New Method for Designing 2-D (Two-Dimensional) IIR Comb Filters

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Abstract: - An elegant method for designing 2-D (Two-Dimensional) Comb Filters is provided. The method is based on an appropriate variable transformation. Practical examples illustrate the validity and the efficiency of the method are also included in this brief.

Key-Words: - Comb Filters, 2-D Comb Filters, Multidimensional Systems, Multidimensional Filters, Filter Design

1 Introduction

Comb filter is the filter that delays a signal and adds it to itself, causing signal assembly and disassembly interference. Comb Filters have frequency response of a series of regularly spaced spikes (like a "Comb"). 2-D and 3-D comb filters have been introduced recently in PAL and NTSC television decoders. 2-D and 3-D Filters Comb Filters are used to separate the encoded signal of the color from the luminance signal when receiving an analog video signal. A Comb Filter enhances, cleans and clarifies the image colors [1]. In [2], the authors presented a framework for temporal analysis of left ventricular (LV) endocardial wall motion they use 2-D recursive comb filtering. In [3] the author presents a special category of 2-D comb filters with fourfold symmetry. In [4], the authors present a design procedure of comb-line bandpass filter with asymmetrical coupled-lines, which has small size in comparison with other planar type filters to take full advantage of reducing the size of RF and microwave components. In [5], the authors propose a flexible hardware-friendly architecture to perform 2-D upscaling and downscaling.

All the previous 2-D designs can be considered as special cases of the new proposed 2-D comb filters while until now there does not exist any systematic way of 2-D comb filter design. The present brief try to cover this blank in the technical literature.

So, in this brief, a new transformation is introduced for the 2-D and 3-D comb filters design. This paper is organized as follows: Section II presents First-Order 2-D IIR Comb Filters design together with a numerical example. In Section III, the design of a family of Second-Order 2-D IIR Comb Filters is presented. Our conclusion and some remarks for multiple Comb frequencies can be found in Section IV.

2 The Proposed Method for First-Order IIR 2-D Comb Filters

Consider the 1-D transfer function

$$H(z^{-1}) = \frac{1 - \rho z^{-1}}{1 - r z^{-1}} \tag{1}$$

with $z^{-1} = e^{j\omega T}$ $-\pi \le \omega \le \pi$, *T* is the Sampling Period, and $0 << \rho < r < 1$



It is apparent that for $\omega \approx 0$ $H(\omega) \approx \frac{1-\rho}{1-r}$ while for all the other frequencies $|H(\omega)| \approx 1$. So, this filter is an all-pass filter that offers a remarkable amplification the for $\omega = 0$ (e.g. DC frequency). The magnitude response is illustrated in Fig.1.a in the case of $\rho = 0.899, r = 0.900$ (T = 1 without loss of generality).



Fig. 1.b Group Delay for the particular 1-D Comb Filter

The Group Delay $\tau = -\frac{\partial ArgH(j\omega)}{\partial \omega}$ can be found equal to $\tau = \frac{\rho(\cos \omega - \rho)}{1 + \rho^2 - 2\rho \cos \omega} - \frac{r(\cos \omega - r)}{1 + r^2 - 2r \cos \omega}$ is depicted in Fig. 1.b and shows almost linear behavior in a big part of the frequency domain. The Comb filter of (1) is presented in [1]. All the previous 2-D designs are special cases of 2-D comb filters while there does not exist any systematic way of 2-D comb filter design. In this section, we extend it to 2-D case as follows:

For the first-order Comb filter of (1) conside the transformation

$$z^{-1} = \frac{z_1^{-1} + z_2^{-1}}{2}$$
(2)

we take

$$H(z^{-1}) = K \frac{2 - \rho(z_1^{-1} + z_2^{-1})}{2 - r(z_1^{-1} + z_2^{-1})}$$

with
$$z_1^{-1} = e^{j\omega_1 T_1}$$
 $-\pi \le \omega_1 \le \pi$, $z_2^{-1} = e^{j\omega_2 T_2}$,
 $-\pi \le \omega_2 \le \pi$

 T_1, T_2 are the sampling periods to horizontal and vertical direction whereas: $0 << \rho < r < 1$



Transformation (2) can be easily extended to a family of transformations:

 $z^{-1} = \frac{\lambda_1 z_1^{-1} + \lambda_2 z_2^{-1}}{\lambda_1 + \lambda_2} \quad \text{with} \quad \lambda_1, \lambda_2 \quad \text{real numbers or}$

simply

$$z^{-1} = \lambda z_1^{-1} + (1 - \lambda) z_2^{-1} \text{ with } 0 < \lambda < 1$$
one obtains

$$H(z_1^{-1}, z_2^{-1}) = \frac{1 - \rho(\lambda z_1^{-1} + (1 - \lambda) z_2^{-1})}{1 - r(\lambda z_1^{-1} + (1 - \lambda) z_2^{-1})}$$
(3)

with
$$z_1^{-1} = e^{j\omega_1 T_1}$$
 $-\pi \le \omega_1 \le \pi$, $z_2^{-1} = e^{j\omega_2 T_2}$,
 $-\pi \le \omega_2 \le \pi$

$$(0 << \rho < r < 1)$$

The stability of the final 2-D filter can be proved using the following Theorem. The Theorem claims that if our 1-D prototype filter is Stable (like in (1)) then the transformation $z^{-1} = C_1 z_1^{-1} + C_2 z_2^{-1}$ with $C_1 + C_2 = 1$ and $C_1 C_2 > 0$ yields a 2-D stable filter.

Theorem. Consider a prototype 1-D BIBO (Bounded Inputer Bounded Output) stable filter with transfer function

$$H(z^{-1}) = \frac{A(z^{-1})}{B(z^{-1})}$$

Under the transformation $z^{-1} = C_1 z_1^{-1} + C_2 z_2^{-1}$ with $C_1 + C_2 = 1$ and $C_1 C_2 > 0$, the prototype 1-D BIBO of (1) gives the 2-D filter

$$H_{2}\left(z_{1}^{-1}, z_{2}^{-1}\right) = \frac{A_{1}\left(z_{1}^{-1}, z_{2}^{-1}\right)}{B_{2}\left(z_{1}^{-1}, z_{2}^{-1}\right)}$$

which is also stable in BIBO sense.

Proof. We have to prove that $B_2(z_1^{-1}, z_2^{-1}) \neq 0$ for every z_1^{-1} and z_2^{-1} inside the unit bi-disk, i.e. for every z_1^{-1} and z_2^{-1} with $|z_1^{-1}| < 1$ and $|z_2^{-1}| < 1$. Assume first that there are some ζ_1^{-1} and ζ_2^{-1} with $|\zeta_1^{-1}| < 1$ and $|\zeta_2^{-1}| < 1$ such that $B_2(\zeta_1^{-1}, \zeta_2^{-1}) = 0$. However, in this case, we have a ζ^{-1} where $\zeta^{-1} = C_1\zeta_1^{-1} + C_2\zeta_2^{-1}$ such that $B(\zeta^{-1}) = 0$, on the other hand, since $|\zeta_1^{-1}| < 1$ and $|\zeta_2^{-1}| < 1$, we have $|\zeta_1^{-1}| = |C_1\zeta_1^{-1} + C_2\zeta_2^{-1}| \le |C_1||\zeta_1^{-1}| + |C_2||\zeta_2^{-1}| < |C_1| + |C_2| =$ $= |C_1 + C_2| = 1$ (since $C_1C_2 > 0$), that makes our 1-D filter with transfer function $H(z^{-1}) = \frac{A(z^{-1})}{B(z^{-1})}$ non-stable (in BIBO sense), but this contradicts the assumption and this completes the Proof.

Numerical Example 1:

Consider without loss of generality T_1, T_2 equal to 1.

Then, for r = 0.9 and $\lambda = \frac{1}{2}$ in (3)

the magnitude response is depicted in Fig.2.a, while the Group Delays

$$\tau_{1} = -\frac{\partial ArgH(j\omega_{1}, j\omega_{2})}{\partial \omega_{1}}, \ \tau_{1} = -\frac{\partial ArgH(j\omega_{1}, j\omega_{2})}{\partial \omega_{2}}$$

are depicted in Fig.2.b and Fig.2.c.



Fig.2.b 1st Group Delay Response for the 2-D Comb Filter



Fig.2.c 2nd Group Delay Response for the 2-D Comb Filter

It is apparent that the family of the filters of (3) offers a remarkable amplification for the 2-D frequency. If amplification of another 2-D frequency $(\omega_1, \omega_2) = (\omega_{10}, \omega_{20}) \neq (0, 0)$ is necessary, a second-order 2-D IIR Comb filter must be used.

