.

4 Reliability indices

The reliability indices are measurement of the safety and availability of the system[16]. To obtain reliability indices of the system, including the reliability indices of the steady state, is an important content in system analysis. In this section we focus our attention on the discussion of some indices of the system such as availability, rate of occurrence of failures and the probability of the repairman working. Although we have established the state equations (5), we cannot determine these indices from it. To get these indices, we return to equations (1) and (2). In what follows we shall get the reliability indices via the Laplace transform method. Since our model describes a practical problem, we can assume that there exists a group nonnegative solution $(p_{0,j}(t), p_{1,j}(t), p_{2,j}(t), p_{3,j}(t,y))$ to equations (1) and (2).

Denote the Laplace transform of the functions by

$$p^*_{0,j}(s) = \int_0^\infty p_{0,j}(t)e^{-st}dt,$$
$$p^*_{1,j}(s) = \int_0^\infty p_{1,j}(t)e^{-st}dt,$$
$$p^*_{2,j}(s) = \int_0^\infty p_{2,j}(t)e^{-st}dt,$$
$$p^*_{3,j}(s,y) = \int_0^\infty p_{3,j}(t,y)e^{-st}dt$$

For equations (1) and (2), taking Laplace transform and using the initial condition (3) we get the following algebraic and differential equations with real parameter $s > 0$:

$$\begin{cases} sp^*_{0,1}(s) + (\varepsilon_0 + \lambda_0)p^*_{0,1}(s) = \varepsilon_1 p^*_{1,1}(s) + 1, \\ sp^*_{1,1}(s) + (\varepsilon_1 + \lambda_1)p^*_{1,1}(s) = \varepsilon_0 p^*_{0,1}(s), \\ sp^*_{2,1}(s) + \gamma p^*_{2,1}(s) = \lambda_1 p^*_{1,1}(s), \\ sp^*_{3,1}(s,y) + \frac{\partial}{\partial y}p^*_{3,1}(s,y) + \mu(y)p^*_{3,1}(s,y) = 0, \\ sp^*_{0,2}(s) + (\varepsilon_0 + c\lambda_0)p^*_{0,2}(s) = \varepsilon_1 p^*_{1,2}(s) \\ \qquad + \int_0^\infty \mu(y)p^*_{3,1}(s,y)dy, \\ sp^*_{1,2}(s) + (\varepsilon_1 + a\lambda_1)p^*_{1,2}(s) = \varepsilon_0 p^*_{0,2}(s), \\ sp^*_{2,2}(s) + b\gamma p^*_{2,2}(s) = a\lambda_1 p^*_{1,2}(s), \\ sp^*_{3,2}(s,y) + \frac{\partial}{\partial y}p^*_{3,2}(s,y) + \mu(y)p^*_{32}(s,y) = 0, \\ \vdots \\ sp^*_{0,j}(s) + (\varepsilon_0 + c^{j-1}\lambda_0)p^*_{0,j}(s) = \varepsilon_1 p^*_{1,j}(s) \\ \qquad + \int_0^\infty \mu(y)p^*_{3,j-1}(s,y)dy, \\ sp^*_{1,j}(s) + (\varepsilon_1 + a^{j-1}\lambda_1)p^*_{1,j}(s) = \varepsilon_0 p^*_{0,j}(s), \\ sp^*_{2,j}(s) + b^{j-1}\gamma p^*_{2,j}(s) = a^{j-1}\lambda_1 p^*_{1,j}(s), \\ sp^*_{3,j}(s,y) + \frac{\partial}{\partial y}p^*_{3,j}(s,y) + \mu(y)p^*_{3,j}(s,y) = 0, \\ \vdots \end{cases}$$

