A Geometrical Approach For Fuzzy DEA Frontiers Using Different T Norms

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Abstract: - Interval DEA frontiers are here used in situations where one input or output is subject to uncertainty in its measurement and is presented as an interval data. We built an efficient frontier without any assumption about the probability distribution function of the imprecise variable. We take into account only the minimum and the maximum values of each imprecise variable. Two frontiers are constructed: the optimistic and the pessimistic ones. We use fuzzy relationships to introduce a new efficiency index based on a set of some Fuzzy T Norms. We will explore only the case where only one single variable presents a certain degree of uncertainty.

Key-Words: - Data Envelopment Analysis; Fuzzy Sets; Interval Data; T Norms; optimistic evaluation; pessimistic evaluation

1 Introduction

Classic Data Envelopment Analysis (DEA) models [1] estimate a non-parametric linear piecewise frontier determined by efficient Decision-Making Units (DMUs). Such models assume that the values involves are known with absolute precision. However, such hypothesis might not be true either due to uncertainty hidden in the measurements or because the data are given in interval format [1].

This paper proposes a method to evaluate efficiency in the case where the data are in interval form. The method uses a geometrical approach in order to build a fuzzy efficient frontier sets. Instead of efficiency score we will attribute to each DMU a membership degree to the frontier which will become a fuzzy set [2].

Alternative solutions for interval data in DEA models can be found in the literature with Fuzzy Linear Programming Problems. Yet another approach found is to present the efficiency measurements in terms of fuzzy functions. We emphasize that the approach followed in this paper uses only the concept of fuzzy sets and T Norms. The scores obtained herebellow are based mainly on geometrical considerations.

2 Literature Review in DEA Models with Uncertainties

A comprehensive literature review on methods used to deal with imprecise DEA data can be found in [3], where data uncertainties are classified into three types: interval data, ordinal data and interval data ratios. The author uses a model called Imprecise Data Envelopment Analysis (IDEA) [4] which treats the three types of data uncertainties using scale transformations.

The IDEA model was used by [5] to deal with uncertain data of two types: interval data and ordinal data. The use of such a linear model is carried out through a change of variable scales turning the non-linear model into a linear programming model. As a result, upper and lower bounds are obtained for the efficiency of each DMU. According to the authors, the use of post DEA models allows a better discrimination among DMUs.

In [6] an extended IDEA model is proposed, which enables not only the use of imprecise data but also the use of weight restrictions in the form of assurance regions or cone-ratios. In that case, the variable limits are changed to data adjustments. Such a model was applied to the efficiency evaluation of a Korean mobile telecommunication company.

In [7] uncertain inputs and outputs are considered as fuzzy sets. Efficiency computations are then carried out by means of linear fuzzy programming. As an alternative approach, the same authors proposed the use of possibility DEA models. A fuzzy variable is associated to a possibility distribution [8] where the fuzzy-DEA scores, although not unique, depend on the level of possibility used.

In [9] the authors employed a DEA model to assess DMUs optimistically. These results were used to determine interval efficiencies by means of new DEA models. Consequently, the efficiency score is not represented by a number, but by an interval. On the other hand, the authors of [10] assessed each DMU pessimistically based on the Inverted DEA model and calculated interval efficiency scores. The authors still considered interval data and proposed a model to evaluate interval efficiency and inefficiency as carried out using crisp data.

In [11] a performance evaluation of University academic departments is carried out. The DEA results on teaching, research and quality are fuzzy numbers. A unique performance score for each department was built using a weighted ordered aggregator. A performance evaluation in the educational field using DEA and Fuzzy sets is done in [12].

The authors in [13] extended the DEA CCR model with fuzzy inputs and outputs to a model named DEARA. This model uses regression analysis concepts and a Fuzzy-DEA model in which the resulting efficiency scores are interval fuzzy evaluations.

In [14], authors proposed a method to measure DMUs efficiencies with fuzzy variables. The fuzzy model then turns out to a family of conventional DEA models based on crisp data using the a-cut approach. According to the authors, the fuzzy efficiency scores obtained are given by interval functions yielding more information to the decision-maker. This approach uses Fuzzy Linear Programming. A similar approach has been used by [15].

