# Some optimal approximate methods with application to thin film flow 

VASILE MARINCA, NICOLAE HERIŞANU<br>Politehnica University of Timişoara<br>Bd.M.Viteazu, 1, 300222 Timişoara,<br>Center for Advanced and Fundamental Technical Research, Romanian Academy, Timisoara Branch, Bd. M.Viteazu, nr.24, 300223 Timişoara<br>ROMANIA<br>vmarinca@mec.upt.ro, herisanu@mec.upt.ro<br>DUMITRU BĂLĂ<br>Faculty of Economics and Business Management<br>Craiova University<br>ROMANIA<br>dumitru_bala@yahoo.com


#### Abstract

In this work a totally analytic solution for this film flow of a fourth grade fluid down a vertical cylinder are obtained using some approximate optimal methods. These approaches do not depend upon any small or large parameters in comparison with other perturbation methods. The solutions obtained using our procedures are in good agreement with the exact solution, which show the applicability of the methods.


Key-Words: - Optimal variational iteration method (OVIM), Optimal parametric iteration method (OPIM), thin film flow, approximate analytic solution

## 1 Introduction

In general it is very difficult to solve nonlinear problems either numerically or theoretically. Approximate analytical and numerical methods are widely used to solve non-linear differential equations modeling physical phenomena. This is because exact solutions of these equations are rare.

Mathematical modeling of many physical systems leads to non-linear ordinary or partial differential equations in various fields of physics and engineering. An effective method is required to analyze the mathematical model which provides solutions conforming to physical reality. In many cases, it is possible to replace a non-linear differential equation by a corresponding linear differential equation that approximates the original non-linear equation closely to give useful results. In general, the study of nonlinear differential equations is restricted to a variety of special classes of equations and the method of solution usually involves a limited number of techniques to achieve analytical approximations to the solutions.

There are some approaches for approximating solutions of a non-linear system. The most common and most widely used methods for non-linear differential equations are the perturbation methods [1], [2]. But almost all perturbation methods are based on such an assumption that a small parameter must exist in an equation. This so-called small parameter assumption
greatly restricts applications of perturbation techniques. As it is well-known, an overwhelming majority of nonlinear problems, especially those having strong nonlinearity, have no small parameters at all. The approximate solutions obtained through perturbation methods are valid, in most cases, only for small values of the parameters. However, there is no criterion on how the small parameters should be.

To overcome the restrictions of perturbation techniques, some non-perturbation methods are developed such as the Adomian decomposition method [3], the weighted linearization method [4], $\delta$-method [5], the bookkeeping artificial parameter method [6], the modified Lindstedt-Poincare method [7], and so on.

In recent years a growing interest toward the application of iterative techniques and homotopy techniques in nonlinear problems appeared in engineering practice [8-15]. In 1987 Mickens [16] proposed an iteration scheme for a nonlinear conservative oscillator. Lim et al. [17] proposed a modified iterative scheme for an oscillator described as

$$
\begin{equation*}
\ddot{\mathrm{x}}+\mathrm{f}(\mathrm{x})=0, \mathrm{x}(0)=\mathrm{A}, \dot{\mathrm{x}}(0)=0 \tag{1}
\end{equation*}
$$

where f is an odd function in the following from:

$$
\begin{align*}
& \ddot{\mathrm{x}}_{\mathrm{k}+1}+\omega^{2} \mathrm{x}_{\mathrm{k}+1}=\mathrm{g}\left(\omega, \mathrm{x}_{\mathrm{k}-1}\right)+  \tag{2}\\
& +\mathrm{g}_{\mathrm{x}}\left(\omega, \mathrm{x}_{\mathrm{k}-1}\right)\left(\mathrm{x}_{\mathrm{k}}-\mathrm{x}_{\mathrm{k}-1}\right), \mathrm{k}=0,1,2, \ldots
\end{align*}
$$

where $\mathrm{g}(\omega, \mathrm{x})=\omega^{2} \mathrm{x}-\mathrm{f}(\mathrm{x}), \omega$ is a priori unknown frequency and $g_{x}=\frac{d g}{d x}$.

Later, Marinca and Herişanu [18] proposed a new iteration method combining Mickens and He's iteration methods. These iteration procedures have been used to solve both nonlinear conservative and nonconservative oscillators. For most conservative oscillators, like some other methods for nonlinear oscillators, the secondorder, even the first order of approximation can give uniformly accurate solutions [19,20].

In 1978, Inokuti et al [21] proposed a general Lagrange multiplier method to solve non-linear problems, which was first proposed to solve problems in quantum mechanics. The main feature of the method is as follows: the solution of a mathematical problem with linearization assumption is used as initial approximation or trial-function and then a more highly precise approximation at some special point can be obtained. Variational iteration method has been favorably applied to various kinds of nonlinear problems. The main property of the method is in its flexibility and ability to solve nonlinear equations accurately and conveniently. Major applications to nonlinear wave equation, nonlinear fractional differential equations, nonlinear oscillations and nonlinear problems arising in various engineering applications are surveyed. The confluence of modern mathematics and symbol computation has posed a challenge to developing technologies capable of handling strongly nonlinear equations which cannot be successfully dealt with by classical methods. Very recently it was recognized that the variational iteration method [22-28] can be an effective procedure for solution various nonlinear problems without usual restrictive assumptions. The method, extensively worked out by numerous authors, has been maturing into a fully fledged theory, more and more merits have been discovered and some modifications are suggested to overcome the demerit arising in the solution procedure. Applications of the method have been employed due to its flexibility, convenience and accuracy. D'Acunto [29] applied the variational iteration method to nonlinear heat transfer, Marinca [30] to nonlinear oscillations, Lu [31] to two-point boundary value problems, Sweilan et al [32] to nonlinear thermoelasticity, Siddiqui et al [33] to nonNewtonian flows, Liu [34] to ion acoustic plasma wave and so on.