Solving the above equations, we derive

$$\begin{cases} p^*_{0,1}(s) = \frac{s+\varepsilon_1+\lambda_1}{(s+\varepsilon_1+\lambda_1)(s+\lambda_0)+(s+\lambda_1)\varepsilon_0}, \\ p^*_{1,1}(s) = \frac{\varepsilon_0}{s+\varepsilon_1+\lambda_1}p^*_{0,1}(s), \\ p^*_{2,1}(s) = \frac{\lambda_1}{s+\gamma}\frac{\varepsilon_0}{s+\varepsilon_1+\lambda_1}p^*_{0,1}(s), \\ p^*_{3,1}(s,y) = (\lambda_0 p^*_{0,1}(s) + \gamma p^*_{2,1}(s))e^{-\int_0^y (s+\mu(\tau))d\tau} \\ = (\lambda_0 + \gamma\frac{\lambda_1}{s+\gamma}\frac{\varepsilon_0}{s+\varepsilon_1+\lambda_1})p^*_{0,1}(s)e^{-\int_0^y (s+\mu(\tau))d\tau}, \\ p^*_{0,2}(s) = \frac{s+\varepsilon_1+a\lambda_1}{(s+\varepsilon_1+a\lambda_1)(s+c\lambda_0)+(s+a\lambda_1)\varepsilon_0} \\ \quad \times(\lambda_0 + \gamma\frac{\lambda_1}{s+\gamma}\frac{\varepsilon_0}{s+\varepsilon_1+\lambda_1})p^*_{0,1}(s)(1 - sG^*(s)), \\ p^*_{1,2}(s) = \frac{\varepsilon_0}{s+\varepsilon_1+a\lambda_1}p^*_{0,2}(s), \\ p^*_{2,2}(s) = \frac{a\lambda_1}{s+b\gamma}\frac{\varepsilon_0}{s+\varepsilon_1+a\lambda_1}p^*_{0,2}(s), \\ p^*_{3,2}(s,y) \\ = (c\lambda_0 p^*_{0,2}(s) + b\gamma p^*_{2,2}(s))e^{-\int_0^y (s+\mu(\tau))d\tau} \\ = (c\lambda_0 + b\gamma\frac{a\lambda_1}{s+b\gamma}\frac{\varepsilon_0}{s+\varepsilon_1+a\lambda_1})p^*_{0,2}(s)e^{-\int_0^y (s+\mu(\tau))d\tau}, \\ \vdots \\ p^*_{0,j}(s) = \frac{s+\varepsilon_1+a^{j-1}\lambda_1}{(s+\varepsilon_1+a^{j-1}\lambda_1)(s+c^{j-1}\lambda_0)+(s+a^{j-1}\lambda_1)\varepsilon_0} \\ \quad \times(c^{j-2}\lambda_0 + b^{j-2}\gamma\frac{a^{j-2}\lambda_1}{s+b^{j-2}\gamma}\frac{\varepsilon_0}{s+\varepsilon_1+a^{j-2}\lambda_1}) \\ \quad \times p^*_{0,j-1}(s)(1 - sG^*(s))^{j-1}, \\ p^*_{1,j}(s) = \frac{\varepsilon_0}{s+\varepsilon_1+a^{j-1}\lambda_1}p^*_{0,j}(s), \\ p^*_{2,j}(s) = \frac{a^{j-1}\lambda_1}{s+b^{j-1}\gamma}\frac{\varepsilon_0}{s+\varepsilon_1+a^{j-1}\lambda_1}p^*_{0,j}(s), \\ p^*_{3,j}(s,y) \\ = (c^{j-1}\lambda_0 p^*_{0,j}(s) + b^{j-1}\gamma p^*_{2,j}(s))e^{-\int_0^y (s+\mu(\tau))d\tau} \\ = (c^{j-1}\lambda_0 + b^{j-1}\gamma\frac{a^{j-1}\lambda_1}{s+b^{j-1}\gamma}\frac{\varepsilon_0}{s+\varepsilon_1+a^{j-1}\lambda_1}) \\ \quad \times p^*_{0,j}(s)e^{-\int_0^y (s+\mu(\tau))d\tau}, \\ \vdots \end{cases} \tag{9}$$

where $G^*(s) = \int_0^\infty e^{-\int_0^y (s+\mu(\tau))d\tau} dy$. Here we point out that all functions $p_{i,j}(s) \geq 0. i = 0, 1, 2, 3, j \in \mathbb{N}$.

4.1 Availability of the system

Availability of the system at time t is the probability of the system in working state, which is defined by

$$A(t) = A_0(t) + A_1(t),$$

where

$$A_0(t) = P\{N(t) = 0\} = P_0(t) = \sum_{j=1}^{\infty} p_{0,j}(t)$$

and

$$A_1(t) = P\{N(t) = 1\} = P_1(t) = \sum_{j=1}^{\infty} p_{1,j}(t).$$

The Laplace transform of $A_0(t)$ is given by

$$A_0^*(s) = \sum_{j=1}^{\infty} p_{0,j}^*(s) = p_{0,1}^*(s) + \sum_{j=2}^{\infty} p_{0,j}^*(s)$$

Substituting the expression of the Laplace transform of the solution, we get an explicit expression of $A^*(s)$.