To measure technical efficiency of DMUs, the authors of [16] relaxed the concept of production frontier and proposed a pair-wise comparison, checking the dominance or non-dominance of each DMU when compared to any other. The use of fuzzy variables to take into account imprecise data yields a fuzzy pair-wise comparison. Such results are represented in matrix form that shows two-way dominance. In other words, efficiency scores are not actually obtained, but only an indication of domination among DMUs.

In [17] fuzzy intervals are used to combine the information given by DEA analytical efficiency scores with subjective efficiency scores. Qualitative and organizational aspects are in fuzzy intervals format. The relationship between this information is given by a fuzzy interval function. Ideally, the two sources of information related to the performance of a DMU can be joined in such a way that the objective DEA aspect is used to control the subjectivity in the expert point of view, and vice-versa. That leads to a modified score set in terms of a fuzzy interval.

The authors [18] suggested a three-stage approach to measure technical efficiency in a fuzzy environment. This approach uses classic DEA techniques and is built on fuzzy parametric programming concepts [19].

In [20] fuzzy sets theory is used in a DEA context. The author uses three types of fuzzy statistics (fuzzy mathematical programming, fuzzy regression and fuzzy entropy) to illustrate the types of decision and solution that can be reached when we have imprecise data and a priori information is uncertain and imprecise. The same author [21] generalizes the nonparametric approach of DEA in both static and dynamic directions by incorporating uncertainties. He addresses an extension of the convex hull method of DEA for determining a production frontier in the presence of demand and supply uncertainty of outputs and inputs.

The authors of [22] used DEA, Fuzzy Sets and AHP for making rankings with incomplete and confidential information. Another work dealing with ranking in a fuzzy context is presented by Wen and Li [23].

An approach based on randomised ranks is presented by [24].

When applications are concerned we may cite a study on the location for the geographic situation of hydroelectric plants [25], the study on the efficiency of Taiwan hotels [26] and a performance assessment of manufacturing enterprises [27].

3 Fuzzy Efficient Frontier

The approach developed here makes no assumption regarding the way each input or output varies. Only maximum and minimum values for each output and each input are required. To obtains the membership degree of each DMU to the fuzzy frontier only geometric relationships are required. The algebraic calculation of those relations uses only classic DEA models. If the variables are in interval form, the exact location of the efficient frontier is unknown. It may be placed between upper and lower bounds. In other words the frontier is not a piecewise linear surface but a region of the space. In the case of one single input and one single output, such a frontier would be a strip. In other words, this frontier is a fuzzy set [2]. To such sets, instead of stating that a single element belongs or not to the set, we consider that all elements belong to it with a certain degree of membership.

In the absence of objective reason to choose among one of the various classical membership functions we will use some geometric measurements in the fuzzy efficient frontier. To do so we need to introduce some concepts:

1. Upper frontier: It is the frontier obtained by a classic DEA model (CCR or BCC) that considers the maximum value of the imprecise output for each DMU. As in terms of production this is the most desirable situation for all DMUs, the frontier so obtained may also be named Optimistic Frontier.
2. Lower frontier: It is the frontier obtained by a classic DEA model (CCR or BCC) that considers the minimum value of the imprecise output for each DMU. Since in terms of production this is the least favourable situation for all DMUs, the frontier so obtained may also be named Pessimistic Frontier.

The definitions hereabove are concerning to the case when the variable in interval form is an output. Moreover, these concepts are similar to those defined by [14]. Those authors have used Fuzzy Linear Programming, and we will use a geometrical approach. The relations derived from our geometrical approach are a generalization of those obtained in [28].

Figure 1 illustrates these concepts considering the BCC DEA model [29]. The interval data DEA frontier comprises the region between the lower and the upper frontiers. In opposition to classic DEA frontier, a DMU cannot be represented as a point in a multidimensional space. Its geometric representation must be a line segment (even in multidimensional cases). In Figure 1, the DMU C is represented by the segment $\overline{P1P2}$. The point P2 corresponds to the lower value of the imprecise output and the point P1 corresponds to the upper value of the imprecise output.