Over the last few years, a number of investigators [35-39] have been involved in the study of nonNewtonian fluids. The study of non-Newtonian fluids is very important in view of its potential applicability in the fields of engineering and technology. Examples of such fluids include blood, drilling muds, clay coating and
other suspensions, certain oils and greases, polymer melts, elastomers and many emulsions.

Non-Newtonian fluids may be defined as fluids for which the shear stress depends on the shear rate, fluids for which the relation between the shear stress and shear rate depend upon the time and fluids which possess both elastic and viscous properties, which are called viscoelastic fluids or elastic-viscous fluids.

It is very difficult to suggest a simple model which exhibits all properties of non-Newtonian fluids. Therefore, several fluid models have been proposed to predict the non-Newtonian behaviour of various types of materials.

The most popular model for non-Newtonian fluid designated the "second-grade" generally involves simple calculations and it is thus feasible to obtain analytical solutions. The "second-grade" fluid describes normal stress effects but it does not exhibit the property of shear thinning or thickening for a steady flow. Due to this fact, some experiments may well be described by the fluids of the order three or four.

The study of such fluids involves the use of nonlinear equations with increasing complexity for systems that involve substantial non-linear analysis.

The model under study in the present paper is of the fourth grade fluid type, and we have applied the Optimal Parametric Iteration Method (OPIM) and the Optimal Variational Iteration Method (OVIM) in order to analyze the non-linear behaviour of a thin film flow down a vertical cylinder. The fluid under investigation introduces greater nonlinearity in the analysis.

The results obtained by these two approximate methods are compared with the exact solution and a very good agreement was observed. These procedures are not valid only for small but also for large parameters. They provide us with a convenient way to control the convergence of approximations, which ensure their applicability and great potential to solve a wide number of non-linear problems in science and engineering.

## 2 Basic Ideas of the Optimal Parametric Iteration Method

We consider the nonlinear differential equation in the following form

$$
\begin{equation*}
\operatorname{Lf}(\eta)+N\left(\eta, f(\eta), f^{\prime}(\eta), f^{\prime}(\eta)\right)=0, \eta \in[a, b] \tag{3}
\end{equation*}
$$

with the initial conditions

$$
\begin{equation*}
f(a)=\alpha, f^{\prime}(b)=\beta \tag{4}
\end{equation*}
$$

where $f^{\prime}=\frac{d f}{d \eta}$, $L$ is a linear operator and $N$ is a nonlinear operator.

Applying the Taylor series theorem for the analytic function $\mathrm{F}\left(\mathrm{x}, \mathrm{x}^{\prime}, \mathrm{x}^{\prime \prime}\right)$ and for any real values $\mathrm{x}_{0}, \alpha, \beta$ and $\gamma$ we have

$$
\begin{align*}
& \mathrm{F}\left(\mathrm{x}_{0}+\alpha, \mathrm{x}_{0}^{\prime}+\beta, \mathrm{x}^{\prime}{ }_{0}+\gamma\right)=\mathrm{F}\left(\mathrm{x}_{0}, \mathrm{x}_{0}^{\prime}, \mathrm{x}^{\prime \prime}{ }_{0}\right)+ \\
& +\alpha \mathrm{F}_{\mathrm{x}}\left(\mathrm{x}_{0}, \mathrm{x}_{0}^{\prime}, \mathrm{x}^{\prime \prime}{ }_{0}\right)+\beta \mathrm{F}_{\mathrm{x}^{\prime}}\left(\mathrm{x}_{0}, \mathrm{x}_{0}, \mathrm{x}^{\prime \prime}{ }_{0}\right)+  \tag{5}\\
& +\gamma \mathrm{F}_{\mathrm{x}^{\prime \prime}}\left(\mathrm{x}_{0}, \mathrm{x}_{0}^{\prime}, \mathrm{x}^{\prime \prime}{ }_{0}\right)+\ldots
\end{align*}
$$

where $\mathrm{F}_{\mathrm{x}}=\frac{\partial \mathrm{F}}{\partial \mathrm{x}}$
Instead of solving the nonlinear differential equation (3), one can solve another equation, making recourse to the following scheme:

$$
\begin{align*}
& L_{n+1}(\eta)+N\left(\eta, f_{n}(\eta), f_{n}^{\prime}(\eta), f_{n} "_{n}(\eta)+\right. \\
& +\alpha_{n}\left(\eta, C_{i}\right) N_{f}\left(\eta, f_{n}(\eta), f_{n}^{\prime}(\eta), f_{n} "_{n}(\eta)\right)+ \\
& +\beta_{n}\left(\eta, C_{j}\right) N_{f^{\prime}}\left(\eta, f_{n}(\eta), f_{n}^{\prime}(\eta), f_{n}(\eta)\right)+  \tag{6}\\
& +\gamma_{n}\left(\eta, C_{k}\right) N_{f^{\prime \prime}}\left(\eta, f_{n}(\eta), f_{n}^{\prime}(\eta), f^{\prime \prime}{ }_{n}(\eta)\right) \\
& f_{n}(a)=0, f_{n}^{\prime}(b)=0
\end{align*}
$$

where the initial approximation $f_{0}(\eta)$ is given by the equation

$$
\begin{equation*}
\operatorname{Lf}_{0}(\eta)=0, f_{0}(a)=\alpha, f_{0}(b)=\beta \tag{7}
\end{equation*}
$$

and $\alpha_{n}\left(\eta, C_{i}\right), \beta_{n}\left(\eta, C_{i}\right)$ and $\gamma_{n}\left(\eta, C_{i}\right)$ are functions of $\eta$ and a number of constants $C_{1}, C_{2}, \ldots$ which are chosen of the same form like $N\left(\eta, f_{n}(\eta), f_{n}^{\prime}(\eta), f^{\prime \prime}{ }_{n}(\eta)\right)$.

In this way we obtain an approximation to the solution of Eq.(3) given by a truncated series (5). To improve the order of convergence of the sequence $f_{n}(\eta)$ as given in Eq.(6) and (7) we propose that the constants $\mathrm{C}_{1}, \mathrm{C}_{2}, \ldots$ which appear in the functions $\alpha_{\mathrm{n}}, \beta_{\mathrm{n}}$ and $\gamma_{\mathrm{n}}$ (and are unknown at this moment) be determined optimally, i.e. the residual functional J given by

$$
\begin{equation*}
J=\int_{a}^{b}\left[L f_{n}(\eta)+N\left(\eta, f_{n}(\eta), f_{n}^{\prime}(\eta), f_{n}^{\prime \prime}(\eta)\right)\right]^{2} d \eta \tag{8}
\end{equation*}
$$

be minimum, i.e.

$$
\begin{equation*}
\frac{\partial \mathrm{J}}{\partial \mathrm{C}_{\mathrm{i}}}=0, \mathrm{i}=1,2, \ldots \tag{9}
\end{equation*}
$$

We remark that the constants $\mathrm{C}_{\mathrm{i}}$ can be determined via various methods, for example the last square method, the Galerkin method, the collocation method and so on. In this way the solution of Eq.(6) and (7) is well determined.

The basic ideas of the proposed procedure are the construction of a new iteration scheme (6) and (7) and the presence of the constants $\mathrm{C}_{\mathrm{i}}, \mathrm{i}=1,2, \ldots$ which lead to an excellent agreement between the approximate and exact solutions.

## 3 Formulation of the Problem

Consider a fourth grade fluid falling on the outside surface on an infinitely long vertical cylinder of radius R. The flow is considered in thin, uniform, axisymmetric film with thickness $\delta$, in contact with stationary air. In cylindrical coordinates, we have [35-37]:

$$
\begin{align*}
& \frac{\partial \mathrm{p}}{\partial \mathrm{r}}=\left(2 \alpha_{1}+\alpha_{2}\right) \frac{1}{\mathrm{r}} \frac{\mathrm{~d}}{\mathrm{dr}}\left[\mathrm{r}\left(\frac{\mathrm{du}}{\mathrm{dr}}\right)^{2}\right]+ \\
& +\frac{4}{\mathrm{r}}\left(\gamma_{3}+\gamma_{4}+\gamma_{5}+\frac{\gamma_{6}}{2}\right) \frac{\mathrm{d}}{\mathrm{dr}}\left[\mathrm{r}\left(\frac{\mathrm{du}}{\mathrm{dr}}\right)^{4}\right]  \tag{10}\\
& \frac{\partial \mathrm{p}}{\partial \mathrm{z}}=\frac{\mu}{\mathrm{r}} \frac{\mathrm{~d}}{\mathrm{dr}}\left(\mathrm{r} \frac{\mathrm{du}}{\mathrm{dr}}\right)+ \\
& \quad+\frac{2}{\mathrm{r}}\left(\beta_{2}+\beta_{3}\right) \frac{\mathrm{d}}{\mathrm{dr}}\left[\mathrm{r}\left(\frac{\mathrm{du}}{\mathrm{dr}}\right)^{3}\right]+\rho \mathrm{g} \tag{11}
\end{align*}
$$

where $\mathrm{p} \neq \mathrm{p}(\mathrm{z})$ is pressure and Eq .(11) further gives:

$$
\begin{align*}
& \mathrm{r} \frac{\mathrm{~d}^{2} \mathrm{u}}{\mathrm{dr}^{2}}+\frac{\mathrm{du}}{\mathrm{dr}}+\frac{2\left(\beta_{2}+\beta_{3}\right)}{\mu}\left[3 \mathrm{r}\left(\frac{\mathrm{du}}{\mathrm{dr}}\right)^{2} \frac{\mathrm{~d}^{2} \mathrm{u}}{\mathrm{dr}^{2}}+\right. \\
& \left.+\left(\frac{\mathrm{du}}{\mathrm{dr}}\right)^{3}\right]+\frac{\rho \mathrm{g}}{\mu} \mathrm{r}=0 \tag{12}
\end{align*}
$$