3 The Proposed Method For Second -Order IIR 2-D Comb Filters

In this session, we extend (3) as follows in order to create a filter for remarkable amplification for the 2-D frequency $(\omega_1, \omega_2) = (\omega_{10}, \omega_{20})$

Consider first

$$H\left(z_{1}^{-1}, z_{2}^{-1}\right) = \frac{\lambda e^{j\omega_{10}T_{1}} + (1-\lambda)e^{j\omega_{20}T_{2}} - \rho(\lambda z_{1}^{-1} + (1-\lambda)z_{2}^{-1})}{\lambda e^{j\omega_{10}T_{1}} + (1-\lambda)e^{j\omega_{20}T_{2}} - r(\lambda z_{1}^{-1} + (1-\lambda)z_{2}^{-1})} \cdot \frac{\lambda e^{-j\omega_{10}T_{1}} + (1-\lambda)e^{-j\omega_{20}T_{2}} - \rho(\lambda z_{1}^{-1} + (1-\lambda)z_{2}^{-1})}{\lambda e^{-j\omega_{10}T_{1}} + (1-\lambda)e^{-j\omega_{20}T_{2}} - r(\lambda z_{1}^{-1} + (1-\lambda)z_{2}^{-1})}$$

(4)

with $0 < \lambda < 1$, $0 < \rho < r < 1$

or

$$H\left(z_{1}^{-1}, z_{2}^{-1}\right) = K \frac{A\left(z_{1}^{-1}, z_{2}^{-1}\right)}{B\left(z_{1}^{-1}, z_{2}^{-1}\right)}$$

where $A(z_1^{-1}, z_2^{-1}) = \lambda^2 + (1 - \lambda)^2 + 2\lambda(1 - \lambda)\cos(\omega_{10}T_1 - \omega_{20}T_2)$ $-2r(\lambda z_1^{-1} + (1 - \lambda)z_2^{-1})(\lambda\cos\omega_{10}T_1 + (1 - \lambda)\cos\omega_{20}T_2) + \rho^2(\lambda z_1^{-1} + (1 - \lambda)z_2^{-1})^2$ and $B(z_1^{-1}, z_2^{-1}) = \lambda^2 + (1 - \lambda)^2 + 2\lambda(1 - \lambda)\cos(\omega_{10}T_1 - \omega_{20}T_2)$ $-2r(\lambda z_1^{-1} + (1 - \lambda)z_2^{-1})(\lambda\cos\omega_{10}T_1 + (1 - \lambda)\cos\omega_{20}T_2) + r^2(\lambda z_1^{-1} + (1 - \lambda)z_2^{-1})^2$

First one has to examine for what (z_1^{-1}, z_2^{-1}) we have

the amplification $\frac{1-\rho}{1-r}$.

So,

$$\frac{\rho \lambda e^{j\omega_{1}T_{1}} + \rho(1-\lambda)e^{j\omega_{2}T_{2}} - \lambda e^{j\omega_{10}T_{1}} - (1-\lambda)re^{j\omega_{20}T_{2}}}{r\lambda e^{j\omega_{1}T_{1}} + r(1-\lambda)e^{j\omega_{2}T_{2}} - \lambda e^{j\omega_{10}T_{1}} - (1-\lambda)re^{j\omega_{20}T_{2}}} = \frac{1-\rho}{1-r}$$
or

$$\frac{\rho\lambda e^{j\omega_{1}T_{1}} + \rho(1-\lambda)e^{j\omega_{2}T_{2}} - \lambda e^{-j\omega_{10}T_{1}} - (1-\lambda)e^{-j\omega_{20}T_{2}}}{r\lambda e^{j\omega_{1}T_{1}} + r(1-\lambda)e^{j\omega_{2}T_{2}} - \lambda e^{-j\omega_{10}T_{1}} - (1-\lambda)e^{-j\omega_{20}T_{2}}} = \frac{1-\rho}{1-r}$$