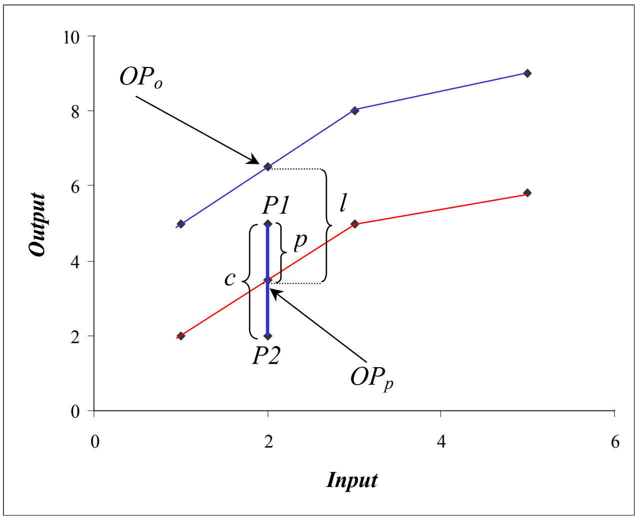

Figure 1: Optimistic and pessimistic frontiers

Also in Figure 1, *OPo* and *OPp* are the projected output on the optimistic and pessimistic frontiers; *c* is the DMU length, i.e., the difference between the optimistic and pessimistic values of the output; *l* is the width of the strip connecting the DMU projections on both frontiers; *p* is the difference between the optimistic output of each DMU and its projection on the pessimistic frontier.

To determine the DMU's membership degrees to the frontier we consider the following cases.

1. Figure 2 shows that DMUs A and F are totally inside the region defining the fuzzy frontier. Such DMUs must have a unitary membership degree the fuzzy frontier.

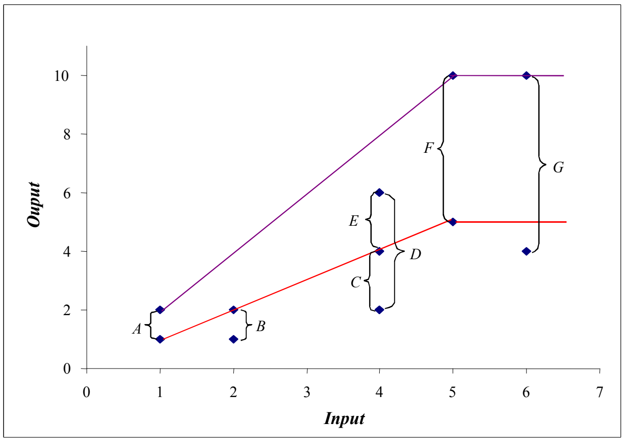

Figure 2: Interval data DEA frontier in a BCC model.

2. DMUs B and C slightly touch the frontier and so their membership degrees are zero.

3. Between those extreme situations, DMUs would have intermediate membership degrees.

a) The segment that represents DMU G covers all the length of the fuzzy frontier. However, its membership degree cannot be one, as it still has a strip outside the fuzzy frontier, i.e., although this DMU totally includes the frontier, it is not totally included there. The ratio p/c is adequate to evaluate the membership degree in situations similar to that of DMU G.

b) An inverse situation is presented by DMU E, which is fully contained in the fuzzy frontier, but does not entirely cover it. Like DMU G, this DMU cannot present a unitary membership degree to the frontier. For such situation the ratio p/l adequately represents the membership degree.

Both ratios above are only adequate in particular situations and lead to meaningless results when used in a different situation. In order to obtain a membership function with properties required in items 1, 2 and 3 (*a* and *b*), we need to combine the two ratios.

Properties 1, 2 and 3 are satisfied if we define a T Norm between the Fuzzy Set defined by the membership p/l and the fuzzy set defined by the membership function p/c.

We will use three T Norm to evaluate the membership degree of a DMU to the Fuzzy Frontier: i) The Product, ii) the Drastic Product and iii) the Min. The graphic representations of these three T Norms can be seen in Figure 3, 4 and 5, respectively.

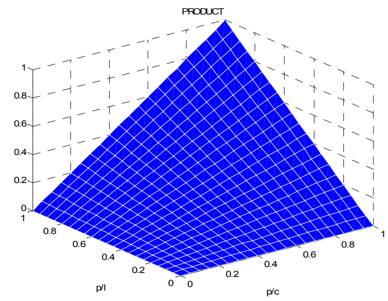

Figure 3. Representation of the T Norm "Product"