The boundary conditions are:

$$
\begin{equation*}
\mathrm{u}(\mathrm{R})=0, \frac{\mathrm{du}(\mathrm{R}+\delta)}{\mathrm{dr}}=0 \tag{13}
\end{equation*}
$$

Defining

$$
\begin{align*}
& \eta=\frac{r}{R}, f=\frac{R}{\mu} u, k=\frac{\rho g R^{3}}{\mu^{2}}, \\
& \beta=\frac{\mu\left(\beta_{2}+\beta_{3}\right)}{R^{4}}, d=1+\frac{\delta}{R} \tag{14}
\end{align*}
$$

Eqs.(21) and (22) reduces to

$$
\begin{gather*}
\eta \frac{d^{2} f}{d \eta^{2}}+\frac{d f}{d \eta}+k \eta+ \\
+2 \beta\left[\left(\frac{d f}{d \eta}\right)^{3}+3 \eta\left(\frac{d f}{d \eta}\right)^{2} \frac{d^{2} f}{d \eta^{2}}\right]=0  \tag{15}\\
f(1)=0, f^{\prime}(d)=0 \tag{16}
\end{gather*}
$$

In accordance with Eq.(15), the linear operator is chosen as

$$
\begin{equation*}
\operatorname{Lf}(\eta)=\eta \frac{\mathrm{d}^{2} \mathrm{f}}{\mathrm{~d} \eta^{2}}+\frac{\mathrm{df}}{\mathrm{~d} \mathrm{\eta}}+\mathrm{k} \mathrm{\eta} \tag{17}
\end{equation*}
$$

and we define a non-linear operator as

$$
\begin{equation*}
\left.N\left(\eta, f(\eta), f^{\prime}(\eta), \mathrm{f}^{\prime \prime}(\eta)\right)=2 \beta\left[\mathrm{f}^{\prime 3}+3 \eta \mathrm{f}^{\prime 2} \mathrm{f}^{\prime}\right]\right] \tag{18}
\end{equation*}
$$

## 4 Solution through Optimal Parametric Iteration Method

The initial approximation $f_{0}(\eta)$ is obtained from Eq.(7):

$$
\begin{align*}
& \eta \mathrm{f}^{\prime \prime}{ }_{0}+\mathrm{f}^{\prime}{ }_{0}+\mathrm{k} \mathrm{\eta}=0  \tag{19}\\
& \mathrm{f}^{\prime}{ }_{0}(\mathrm{~d})=0 \tag{20}
\end{align*}
$$

It is obtained:

$$
\begin{equation*}
{f^{\prime}}_{0}(\eta)=\frac{k}{2}\left(\frac{d^{2}}{\eta}-\eta\right) \tag{21}
\end{equation*}
$$

The first-order approximate solution $f_{1}(\eta)$ is obtained from Eq.(6) for $\mathrm{n}=0$ :

$$
\begin{align*}
& \eta \mathrm{f}^{\prime \prime}+\mathrm{f}^{\prime}{ }_{1}+\mathrm{k} \mathrm{\eta}+2 \beta\left[\mathrm{f}_{0}^{\prime 3}+3 \eta \mathrm{f}_{0}^{\prime 2} \mathrm{f}^{\prime \prime}{ }_{0}+\right. \\
& +\alpha\left(\eta, \mathrm{C}_{\mathrm{i}}\right)\left(3 \mathrm{f}^{\prime 2}+6 \eta \mathrm{f}_{0}{ }_{0} \mathrm{f}^{\prime \prime}{ }_{0}\right)+  \tag{22}\\
& \left.+\gamma\left(\eta, \mathrm{C}_{\mathrm{j}}\right)\left(3 \eta \mathrm{f}^{\prime 2}\right)\right]=0 \\
& \mathrm{f}^{\prime}{ }_{1}(\mathrm{~d})=0 \tag{23}
\end{align*}
$$

The Eq.(22) can be written in the following form

$$
\begin{align*}
& \left(\eta f_{1}^{\prime}\right)^{\prime}+\left(\frac{\mathrm{k}}{2} \eta^{2}\right)^{\prime}+2 \beta\left(\eta \mathrm{f}_{0}^{3}\right)^{\prime}+  \tag{24}\\
& +6 \beta\left[\alpha\left(\eta, \mathrm{C}_{\mathrm{i}}\right)\left(\eta \mathrm{f}_{0}^{\prime 2}\right)^{\prime}+\gamma\left(\eta, \mathrm{C}_{\mathrm{j}}\right)\left(\eta \mathrm{f}_{0}^{\prime 2}\right)\right]=0
\end{align*}
$$

The Eq.(24) can be easy solved if the functions $\alpha\left(\eta, C_{i}\right)$ and $\gamma\left(\eta, C_{i}\right)$ are chosen such as

$$
\begin{align*}
& \alpha\left(\eta, C_{i}\right)\left(\eta f_{0}^{\prime 2}\right)^{\prime}+\gamma\left(\eta, C_{j}\right)\left(\eta f_{0}^{\prime 2}\right)= \\
& =\frac{1}{3}\left(C_{1} \eta f_{0}^{\prime \mathrm{i}}+C_{2} \eta f_{0}^{\prime j}+C_{3} \eta f_{0}^{\prime k}\right)^{\prime} \tag{25}
\end{align*}
$$

where $C_{1}, C_{2}$ and $C_{3}$ are unknown constants and $i, j, k$ are integers.