Using now:
$$c = \frac{1-\lambda}{\lambda}$$

$$\frac{\rho e^{j\omega_{1}T_{1}} + \rho c e^{j\omega_{2}T_{2}} - e^{j\omega_{10}T_{1}} - c e^{j\omega_{20}T_{2}}}{r e^{j\omega_{1}T_{1}} + r c e^{j\omega_{2}T_{2}} - e^{j\omega_{10}T_{1}} - c e^{j\omega_{20}T_{2}}} = \frac{1 - \rho}{1 - r}$$
(5.1)

$$\frac{\rho e^{j\omega_1 T_1} + \rho c e^{j\omega_2 T_2} - e^{-j\omega_1 \sigma_1} - c e^{-j\omega_2 \sigma_2}}{r e^{j\omega_1 T_1} + r c e^{j\omega_2 T_2} - e^{-j\omega_1 \sigma_1} - c e^{-j\omega_2 \sigma_2}} = \frac{1 - \rho}{1 - r}$$
(5.2)

Two cases exist:

a)
$$c = 1$$
, that means $\lambda = \frac{1}{2}$ and
b) $c \neq 1$, that means $\lambda \neq \frac{1}{2}$

a) The first case yields the two equations:

$$\frac{e^{j\omega_{0}T_{1}} + e^{j\omega_{2}T_{2}} - \rho e^{j\omega_{0}T_{1}} - \rho r e^{j\omega_{20}T_{2}}}{e^{j\omega_{0}T_{1}} + e^{j\omega_{2}T_{2}} - r e^{j\omega_{10}T_{1}} - r e^{j\omega_{20}T_{2}}} = \frac{1-\rho}{1-r}$$
(6.1)

$$\frac{e^{j\omega_{1}T_{1}} + e^{j\omega_{2}T_{2}} - re^{-j\omega_{10}T_{1}} - \rho re^{-j\omega_{20}T_{2}}}{e^{j\omega_{1}T_{1}} + e^{j\omega_{2}T_{2}} - re^{-j\omega_{10}T_{1}} - re^{-j\omega_{20}T_{2}}} = \frac{1-\rho}{1-r}$$
(6.2)

Using (6.1) one obtains the "comb" frequencies $\omega_1 = \omega_{10}, \omega_2 = \omega_{20}$ and the symmetric solution

$$\omega_1 = \frac{T_2}{T_1} \omega_{20}, \omega_2 = \frac{T_1}{T_2} \omega_{10}$$

while from (6.2) two other couple of Comb frequencies, i.e. $\omega_1 = -\omega_{10}, \omega_2 = -\omega_{20}$

$$\omega_1 = -\frac{T_2}{T_1} \omega_{20}, \omega_2 = -\frac{T_1}{T_2} \omega_{10}$$

are obtained.

b) The second case yields also two equations:

$$\frac{\rho e^{j\omega_{1}T_{1}} + \rho c e^{j\omega_{2}T_{2}} - e^{j\omega_{1}\sigma_{1}} - c e^{j\omega_{2}\sigma_{2}}}{r e^{j\omega_{1}T_{1}} + r c e^{j\omega_{2}T_{2}} - e^{j\omega_{1}\sigma_{1}} - c e^{j\omega_{2}\sigma_{2}}} = \frac{1 - \rho}{1 - r}$$

$$\frac{\rho e^{j\omega_{1}T_{1}} + \rho c e^{j\omega_{2}T_{2}} - e^{-j\omega_{1}\sigma_{1}} - c e^{-j\omega_{2}\sigma_{2}}}{r e^{j\omega_{1}T_{1}} + r c e^{j\omega_{2}T_{2}} - e^{-j\omega_{1}\sigma_{1}} - c e^{-j\omega_{2}\sigma_{2}}} = \frac{1 - \rho}{1 - r}$$

$$(7.1)$$

$$(7.2)$$
with $c \neq 1$.

Hence, from (7.1) we find the comb frequencies $\omega_1 = \omega_{10}, \omega_2 = \omega_{20}$, and from (7.2) the comb frequencies $\omega_1 = -\omega_{10}, \omega_2 = -\omega_{20}$

It is obvious for our 2-D IIR filter that we can use only the case b) since the amplification of the "symmetric frequencies" $(\omega_1 = \frac{T_2}{T_1} \omega_{20}, \omega_2 = \frac{T_1}{T_2} \omega_{10})$ is not required.