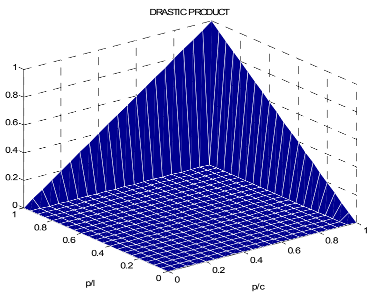

Figure 4. Representation of the T Norm "Drastic Product"

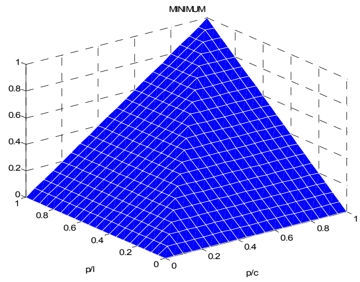

Figure 5. Representation of the T Norm "Min"

Expression (1) $\wp_P$ is the membership degree using the Product:

$$\wp_P = \frac{p^2}{lc} \qquad (1)$$

Expression (2) $\wp_D$ is the membership degree using the Drastic Product:

$$\wp_D = \begin{cases} 0, \text{ if } p<c \text{ and } p<l \\ p/c, \text{ if } p=l \\ p/l, \text{ if } p=c \end{cases} \qquad (2)$$

Expression (3) $\wp_M$ is the membership degree using the Min:

$$\wp_M = Min\left(p/c, p/l\right) \qquad (3)$$

These expressions may be used only if the uncertainty of the output is not null, to avoid division by zero. In other words, expression (1), (2) and (3) are not valid if a DMU has no uncertainty in its output.

Table 1 shows the results of membership degrees calculations for the DMUs of Figure 2, where *Op* and *Oo* are the output values for the pessimistic and optimistic frontiers and I is the input value.

Table 1: Membership degrees regarding the fuzzy DEA frontier.

DMU	*I*	O_p	O_o	*c*	*l*	*p*	$\wp_P$	$\wp_D$	$\wp_M$
A	1	1	2	1	1	1	1,00	1,00	1,00
B	2	1	2	1	2	0	0,00	0,00	0,00
C	4	2	4	2	4	0	0,00	0,00	0,00
D	4	2	6	4	4	2	0,25	0,00	0,50
E	4	4	6	2	4	2	0,50	0,50	0,50
F	5	5	10	5	5	5	1,00	1,00	1,00
G	6	4	10	6	5	5	0,83	0,83	0,83

From the algebraic properties of the T Norms follows that $\wp_d \le \wp_P \le \wp_M$.

From now on we will only use the T Norm Product. The reason for this choice is that the value of $\wp_P$ is not largest nor the smallest value of the membership degree. In other words, is not too much benevolent, neither very aggressive.

4 Algebraic Calculation of the Membership Degree

The previous calculations are based on a geometrical definition, which is feasible only for very simple models. In order to obtain an expression that might be used for multidimensional general models, in which only one output is imprecise, it is essential to change the geometric terms in equation (1) into variables that might be derived from the classic DEA models.

For the case of one imprecise output, considering the classic DEA definitions for output oriented models and also remembering that for that case the efficiencies are greater than one (BCC DEA model), equations (4) and (5) can be rewritten for *Op* and *Oo*, that are the output values for the pessimistic and optimistic frontiers, where

· *POp* and *POo* are the output targets on the pessimistic and optimistic frontiers, i.e., the projected output on the optimistic and pessimistic frontiers;

· *Effp* is the efficiency calculated using the lower output values, i.e., the efficiency related to the pessimistic frontier;

· *Effo* is the efficiency calculated using the upper output values, i.e., the efficiency related to the optimistic frontier.

$$Eff_p = \frac{PO_p}{O_p} \qquad (4)$$

$$Eff_o = \frac{PO_o}{O_o} \qquad (5)$$

With the purpose of avoiding misunderstandings, *Effo* and *Effp* should not be named optimistic and pessimistic efficiencies, as there is no guarantee that $Eff_o \ge Eff_p$.