From Eq.(25) it is obtained:

$$
\begin{align*}
& \alpha\left(\eta, C_{i}\right)=\frac{1}{6}\left(\mathrm{C}_{1} \mathrm{f}_{0}^{\mathrm{i}-2}+\mathrm{C}_{2} \mathrm{f}_{0}^{\mathrm{j}-2}+\mathrm{C}_{3} \mathrm{f}_{0}^{\prime \mathrm{k}-2}\right)  \tag{26}\\
& \gamma\left(\eta, \mathrm{C}_{\mathrm{i}}\right)=\frac{1}{6 \eta}\left(\mathrm{C}_{1} \mathrm{f}_{0}^{\mathrm{i}-2}+\mathrm{C}_{2} \mathrm{f}_{0}^{\mathrm{j}-2}+\mathrm{C}_{3} \mathrm{f}_{0}^{\mathrm{k}-2}\right) \tag{27}
\end{align*}
$$

From Eqs.(23), (24), (26) and (27) we obtain

$$
\begin{align*}
& \mathrm{f}_{1}^{\prime}(\eta)=\frac{\mathrm{k}}{2}\left(\frac{\mathrm{~d}^{2}}{\eta}-\eta\right)-2 \beta\left[\mathrm{f}_{0}^{3}(\eta)+\right.  \tag{28}\\
& \left.+\mathrm{C}_{1} \mathrm{f}^{\mathrm{f}_{0}^{\mathrm{i}}}+\mathrm{C}_{2} \mathrm{f}_{0}^{\prime \mathrm{j}}+\mathrm{C}_{3} \mathrm{f}_{0}^{\mathrm{k}}\right]
\end{align*}
$$

For $\mathrm{i}=4, \mathrm{j}=5, \mathrm{k}=6$ and using Eq.(21), from Eq.(28) we obtain

$$
\begin{aligned}
& \mathrm{f}_{1}^{\prime}(\eta)=\frac{\mathrm{k}}{2}\left(\frac{\mathrm{~d}^{2}}{\eta}-\eta\right)-\frac{1}{4} \beta \mathrm{k}^{3}\left(\frac{\mathrm{~d}^{2}}{\eta}-\eta\right)^{3}- \\
& -\frac{1}{8} \beta \mathrm{k}^{4} \mathrm{C}_{1}\left(\frac{d^{2}}{\eta}-\eta\right)^{4}-\frac{1}{16} \beta \mathrm{k}^{5} \mathrm{C}_{2}\left(\frac{d^{2}}{\eta}-\eta\right)^{5}-
\end{aligned}
$$

$$
\begin{equation*}
-\frac{1}{32} \beta k^{6} C_{3}\left(\frac{d^{2}}{\eta}-\eta\right)^{6} \tag{29}
\end{equation*}
$$

The residual functional J given by Eq.(8) becomes:

$$
\begin{align*}
& \mathrm{J}=\int_{1}^{\mathrm{d}}\left\{\eta \mathrm{f}^{\prime \prime}(\eta)+\mathrm{f}_{1}^{\prime}(\eta)+\mathrm{k} \mathrm{\eta}+2 \beta\left[\mathrm{f}_{1}^{\prime 3}(\eta)+\right.\right.  \tag{30}\\
& \left.\left.+3 \eta \mathrm{f}_{1}^{\prime 2}(\eta) \mathrm{f}^{\prime \prime}(\eta)\right]\right\}^{2} \mathrm{~d} \eta
\end{align*}
$$

From Eqs.(9) for $\beta=\mathrm{k}=1$ we obtain

$$
\begin{align*}
& C_{1}=-0.00135812 ; C_{2}=-5.98314207 \\
& C_{3}=0.02175281 \tag{31}
\end{align*}
$$

Therefore, the explicit analytic expression given by Eq.(29) of the first-order approximate solution becomes:

$$
\begin{align*}
& \mathrm{f}_{1}(\eta)=0.5\left(\frac{\mathrm{~d}^{2}}{\eta}-\eta\right)-0.25\left(\frac{\mathrm{~d}^{2}}{\eta}-\eta\right)^{3}+ \\
& +0.000169765\left(\frac{d^{2}}{\eta}-\eta\right)^{4}+ \\
& +0.373946379\left(\frac{d^{2}}{\eta}-\eta\right)^{5}-  \tag{32}\\
& -0.000679775\left(\frac{d^{2}}{\eta}-\eta\right)^{6}
\end{align*}
$$

In [37], $\beta \geq 0.3$ is considered a parameter corresponding to strong nonlinearity.

In tables 1 and 2 is presented a comparison between the present solution obtained from formula (32) and the exact solution of Eq.(15).