In order to calculate the steady state availability of $A(t)$, let us at first estimate $p_{0,j}^*(s)$. Observe that for $j \geq 2$,

$$\begin{aligned} &\frac{p_{0,j}^*(s)}{p_{0,j-1}^*(s)} \\ = &\frac{s+\varepsilon_1+a^{j-1}\lambda_1}{(s+\varepsilon_1+a^{j-1}\lambda_1)(s+c^{j-1}\lambda_0)+(s+a^{j-1}\lambda_1)\varepsilon_0} \\ &(c^{j-2}\lambda_0+\frac{\gamma}{s+b^{j-2}\gamma}\frac{a^{j-2}\lambda_1\varepsilon_0}{s+\varepsilon_1+a^{j-2}\lambda_1})(1-sG^*(s))^{j-1} \end{aligned}$$

and hence

$$\begin{aligned} &\frac{p_{0,j}^*(s)}{p_{0,1}^*(s)} \\ = &\prod_{k=1}^{j-1}\frac{s+\varepsilon_1+a^{k}\lambda_1}{(s+\varepsilon_1+a^{k}\lambda_1)(s+c^{k}\lambda_0)+(s+a^{k}\lambda_1)\varepsilon_0} \\ &(c^{k-1}\lambda_0+\frac{\gamma}{s+b^{k-1}\gamma}\frac{a^{k-1}\lambda_1\varepsilon_0}{s+\varepsilon_1+a^{k-1}\lambda_1})(1-sG^*(s))^{k}. \end{aligned}$$

Using inequality

$$0 < 1 - sG^*(s) = 1 - s\int_0^\infty e^{-\int_0^r (s+\mu(y))dy} dr < 1$$

and the conditions $a > 1$, $b \geq 1$ and $c > 1$ we have estimate

$$\begin{aligned} &\frac{p_{0,j}^*(s)}{p_{0,1}^*(s)} \\ < &\prod_{k=1}^{j-1}\frac{s+\varepsilon_1+a^{k}\lambda_1}{(s+\varepsilon_1+a^{k}\lambda_1)(s+c^{k}\lambda_0)+(s+a^{k}\lambda_1)\varepsilon_0} \\ &\times\frac{c^{k-1}\lambda_0(s+\varepsilon_1+a^{k-1}\lambda_1)+a^{k-1}\lambda_1\varepsilon_0}{s+\varepsilon_1+a^{k-1}\lambda_1} \\ \leq &\frac{1}{(c^{j-1}\lambda_0)}(\lambda_0+\varepsilon_0). \end{aligned}$$

Then

$$\sum_{j=2}^{\infty} p_{0,j}^*(s) \leq \sum_{j=2}^{\infty} \frac{1}{c^{j-1}\lambda_0}(\lambda_0+\varepsilon_0)p_{0,1}^*(s). \quad (10)$$

From (9) we have

$$p_{0,1}^*(s) = \frac{s+\varepsilon_1+\lambda_1}{(s+\varepsilon_1+\lambda_1)(s+\lambda_0)+(s+\lambda_1)\varepsilon_0}.$$

Using above and (10), for $s > 0, a > 1, c > 1$ and $b \geq 1$, the Laplace transform of $A_0(t)$ has estimate

$$\begin{aligned} A_0^*(s) &= p_{0,1}^*(s) + \sum_{j=2}^{\infty} p_{0,j}^*(s) \\ &\leq p_{0,1}^*(s)\left\{1+\frac{\lambda_0+\varepsilon_0}{(c-1)\lambda_0}\right\} \end{aligned}$$

These yields

$$\lim_{s\to 0} sA_0^*(s) = 0.$$

For $i = 1$, from (9) we have

$$p_{1,1}^*(s) = \frac{\varepsilon_0}{s+\varepsilon_1+\lambda_1}p_{0,1}^*(s),$$

$$p_{1,j}^*(s) = \frac{\varepsilon_0}{s+\varepsilon_1+a^{j-1}\lambda_1}p_{0,j}^*(s).$$