Therefore the 2-D IIR Comb Filter in design is given by (4) that can be also rewritten as

$$H(z_1^{-1}, z_2^{-1}) = \frac{A(z_1^{-1}, z_2^{-1})}{B(z_1^{-1}, z_2^{-1})}$$

with

$$A\left(z_{1}^{-1}, z_{2}^{-1}\right) = \lambda^{2} + (1-\lambda)^{2} + 2\lambda(1-\lambda)\cos(\omega_{10}T_{1} - \omega_{20}T_{2})$$
$$-2r(\lambda z_{1}^{-1} + (1-\lambda)z_{2}^{-1})(\lambda\cos(\omega_{10}T_{1}) + (1-\lambda)\cos(\omega_{20}T_{2})) + \rho^{2}(\lambda z_{1}^{-1} + (1-\lambda)z_{2}^{-1})^{2}$$

and

 $B\left(z_{1}^{-1}, z_{2}^{-1}\right) = \lambda^{2} + (1 - \lambda)^{2} + 2\lambda(1 - \lambda)\cos(\omega_{10}T_{1} - \omega_{20}T_{2})$ $-2r(\lambda z_{1}^{-1} + (1 - \lambda)z_{2}^{-1})(\lambda\cos(\omega_{10}T_{1}) + (1 - \lambda)\cos(\omega_{20}T_{2})) + r^{2}(\lambda z_{1}^{-1} + (1 - \lambda)z_{2}^{-1})^{2}$

with $0 \ll \rho \ll r \ll 1$, $0 \ll \lambda \ll 1$ and $\lambda \neq 0.5$

The stability of the final 2-D filter can be proved using the Theorem of Section II.

With the notation $c = \frac{1-\lambda}{\lambda}$ now, a further simplification of the second-order 2-D IIR Comb Filter transfer function can be formulated



Numerical Example 2:

Consider the 2-D IIR Comb Filter of (8). Suppose that we want the amplification of $\omega_{10} = \frac{\pi}{2}, \omega_{20} = \frac{\pi}{4}$ (and of course the symmetric $\omega_{10} = -\frac{\pi}{2}, \omega_{20} = -\frac{\pi}{4}$). Choose for example c = 2, r = 0.9. Having without loss of generality $T_1, T_2 = 1$, one obtains

$$H\left(z_{1}^{-1}, z_{2}^{-1}\right) = \frac{(\rho z_{1}^{-1} + \rho 2 z_{2}^{-1} + 3j)(\rho z_{1}^{-1} + \rho 2 z_{2}^{-1} - 3j)}{(r z_{1}^{-1} + r 2 z_{2}^{-1} + 3j)(r z_{1}^{-1} + r 2 z_{2}^{-1} - 3j)}$$

$$z_{1}^{-1} = e^{j\omega_{1}} -\pi \le \omega_{1} \le \pi , \quad z_{2}^{-1} = e^{j\omega_{2}} , \quad -\pi \le \omega_{2} \le \pi$$

Fig.3.a depicts the magnitude response while Fig.3.b and Fig.3.c depict the Group Delays

$$\tau_1 = -\frac{cArgH(j\omega_1, j\omega_2)}{\partial \omega_1}, \ \tau_1 = -\frac{cArgH(j\omega_1, j\omega_2)}{\partial \omega_2}$$



Fig. 3.a: Magnitude Response 2-D Comb Filter (2nd Order)



Fig.3.b 1st Group Delay Response for the 2-D Comb Filter (2nd order)



Fig.3.c 2^{nd} Group Delay Response for the 2-D Comb Filter (2^{nd} order)

4 Conclusion

A method for designing 2-D comb filters has been presented and illustrated through specific numerical examples. The method is based on the appropriate transformation $z^{-1} = \lambda z_1^{-1} + (1-\lambda) z_2^{-1}$ with $0 < \lambda < 1$. A Theorem regarding the stability of our 2-D Comb Filters is also stated and proven. Numerical examples illustrate the validity and the efficiency of the method. 2-D filters with several Comb frequencies can be easily implemented by cascade design, while by using further transformations like $z_1^{-1} = z_1^{-P_1}$ and $z_2^{-1} = z_2^{-P_2}$ where P_1, P_2 are positive integers, except the Comb frequencies $\omega_1 = \pm \omega_{10}, \omega_2 = \pm \omega_{20}$ the following comb frequencies

$$\omega_1 = \pm \frac{k_1}{P_1} \omega_{10}, \omega_2 = \pm \frac{k_2}{P_2} \omega_{20}$$

 $k_1 = 1, 2, \dots, P_1$ and $k_2 = 1, 2, \dots, P_2$

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are obtained

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