It can be seen that the solution obtained through the present method is identical with that given by exact solution, demonstrating a very good accuracy.

| $\eta$ | $\bar{f}^{\prime}$ given by <br> Eq.(32) | $\mathrm{f}^{\prime}$ exact |
| :--- | :--- | :--- |
| 1 | 0.020183555 | 0.020183555 |
| 1.005 | 0.015105047 | 0.015105047 |
| 1.007 | 0.013079437 | 0.013079437 |
| 1.01 | 0.010047475 | 0.010047475 |
| 1.0105 | 0.009542918 | 0.009542918 |
| 1.0108 | 0.009240289 | 0.009240289 |

Table 1: Comparison between the present solution (32) and the exact solution for $\mathrm{d}=1.02$

| $\eta$ | f.'given by <br> $(46)$ | f'exact $^{\prime}$ |
| :--- | :--- | :--- |
| 1 | 0.040665504 | 0.040665504 |
| 1.008 | 0.032439661 | 0.032439661 |
| 1.016 | 0.024254926 | 0.024254926 |
| 1.023 | 0.017131195 | 0.017131195 |
| 1.03 | 0.010046515 | 0.010046515 |
| 1.038 | 0.00200191 | 0.00200191 |

Table 2: Comparison between the present solution (32) and the exact solution for $\mathrm{d}=1.04$

## 5 Basic ideas of the variational iteration method [24] and optimal variational iteration method [40]

Consider the following general non-linear system

$$
\begin{equation*}
\mathrm{Lu}+\mathrm{Nu}=\mathrm{g}(\mathrm{x}) \tag{33}
\end{equation*}
$$

where L is a linear operator and N is a non-linear operator.

Assuming $\mathrm{u}_{0}(\mathrm{x})$ is the solution of $\mathrm{Lu}=0$, one can write down an expression to correct the value of some special point, for example $x=1$

$$
\begin{equation*}
\mathrm{u}_{\text {correction }}(1)=\mathrm{u}_{0}(1)+\int_{0}^{1} \lambda\left(\mathrm{Lu}_{0} \mathrm{Nu}_{0}-\mathrm{g}(\mathrm{x})\right) \mathrm{dx} \tag{34}
\end{equation*}
$$

where $\lambda$ is a general Lagrange multiplier [21] which can be identified optimally via variational theory, the second term on the right is called the correction. J.H.He [22-24] modified the above method into an iteration method in the following way:

$$
\begin{equation*}
\mathrm{u}_{\mathrm{n}+1}\left(\mathrm{x}_{0}\right)=\mathrm{u}_{\mathrm{n}}\left(\mathrm{x}_{0}\right)+\int_{0}^{\mathrm{x}_{0}} \lambda\left(\mathrm{Lu}_{\mathrm{n}}+\mathrm{N} \widetilde{\mathrm{u}}_{\mathrm{n}}-\mathrm{g}\right) \mathrm{dx} \tag{35}
\end{equation*}
$$

with $\mathrm{u}_{0}(\mathrm{x})$ as initial approximation with possible unknowns, and $\widetilde{\mathrm{u}}_{\mathrm{n}}$ is considered as a restricted variation, i.e. $\delta \widetilde{u}_{n}=0$. For arbitrary $\mathrm{x}_{0}$ we can rewrite Eq.(35) as follows:

$$
\begin{equation*}
\mathrm{u}_{\mathrm{n}+1}(\mathrm{x})=\mathrm{u}_{\mathrm{n}}(\mathrm{x})+\int_{0}^{\mathrm{x}} \lambda\left(\mathrm{Lu}_{\mathrm{n}}(\mathrm{~s})+\mathrm{N} \widetilde{\mathrm{u}}_{\mathrm{n}}(\mathrm{~s})-\mathrm{g}(\mathrm{~s})\right) \mathrm{ds} \tag{36}
\end{equation*}
$$

Eq.(36) is called a correction functional. The variational iteration method has been shown to solve effectively, easily and accurately a large class of nonlinear problems with approximations converging rapidly to accurate solutions. For linear problems, exact solutions can be obtained by only one iteration step due to the fact that the Lagrange multiplier can be exactly identified.

Considering the following example

$$
\begin{equation*}
\mathrm{f}\left(\mathrm{x}, \mathrm{u}(\mathrm{x}), \mathrm{u}^{\prime}(\mathrm{x}), \mathrm{u}^{\prime \prime}(\mathrm{x})=0\right. \tag{37}
\end{equation*}
$$

where we consider the function $f$ twice differentiable with respect to all arguments and we assume that the boundary conditions are of the form

$$
\begin{equation*}
u(0)=u_{0}, u^{\prime}(0)=v_{0} \tag{38}
\end{equation*}
$$

Its correction functional can be written down as follows

$$
\begin{equation*}
\mathrm{u}_{\mathrm{n}+1}(\mathrm{x})=\mathrm{u}_{\mathrm{n}}(\mathrm{x})+\int_{0}^{\mathrm{x}} \lambda\left[\mathrm{f}\left(\mathrm{~s}, \mathrm{u}_{\mathrm{n}}(\mathrm{~s}), \mathrm{u}_{\mathrm{n}}^{\prime}(\mathrm{s}), \mathrm{u}_{\mathrm{n}}^{\prime \prime}(\mathrm{s})\right) \mathrm{ds}\right. \tag{39}
\end{equation*}
$$