Using previous estimate yields

$$p_{1,j}^*(s) \leq \frac{\varepsilon_0}{s+\varepsilon_1+a^{j-1}\lambda_1}\frac{\lambda_0+\varepsilon_0}{c^{j-1}\lambda_0}p_{0,1}^*(s).$$

Therefore, the Laplace transform of $A_1(t)$ has estimate

$$\begin{aligned} A_1^*(s) &= p_{1,1}^*(s) + \sum_{j=2}^{\infty} p_{1,j}^*(s) \\ &= \frac{\varepsilon_0}{s+\varepsilon_1+\lambda_1}p_{0,1}^*(s) + \sum_{j=2}^{\infty}\frac{\varepsilon_0 p_{0,j}^*(s)}{s+\varepsilon_1+a^{j-1}\lambda_1} \\ &\leq \left(\frac{\varepsilon_0}{\lambda_1}+\sum_{j=2}^{\infty}\frac{\varepsilon_0}{a^{j-1}\lambda_1}\frac{(\lambda_0+\varepsilon_0)}{c^{j-1}\lambda_0}\right)p_{0,1}^*(s) \end{aligned}$$

this implies

$$\lim_{s\to 0} sA_1^*(s) = 0.$$

According to the Tauberian theorem (see, [15]), the availability of the steady state of the system is given by

$$A = \lim_{t\to\infty} A(t) = \lim_{s\to 0} sA^*(s) = 0. \qquad (11)$$

This shows that when $a > 1, c > 1$ and $b \geq 1$, the availability of the system will tend to zero after a long time running, which means that the deteriorating system will become completely unavailable. This result is consistent with the practical situations. This is because the system after repair is not "as good as new".

4.2 Working probability of the repairman

The probability of the repairman working at time t is $P_W(t) = P_3(t)$. Here we are mainly interested in the steady state probability of the repairman, which is the asymptotic behavior of $P_W(t)$

$$\lim_{t\to\infty} P_W(t) = P_W.$$

Although we have shown that if the steady state of the system exists and it holds that $P_W = \frac{m_d+m_{f_0}}{\mu}$, the terms m_d and m_{f_0} are unknown.

To determine the exact value of P_W, firstly we calculate the waiting repaired probability of the deteriorating system

$$P_2(t) = P\{N(t) = 2\} = \sum_{j=1}^{\infty} p_{2,j}(t).$$

According to (9) we have

$$p_{2,1}^*(s) = \frac{\lambda_1}{s+\gamma}\frac{\varepsilon_0}{s+\varepsilon_1+\lambda_1}p_{0,1}^*(s),$$

$$p_{2,j}^*(s) = \frac{a^{j-1}\lambda_1}{s+b^{j-1}\gamma}\frac{\varepsilon_0}{s+\varepsilon_1+a^{j-1}\lambda_1}p_{0,j}^*(s).$$

Thus, for $s > 0, a > 1, c > 1$ and $b \geq 1$, the Laplace transform of $P_2(t)$ has estimate

$$\begin{aligned} P_2^*(s) &= p_{2,1}^*(s) + \sum_{j=2}^{\infty} p_{2,j}^*(s) \\ &= \frac{\lambda_1}{s+\gamma}\frac{\varepsilon_0}{s+\varepsilon_1+\lambda_1}p_{0,1}^*(s) \\ &\quad + \sum_{j=2}^{\infty}\frac{a^{j-1}\lambda_1}{s+b^{j-1}\gamma}\frac{\varepsilon_0}{s+\varepsilon_1+a^{j-1}\lambda_1}p_{0,j}^*(s) \\ &\leq \frac{\lambda_1}{\gamma}\frac{\varepsilon_0}{\varepsilon_1+\lambda_1}p_{0,1}^*(s) + \frac{\varepsilon_0}{\gamma}\sum_{j=2}^{\infty}\frac{1}{b^{j-1}}p_{0,j}^*(s) \\ &\leq \left\{\frac{\lambda_1}{\gamma}\frac{\varepsilon_0}{\varepsilon_1+\lambda_1} + \frac{\varepsilon_0}{\gamma}\sum_{j=2}^{\infty}\frac{1}{b^{j-1}}\frac{(\lambda_0+\varepsilon_0)}{c^{j-1}\lambda_0}\right\}p_{0,1}^*(s) \\ &= \left\{\frac{\lambda_1\varepsilon_0}{\gamma(\varepsilon_1+\lambda_1)} + \frac{\varepsilon_0(\lambda_0+\varepsilon_0)}{\gamma\lambda_0(bc-1)}\right\}p_{0,1}^*(s). \end{aligned}$$