From the geometrical representation we easily obtain $l = PO_o - PO_p = O_o Eff_o - O_p Eff_p$ and $c = O_o - O_p$. In a situation where the DMU is partially contained by the fuzzy frontier, *p* is the difference between the optimistic output and the output target on the pessimistic frontier, which is a positive number. If the DMU is totally outside the fuzzy frontier (except by a possible single point), the difference above is negative or zero. In this situation the membership degree must be zero, and to obtain this result p must also equal zero. Expression (6) formalizes the equation for *p*.

$$p = \begin{cases} O_o - O_p Eff_p, & if\, O_o - O_p Eff_p \ge 0 \\ 0, & otherwise \end{cases} \qquad (6)$$

From the previous equations, it is possible to derive the expression that represents algebraically the membership degree $\wp_P$, which is shown in (7).

$$\wp_P = \begin{cases} \dfrac{\left(O_o - O_p Eff_p\right)^2}{\left(O_o Eff_o - O_p Eff_p\right)\left(O_o - O_p\right)}, & if\, O_o - O_p Eff_p \ge 0 \\ 0, & otherwise \end{cases} \qquad (7)$$

Expression (7) is valid when we deal with only one output and this output is uncertain. Multidimensional models should be developed in future works.

Table 2 details the algebraic calculation of $\wp_P$. It should be noticed that, due to the output orientation model, the inefficient DMUs produce an efficient score greater than one.

Table 2: Computed values based on expressions (4) to (7).

DMU	*I*	O_p	O_o	Eff_p	Eff_o	*C*	*l*	*p*	$\wp$
A	1	1	2	1,00	1,00	1	1	1	1,00
B	2	1	2	2,00	2,00	1	2	0	0,00
C	4	2	4	2,00	2,00	2	4	0	0,00
D	4	2	6	2,00	1,33	4	4	2	0,25
E	4	4	6	1,00	1,33	2	4	2	0,50
F	5	5	10	1,00	1,00	5	5	5	1,00

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G	6	4	10	1,25	1,00	6	5	5	0,83

5 Comparing this approach with Despotis and Smirlis Model

Despotis and Smirlis [5] model, that we call from now on D&S model, has one common point with our model: it also deals with upper and lower bounds for data. For that reason, we will compare our model only with the D&S model. Such a model defines upper and lower bounds for the efficiency, but does not define a Fuzzy Frontier. Model (8) and (9) show the linear programs for these two efficiency bounds.

$$
\begin{aligned}
\max \quad & H_{j_0} = \sum_{r=1}^{s} u_r y_{rj_0}^{\mathrm{U}} \\
\text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{ij_0}^{\mathrm{L}} = 1, \\
& \sum_{r=1}^{s} u_r y_{rj_0}^{\mathrm{U}} - \sum_{i=1}^{m} v_i x_{ij_0}^{\mathrm{L}} \leqslant 0, \\
& \sum_{r=1}^{s} u_r y_{rj}^{\mathrm{L}} - \sum_{i=1}^{m} v_i x_{ij}^{\mathrm{U}} \leqslant 0, \qquad j = 1, \ldots, n; \quad j \neq j_0, \\
& u_r, v_i \geqslant \varepsilon \quad \forall r, i.
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
\max \quad & F_{j_0} = \sum_{r=1}^{s} u_r y_{rj_0}^{\mathrm{L}} \\
\text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{ij_0}^{\mathrm{U}} = 1, \\
& \sum_{r=1}^{s} u_r y_{rj_0}^{\mathrm{L}} - \sum_{i=1}^{m} v_i x_{ij_0}^{\mathrm{U}} \leqslant 0, \\
& \sum_{r=1}^{s} u_r y_{rj}^{\mathrm{U}} - \sum_{i=1}^{m} v_i x_{ij}^{\mathrm{L}} \leqslant 0, \qquad j = 1, \ldots, n; \quad j \neq j_0 \\
& u_r, v_i \geqslant \varepsilon \quad \forall r, i.
\end{aligned}
\tag{9}
$$

For comparison between the D&S model and our model in models (8) and (9) we will assume that only the output is imprecise and the input is an exact one.

In these models, to calculate efficiency upper bounds, the outputs are adjusted at the upper bound of the interval for the observed DMU and they are adjusted at the lower bounds for other DMUs. To calculate efficiency lower bounds, the outputs are adjusted at the lower bound of the interval for the observed DMU and at the upper bound for other DMUs.

As in our model the upper frontier is defined by all the upper bound of the output, D&S model is more benevolent than our model, when regarding the upper bounds. For lower bounds, we have an inverse situation.