With the notation

$$
\begin{align*}
& \mathrm{F}\left(\mathrm{x}, \mathrm{u}(\mathrm{x}), \mathrm{u}^{\prime}(\mathrm{x}), \mathrm{u}^{\prime \prime}(\mathrm{x})\right)=  \tag{40}\\
& =\lambda \mathrm{f}\left(\mathrm{x}, \mathrm{u}(\mathrm{x}), \mathrm{u}^{\prime}(\mathrm{x}), \mathrm{u}^{\prime \prime}(\mathrm{x})\right)
\end{align*}
$$

the variation of the functional

$$
\begin{equation*}
\mathrm{v}=\int_{0}^{\mathrm{x}} \mathrm{~F}\left(\mathrm{~s}, \mathrm{u}(\mathrm{~s}), \mathrm{u}^{\prime}(\mathrm{s}), \mathrm{u}^{\prime \prime}(\mathrm{s})\right) \mathrm{ds} \tag{41}
\end{equation*}
$$

symbolized by $\delta \mathrm{v}$, becomes [41]:

$$
\begin{align*}
& \delta \mathrm{v}=\delta \int_{0}^{\mathrm{x}} \mathrm{~F}\left(\mathrm{~s}, \mathrm{u}(\mathrm{~s}), \mathrm{u}^{\prime}(\mathrm{s}), \mathrm{u}^{\prime \prime}(\mathrm{s})\right) \mathrm{ds}= \\
& =\left.\left(\mathrm{F}_{\mathrm{u}^{\prime}}-\frac{\mathrm{d}}{\mathrm{ds}} \mathrm{~F}_{\mathrm{u}^{\prime \prime}}\right) \delta \mathrm{u}\right|_{0} ^{\mathrm{x}}+\left.\mathrm{F}_{\mathrm{u}^{\prime \prime}} \delta \mathrm{u}^{\prime}\right|_{0} ^{\mathrm{x}}+  \tag{42}\\
& +\int_{0}^{\mathrm{x}}\left(\mathrm{~F}_{\mathrm{u}}-\frac{\mathrm{d}}{\mathrm{ds}} \mathrm{~F}_{\mathrm{u}^{\prime}}+\frac{\mathrm{d}^{2}}{\mathrm{ds}^{2}} \mathrm{~F}_{\mathrm{u}^{\prime \prime}}\right) \delta \mathrm{uds}
\end{align*}
$$

Making the above functional stationary, noticing that $\delta \widetilde{\mathrm{u}}=0$ :

$$
\begin{align*}
& \delta u_{n+1}(x)=\delta u_{n}(x)+ \\
& +\delta \int_{0}^{x} F\left(s, u_{n}(s), u_{n}^{\prime}(s), u_{n}^{\prime \prime}(s)\right) d s=\delta u_{n}(x)+ \\
& +\left.\left(F_{u^{\prime}}-\frac{d}{d s} F_{u^{\prime}}\right) \delta u_{n}(s)\right|_{0} ^{x}+\left.F_{u^{\prime \prime}} \delta u_{n}^{\prime}(s)\right|_{0} ^{x}+ \\
& +\int_{0}^{x}\left[F_{u}\left(s, u_{n}(s), u_{n}^{\prime}(s), u_{n}^{\prime \prime}(s)\right)-\right.  \tag{44}\\
& -\frac{d}{d s} F_{u^{\prime}}\left(s, u_{n}(s), u_{n}^{\prime}(s), u_{n}^{\prime \prime}(s)\right)+ \\
& \left.+\frac{d^{2}}{d s^{2}} F_{u}^{\prime \prime}\left(s, u_{n}(s), u_{n}^{\prime}(s), u_{n}^{\prime \prime}(s)\right)\right] d s
\end{align*}
$$

yields the following stationary conditions

$$
\begin{aligned}
& \delta \mathrm{u}_{\mathrm{n}}: \mathrm{F}_{\mathrm{u}}\left(\mathrm{~s}, \mathrm{u}_{\mathrm{n}}, \mathrm{u}_{\mathrm{n}}^{\prime}, \mathrm{u}_{\mathrm{n}}^{\prime \prime}\right)-\frac{\mathrm{d}}{\mathrm{ds}} \mathrm{~F}_{\mathrm{u}^{\prime}}\left(\mathrm{s}, \mathrm{u}_{\mathrm{n}}, \mathrm{u}_{\mathrm{n}}^{\prime}, \mathrm{u}_{\mathrm{n}}^{\prime \prime}\right)+ \\
& +\frac{\mathrm{d}^{2}}{\mathrm{ds}^{2}} \mathrm{~F}_{\mathrm{u}^{\prime \prime}}\left(\mathrm{s}, \mathrm{u}_{\mathrm{n}}, \mathrm{u}_{\mathrm{n}}^{\prime}, \mathrm{u}_{\mathrm{n}}^{\prime \prime}\right)=0 \\
& \delta \mathrm{u}_{\mathrm{n}}^{\prime}:\left.\mathrm{F}_{\mathrm{u}^{\prime \prime}}\left(\mathrm{s}, \mathrm{u}_{\mathrm{n}}, \mathrm{u}_{\mathrm{n}}^{\prime}, \mathrm{u}^{\prime \prime}{ }_{\mathrm{n}}\right)\right|_{0} ^{\mathrm{x}}=0 \\
& \delta \mathrm{u}_{\mathrm{n}}: 1+\left.\left(\mathrm{F}_{\mathrm{u}^{\prime}}-\frac{\mathrm{d}}{\mathrm{ds}} \mathrm{~F}_{\mathrm{u}^{\prime \prime}}\right)\right|_{0} ^{\mathrm{x}}=0 \\
& \delta \widetilde{\mathrm{u}}_{\mathrm{n}}=0
\end{aligned}
$$