So we also have

$$P_2 = \lim_{t\to\infty} P_2(t) = \lim_{s\to 0} sP_2^*(s) = 0.$$

This means that when $a > 1, c > 1$ and $b \geq 1$, the probability of the system waiting for repair will tend to zero after a long time running. This is because the system after repair is not "as good as new", the frequency of system failures becomes higher, and the failed system will be repaired at once due to a delayed vacation.

According to the normal condition (4) we have

$$P_0(t) + P_1(t) + P_2(t) + P_3(t) = 1$$

and $A(t) = P_0(t) + P_1(t)$. Therefore,

$$\begin{aligned} P_W &= \lim_{t\to\infty} P_W(t) \\ &= \lim_{t\to\infty}[1 - A(t) - P_2(t)] = 1. \end{aligned}$$

The probability of the repairman working at t tend to 1 after a long time running, it is because the system after repair is not "as good as new".

Further we consider more cases in detail. We split $P_W(t)$ into two parts: $P_{W_0}(t)$ and $P_{W_2}(t)$, i.e.,

$$P_W(t) = P_{W_0}(t) + P_{W_2}(t)$$

where $P_{W_0}(t)$ is the probability of the repairman working at t resulted in the system fails when the repairman is preparing vacation; $P_{W_2}(t)$ is the probability of the repairman working at t after his vacation.

Since the probability of the repairman working at time t is

$$P_W(t) = P_3(t) = \sum_{j=1}^{\infty}\int_0^{\infty} p_{3,j}(t,y)dy$$

Its Laplace transform is given by

$$P_W^*(s) = \left\{\lambda_0 p_{0,1}^*(s) + \gamma p_{2,1}^*(s)\right\} G^*(s) + \sum_{j=2}^{\infty}\left\{c^{j-1}\lambda_0 p_{0,j}^*(s) + b^{j-1}\gamma p_{2,j}^*(s)\right\} G^*(s)$$

where we have used the boundary conditions in (2), so the Laplace transform of $P_{W_0}(t)$ and $P_{W_2}(t)$ are respectively

$$P_{W_0}^*(s) = \lambda_0 \sum_{j=1}^{\infty} c^{j-1} p_{0,j}^*(s) G^*(s)$$

and

$$P_{W_2}^*(s) = \gamma \sum_{j=1}^{\infty} b^{j-1} p_{2,j}^*(s) G^*(s).$$

Using the expression of $p_{2,j}^*(s)$ in (9) and the estimates for $p_{0,j}^*(s)$, we get the following estimate about $P_{W_2}^*(s)$:

$$\begin{aligned} & P_{W_2}^*(s) \\ = \ & \gamma \frac{\lambda_1}{s+\gamma} \frac{\varepsilon_0}{s+\varepsilon_1+\lambda_1} p_{0,1}^*(s) G^*(s) \\ + \ & \sum_{j=2}^{\infty} b^{j-1}\gamma \frac{a^{j-1}\lambda_1}{s+b^{j-1}\gamma} \frac{\varepsilon_0 p_{0,j}^*(s)}{s+\varepsilon_1+a^{j-1}\lambda_1} G^*(s) \\ \leq \ & \varepsilon_0 \left\{ p_{0,1}^*(s) + \sum_{j=2}^{\infty} p_{0,j}^*(s) \right\} G^*(s) \\ \leq \ & \frac{\varepsilon_0}{\lambda_0} \left\{ 1 + \frac{1}{\lambda_0}(\lambda_0+\varepsilon_0)\frac{1}{c-1} \right\} G^*(s). \end{aligned}$$

Therefore, we have

$$\lim_{s\to 0} s P_{W_2}^*(s) = 0.$$

Note that $P_W^*(s) = P_{W_0}^*(s) + P_{W_2}^*(s)$ and

$$\lim_{t\to\infty} P_{W_2}(t) = \lim_{s\to 0} s P_{W_2}^*(s) = 0.$$

So we have

$$\lim_{t\to\infty} P_{W_0}(t) = \lim_{s\to 0} s P_{W_0}^*(s) = 1.$$

The calculation above shows that the probability of the repairman working at t after his vacation will tend to zero and the working probability of the repairman during preparing for his vacation will tend to 1 when the system has a long time running. This means that when the system has run a long period, the repairman will have no time for his vacation. The failed system will be repaired at once. Such a result is also consistent with the practical situations.