Table 3: Calculated values for both models

DMU	I	O_p	O_o	c	l	p	$\wp_P$	$\wp_d$	$\wp_M$	Upper	Lower
A	1	1	2	1	1	1	1	1	1	1,000000	1,000000
B	2	1	2	1	2	0	0	0	0	1,000000	0,500000
C	4	2	4	2	4	0	0	0	0	1,000000	0,250000
D	4	2	6	4	4	2	0,25	0	0,5	1,000000	0,250000
E	4	4	6	2	4	2	0,5	0,5	0,5	1,000000	0,500000
F	5	5	10	5	5	5	1	1	1	1,000000	0,575000
G	6	4	10	6	5	5	0,83	0,83	0,83	1,000000	0,33333

Furthermore, D&S model defines as E^{++} DMU every DMU efficient in the upper and lower bound. A DMU efficient in the upper bound but inefficient in the lower bound is an E^{+}. DMUs not efficient in any frontier are called E^{-} DMU.

According to these definitions DMU A is E^{++} and all the others are E^{+}. We shall note that this model does not rank DMUs directly. We may rank E^{+} DMUs using their lower bound efficiency. Our model directly ranks all DMUs using a membership function. Nonetheless, in our model ties may occur.

Table 4 shows a comparison between the rankings by our model and the ranking by D&S model.

Table 3: Comparing the models

DMU	Ranking using			
	$\wp_P$	$\wp_d$	$\wp_M$	D&S
A	1	1	1	1
B	6	5	6	3
C	6	5	6	6

D	4	5	4	6
E	4	4	4	3
F	1	1	1	2
G	3	3	3	5

When comparing the two models some conclusions arise. Our model has a geometric and an algebraic approach. D&S has only an algebraic approach.

On the other hand, D&S model may deal with interval and precise data in all output and inputs. At the moment our model allows only one output, or input (see next section) to be imprecise. Furthermore, in our model the imprecise variable cannot have any precise value.

From an algebraic point of view, our model may be solved using a DEA solver twice. We have used SIAD [30]. In D&S model we need to use a DEA solver 2n times, n being the number DMUs.

6 Fuzzy Frontier with One Imprecise Input

The case of one imprecise input may be analysed in a way similar to that of one imprecise output. In that case, the optimistic input, *Io*, is the smallest value of the input, and the pessimistic input, *Ip* is the largest one. An optimistic frontier is obtained when optimistic inputs are considered for all DMUs and, conversely, a pessimistic frontier is characterized when pessimistic inputs are assumed for all DMUs. Figure 3 depicts the optimistic and pessimistic frontiers for the case of one imprecise input. In this figure, I_o, I_p, PI_o and PI_p represent, respectively, the imprecise input optimistic and pessimistic values and the input target values on the optimistic and pessimistic frontiers. Conversely for the output-oriented situation, now the line segment representing the DMU with imprecise input value lies in the horizontal position.

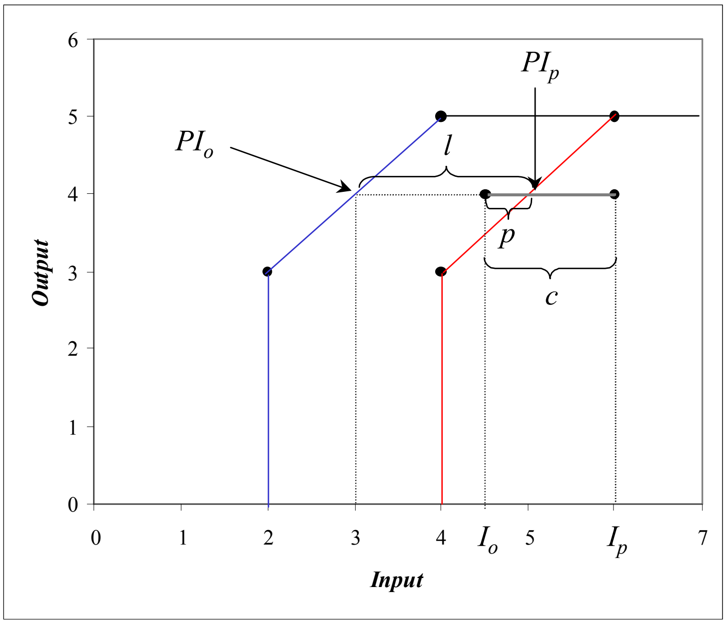

Figure 3: Optimistic and pessimistic frontiers for the input oriented BCC model.