4.3 Delayed rate and rate of occurrence of failures

In this paper we introduced the strategy of delayed vacation. However, when the repairman is on his vacation and the system fails, the repair of the system will be delayed. We need to calculate the delayed rate of the system at time t, which is defined by

$$m_d(t) = \sum_{j=1}^{\infty} b^{j-1}\gamma p_{2,j}(t).$$

At the same time, we also calculate rate of occurrence of failures, $m_f(t)$, the rate of occurrence of failures at time t.

Here we also split $m_f(t)$ into two parts $m_{f_0}(t)$ and $m_{f_1}(t)$, i.e.,

$$m_f(t) = m_{f_0}(t) + m_{f_1}(t)$$

where $m_{f_0}(t)$ is called the rate of occurrence of failures at time t when the repairman is in system; $m_{f_1}(t)$ is called the rate of occurrence of failures at time t when the repairman is on vacation.

4.3.1 Delayed rate

The Laplace transform of delayed rate is

$$m_d^*(s) = \gamma p_{2,1}^*(s) + \sum_{j=2}^{\infty} b^{j-1}\gamma p_{2,j}^*(s).$$

We have proved that

$$P_{W_2}^*(s) = \gamma p_{2,1}^*(s) G^*(s) + \sum_{j=2}^{\infty} b^{j-1}\gamma p_{2,j}^*(s) G^*(s).$$

Obviously, $G^*(s) m_d(s) = P_{W_2}^*(s)$. By expected value

$$\int_0^\infty t\mu(t) e^{-\int_0^t \mu(y)dy} dt = \frac{1}{\mu}$$

and the equality

$$\begin{aligned} \int_0^\infty t\mu(t) e^{-\int_0^t \mu(y)dy} dt &= -\int_0^\infty t de^{-\int_0^t \mu(y)} dt \\ = \int_0^\infty e^{-\int_0^t \mu(y)dy} dt &= \frac{1}{\mu} \end{aligned}$$

and

$$G^*(s) = \int_0^\infty e^{-\int_0^t (s+\mu(y))dy} dt,$$

it has $G^*(0) = \frac{1}{\mu}$. Therefore, it holds that

$$m_d = \lim_{s\to 0} s m_d^*(s) = \frac{P_{W_2}}{G^*(0)} = 0. \tag{12}$$

This calculation shows that for the deteriorating system, after a long time running, the normal working time of the system might be less than the time of repairman preparing for his vacation. The failed system will be repaired at once. The delayed rate tends to be zero means that the delayed repair of the system does not occur.

4.3.2 Rate of occurrence of failures

According to the calculation method of the rate of occurrence of failures (see, [17]), $m_{f_0}(t)$ and $m_{f_1}(t)$ are of the form

$$\begin{aligned} m_{f_0}(t) &= \sum_{j=1}^{\infty} c^{j-1}\lambda_0 p_{0,j}(t), \\ m_{f_1}(t) &= \sum_{j=1}^{\infty} a^{j-1}\lambda_1 p_{1,j}(t). \end{aligned} \quad (13)$$

The Laplace transform of $m_{f_0}(t)$ is

$$m^*_{f_0}(s) = \sum_{j=1}^{\infty} c^{j-1}\lambda_0 p^*_{0,j}(s)$$

$$= \lambda_0 p^*_{0,1}(s) + \sum_{j=2}^{\infty} c^{j-1}\lambda_0 p^*_{0,j}(s).$$