We will use a membership degree based on the T Norm Product, in a similar way to the case of an imprecise output. Expression (10) presents the membership degree for that situation.

$$\wp = \begin{cases} \dfrac{\left(I_p Eff_p - I_o\right)^2}{\left(I_p Eff_p - I_o Eff_o\right)\left(I_p - I_o\right)}, & if\, I_p Eff_p - I_o \geq 0 \\ 0, & otherwise \end{cases} \quad (10)$$

7 Conclusion

The approach proposed in this paper in order to incorporate uncertainties in classic DEA models has the advantage of neither using any particular probability distribution for the variable uncertainties nor a fuzzy function for them. Besides, it is at the same time mathematically simple, since the results are obtained by simple algebraic calculations (after calculating DEA classic frontiers, in opposition to the change of variable used in Despotis and Smirlis

[5] and without the need of using fuzzy programming.

The location of the interval DEA frontier allows the geometrically building of a membership function and consequently obtaining a fuzzy result that uses the membership concept without the need of using the classical membership functions. As a matter of fact, the geometrical considerations used to define the membership index implicitly employed uniform membership functions. Those functions have constant values. One of them is equal to the inverse of the length of the DMU representative segment. The other one has is value inverse to the length of the segment determined by the pessimistic and optimistic targets.

In this paper we have dealt only with a DEA model with a single input and a single output. DEA is for its very nature a multidimensional tool. However, models with a single input and a single output are often used to derive new concepts for further generalizations. For instance, a general theorem was proved first for the single input and single output case in [31]. As a future work, we intent to generalize the model presented in this paper for multidimensional cases.

A possible application of our results is for machine tools evaluation [32], taking into account uncertainties in the measurements.

Acknowledgements

We acknowledge the financial support of CNPq (Brazilian Ministry of Science and Technology).

References:

[1] W. W. Cooper, L. Seiford, K. Tone, *Data envelopment analysis: A comprehensive text with models, applications, references and DEA-solver software*. Boston: Kluwer, 2000.

[2] L. Zadeh, Fuzzy Sets, *Information and Control,* Vol. 8, 1965, pp. 338-353.

[3] J. Zhu, Imprecise data envelopment analysis (IDEA): A review and improvement with an application, *European Journal of Operational Research,* Vol. 144, 2003, pp. 513-529.

[4] W. W. Cooper, K. S. Park, G. Yu, IDEA and AR-IDEA: Models for dealing with imprecise data in DEA, *Management Science,* Vol. 45, 1999, pp. 597-607.

[5] D. K. Despotis, Y. G. Smirlis, Data envelopment analysis with imprecise data, *European Journal of Operational Research,* Vol. 140, 2002, pp. 24-36.

[6] W. W. Cooper, K. S. Park, G. Yu, An illustrative application of idea (imprecise Data Envelopment Analysis) to a Korean mobile telecommunication company, *Operations Research,* Vol. 49, 2001, pp. 807-820.

[7] S. Lertworasirikul, S. C. Fang, J. A. Joines, H. L. W. Nuttle, Fuzzy data envelopment analysis (DEA): a possibility approach, *Fuzzy Sets and Systems,* Vol. 139, 2003, pp. 379-394.

[8] L. Zadeh, Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets and Systems,* Vol. 1, 1978, pp. 3-28.

[9] T. Entani, Y. Maeda, H. Tanaka, Dual models of interval DEA and its extensions to interval data, *European Journal of Operational Research,* Vol. 136, 2002, pp. 32-45.

[10] Y. Yamada, T. Matui, M. Sugiyama, New analysis of efficiency based on DEA, *Journal of the Operations Research Society of Japan,* Vol. 37, 1994, pp. 158-167.

[11] A. L. M. Lopes, E. A. Lanzer, Data envelopment analysis – DEA and fuzzy sets to assess the performance of academic departments: a case study at Federal University of Santa Catarina – UFSC, *Pesquisa Operacional,* Vol. 22, 2002, pp. 217-230.

[12] E. K. Maragos, D. K. Despotis, The evaluation of the efficiency with data envelopment analysis in case of missing values: A fuzzy approach, *WSEAS Transactions on Mathematics,* Vol. 3, 2004, pp. 656-663.

[13] P. Guo, H. Tanaka, Fuzzy DEA: a perceptual evaluation method, *Fuzzy Sets and Systems,* Vol. 119, 2001, pp. 149-160.

[14] C. Kao, S. T. Liu, Fuzzy efficiency measures in data envelopment analysis, *Fuzzy Sets and Systems,* Vol. 113, 2000, pp. 427-437.

[15] P. H. Lin, Multiple criteria ranking by fuzzy data envelopment analysis, *WSEAS Transactions on Computers,* Vol. 5, 2006, pp. 810-816.

[16] K. Triantis, P. V. Eeckaut, Fuzzy Pair-wise Dominance and Implications for Technical Efficiency Performance Assessment, *Journal of Productivity Analysis,* Vol. 13, 2000, pp. 207-230.

[17] J. L. Hougaard, Fuzzy scores of technical efficiency, *European Journal of Operational Research,* Vol. 115, 1999, pp. 529-541.

[18] K. Triantis, O. Girod, A Mathematical Programming Approach for Measuring Technical Efficiency in a Fuzzy Environment, *Journal of Productivity Analysis,* Vol. 10, 1998, pp. 85-102.

[19] C. Carlsson, P. Korhonen, A Parametric Approach to Fuzzy Linear Programming, *Fuzzy Sets and Systems,* Vol. 20, 1986, pp. 17-33.

[20] J. K. Sengupta, A fuzzy systems approach in data envelopment analysis, *Computers & Mathematics with Applications,* Vol. 24, 1992, pp. 8-9.

[21] J. K. Sengupta, Nonparametric efficiency analysis under uncertainty using data envelopment analysis, *International Journal of Production Economics,* Vol. 95, 2005, pp. 39-49.

[22] L. C. Ma, H. L. Li, A fuzzy ranking method with range reduction techniques, *European Journal of Operational Research,* Vol. 184, 2008, pp. 1032-1043.

[23] M. Wen, H. Li, Fuzzy data envelopment analysis (DEA): Model and ranking method, *Journal of Computational and Applied Mathematics,* Vol. 223, 2009, pp. 872-878.

[24] A. P. Sant´anna, Data envelopment analysis of randomized ranks, *Pesquisa Operacional,* Vol. 22, 2002, pp. 203-215.

[25] L. A. F. P. Sant'Anna, A. P. Sant'Anna, A probabilistic approach to evaluate the exploitation of the geographic situation of hydroelectric plants, *Energy Policy,* Vol. 36, 2008, pp. 2320-2329.

[26] C. C. Shen, K. L. Hsieh, Incorporating Fuzzy Theory and DEA into Performance Evaluation: The Case of Taiwan's International Hotels, *WSEAS Transactions on Systems,* Vol. 5, 2006, pp. 2666-2671.

[27] Y. M. Wang, Y. Luo, L. Liang, Fuzzy data envelopment analysis based upon fuzzy arithmetic with an application to performance assessment of manufacturing enterprises, *Expert Systems with Applications,* Vol. 36, 2009, pp. 5205-5211.

[28] J. C. C. B. Soares de Mello, E. G. Gomes, L. Angulo-Meza, L. Biondi Neto, A. P. Sant´anna, Fronteiras DEA Difusas, *Investigação Operacional,* Vol. 25, 2005, pp. 85-103.

[29] R. D. Banker, A. Charnes, W. W. Cooper, Some models for estimating technical scale inefficiencies in data envelopment analysis, *Management Science* Vol. 30, 1984, pp. 1078-1092.

[30] L. Angulo-Meza, L. Biondi Neto, J. C. C. B. Soares de Mello, E. G. Gomes, P. H. G. Coelho, "Free software for Decision Analysis a software package for Data Envelopment models," in *ICEIS 2005 - Proceedings of the 7th International Conference on Enterprise Information Systems*, Miami, 2005, pp. 207-212.

[31] M. P. E. Lins, E. G. Gomes, J. C. C. B. Soares de Mello, A. J. R. Soares de Mello, Olympic ranking based on a zero sum gains DEA model, *European Journal of Operational Research,* Vol. 148, 2003, pp. 312-322.

[32] J. C. C. B. Soares de Mello, E. G. Gomes, L. Angulo-Meza, F. R. Leta, DEA Advanced Models for Geometric Evaluation of used Lathes, *WSEAS Transactions on Systems,* Vol. 7, 2008, pp. 500-520.