Since

$$P^*_{W_0}(s) = \sum_{j=1}^{\infty} c^{j-1}\lambda_0 p^*_{0,j}(s) G^*(s) = m^*_{f_0}(s) G^*(s),$$

so we have

$$m_{f_0} = \lim_{s\to 0} s m^*_{f_0}(s) = \frac{P_{W_0}}{G^*(0)} = \mu.$$

While the Laplace transform of $m_{f_1}(t)$ is

$$m^*_{f_1}(s) = \sum_{j=1}^{\infty} a^{j-1}\lambda_1 p^*_{1,j}(s),$$

from (9) we have

$$p^*_{1,1}(s) = \frac{\varepsilon_0}{s+\varepsilon_1+\lambda_1} p^*_{0,1}(s)$$

and

$$p^*_{1,j}(s) = \frac{\varepsilon_0}{s+\varepsilon_1+a^{j-1}\lambda_1} p^*_{0,j}(s).$$

Thus for $s > 0, a > 1, c > 1$ and $b \geq 1$ we have estimate

$$\begin{aligned} m^*_{f_1}(s) &= \sum_{j=1}^{\infty} a^{j-1}\lambda_1 p^*_{1,j}(s) \\ &= \lambda_1 p^*_{1,1}(s) + \sum_{j=2}^{\infty} a^{j-1}\lambda_1 p^*_{1,j}(s) \\ &= \frac{\lambda_1\varepsilon_0}{s+\varepsilon_1+\lambda_1} p^*_{0,1}(s) \\ &\quad + \sum_{j=2}^{\infty} \frac{a^{j-1}\lambda_1\varepsilon_0}{s+\varepsilon_1+a^{j-1}\lambda_1} p^*_{0,j}(s) \\ &\leq \varepsilon_0 \left\{ p^*_{0,1}(s) + \sum_{j=2}^{\infty} p^*_{0,j}(s) \right\} \\ &= \varepsilon_0 A^*_0(s) \end{aligned}$$

and hence $m_{f_1} = \lim_{s\to 0} s m^*_{f_1}(s) = 0$. So it holds that

$$m_f = m_{f_0} + m_{f_1} = \mu.$$

The above calculation again verifies the equality

$$P_W = \frac{m_d + m_{f_0}}{\mu} = 1.$$

The result of calculation above shows that after the system has a long working time, the rate of occurrence of failures m_{f_0} as the repairman is in system will be constant; the rate of occurrence of failures when the repairman is on vacation will be 0. The system is no longer valid, the failure probability m_f of the system at time t might be nonzero constant. The repairman will be kept in the system to repair.

The result of our calculation is different from that in [13], where the author asserted that the rate of occurrence of failures is constant zero with unproved. Here we shows that $m_f = \mu$. This is right and consistent with the practical situations, because the failed system must be repaired at once.

5 Conclusion

In the present paper we studied reliability of a class of deteriorating system under the strategy of delayed vacation. The reliability problem of the deteriorating system becomes so important in engineering. This is because the electronic products are extensively used. This problem has been studied from different aspects, for examples, a priority of repair policy used [18] according to the importance of different components in the deteriorating system, deteriorating of standby system considered in [19]. The system modeling of the repairman with vacation strategy is a new content, which usually adopts the vacation strategy as the system starts to work. As the system's failure is random, it is probable the system fails in a short time. In order to solve this question, we proposed the strategy of delayed vacation. The purpose is to improve reliability of the system.

Based on the strategy of delayed vacation, we deduced the mathematical model for such a deteriorating system under the general distribution obeyed by repair time. Using the supplementary variable method, we established partial differential equations model in cycles and state model. In addition, we studied reliability indices of the system, such as availability of the system, failure rate as well as the probability of repair working. The results show that

1) The strategy of delayed vacation can enhance the reliability of the system, but it cannot change the nature of the deteriorating system;

2) When the time goes to sufficient long, the availability of the system becomes very small; the working probability of the repairman will tend to be 1. This is just a character of the deteriorating system.

3) The working probability of the repairman in preparing for vacation will tend to be 1, and the working probability of the repairman after his vacation will tend to be 0.

4) The rate of occurrence of failure of the system is a nonzero constant, which corrected a result in previous work [13].

It is well known that for a deteriorating system, the more the system is repaired, the more quickly system fails. So, with time development, the system will be invalid state and the repairman will be in working state. From engineering point of view, it will lead to much higher costs in system maintenance [20]. Therefore, a renew policy or replacement policy is necessary in practice. We will study this question in the further work.

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