The Lagrange multiplier therefore can be readily identified from Eqs.(45). For Eqs.(15) and (16), the
correction functional becomes

$$
\begin{align*}
& \mathrm{f}_{\mathrm{n}+1}(\eta)=\mathrm{f}_{\mathrm{n}}(\eta)+\int_{\mathrm{d}}^{\eta} \lambda(\mathrm{s}, \eta)\left\{s \mathrm{f}_{\mathrm{n}}^{\prime \prime}(\mathrm{s})+\widetilde{\mathrm{f}}_{\mathrm{n}}^{\prime}(\mathrm{s})+\right.  \tag{46}\\
& \left.+\mathrm{ks}+2 \beta\left[\widetilde{\mathrm{f}}_{\mathrm{n}}^{\prime 3}(\mathrm{~s})+3 \mathrm{~s} \widetilde{\mathrm{f}}_{\mathrm{n}}^{\prime 2} \hat{\mathrm{f}}_{\mathrm{n}}^{\prime \prime}(\mathrm{s})\right]\right\} \mathrm{ds}
\end{align*}
$$

where ${ }^{\prime}=\frac{\mathrm{d}}{\mathrm{ds}}$.
The stationary conditions (45) in the case of Eq.(15) are $\left(\delta \widetilde{\mathrm{f}}_{\mathrm{n}}^{\prime}=\delta \widetilde{\mathrm{f}}_{\mathrm{n}}^{\prime \prime}=\delta \mathrm{f}_{\mathrm{n}}(0)=0\right)$ :

$$
\begin{align*}
& \frac{\partial \lambda(s, \eta)}{\partial s}+s \frac{\partial^{2} \lambda(s, \eta)}{\partial s^{2}}=0 \\
& {\left[1-\lambda(s, \eta)-s \frac{\partial \lambda(s, \eta)}{\partial s}\right] s=\eta=0} \tag{47}
\end{align*}
$$

$$
\lambda(s, \eta) \mid s=\eta=0
$$

The Lagrange multiplier is

$$
\begin{equation*}
\lambda(s, \eta)=1-\frac{\eta}{s} \tag{48}
\end{equation*}
$$

As a result, we obtain the following iteration formula

$$
\begin{align*}
& \mathrm{f}_{\mathrm{n}+1}(\eta)=\mathrm{f}_{\mathrm{n}}(\eta)+\int_{\mathrm{d}}^{\eta}\left(1-\frac{\eta}{\mathrm{s}}\right)\left\{\mathrm{sf}_{\mathrm{n}}^{\prime \prime}(\mathrm{s})+\mathrm{f}_{\mathrm{n}}^{\prime}(\mathrm{s})+\right. \\
& \left.+\mathrm{ks}+2 \beta\left[\mathrm{f}_{\mathrm{n}}^{\prime 3}(\mathrm{~s})+3 \mathrm{sf}_{\mathrm{n}}^{\prime 2}(\mathrm{~s}) \mathrm{f}_{\mathrm{n}}^{\prime \prime}(\mathrm{s})\right]\right\} \mathrm{ds} \tag{49}
\end{align*}
$$

Integrating part by part, from Eq.(49) we obtain the following two identities

$$
\begin{align*}
& \int_{d}^{\eta}\left(1-\frac{\eta}{s}\right)\left[s f_{n}^{\prime \prime}(s)+f_{n}^{\prime}(s)\right] d s=-\eta \int_{d}^{\eta} \frac{1}{s} f_{n}^{\prime}(s) d s \\
& \int_{d}^{\eta}\left(1-\frac{\eta}{s}\right)\left[s f_{n}^{\prime 3}(s)+3 s f_{n}^{\prime 2}(s) f_{n}^{\prime \prime}(s)\right] d s= \\
& =-\eta \int_{d}^{\eta} \frac{1}{s} f_{n}^{\prime 3}(s) d s \tag{51}
\end{align*}
$$

From Eqs. (50), (51) and (49) we obtain

$$
\begin{align*}
& f_{n+1}(\eta)=f_{n}(\eta)-\eta \int_{d}^{\eta} \frac{1}{s}\left[f_{n}^{\prime}(s)+2 \beta f_{n}^{\prime 3}(s)\right]-  \tag{52}\\
& -\frac{1}{2} k(d-\eta)^{2}
\end{align*}
$$

By differentiating into Eq.(52), we obtain the following iteration formula

$$
\begin{align*}
& f_{n+1}^{\prime}(\eta)=k(d-\eta)-2 \beta f_{n}^{\prime 3}(\eta)- \\
& -\int_{d}^{\eta} \frac{1}{s}\left[f_{n}^{\prime}(s)+2 \beta f_{n}^{\prime 3}(s)\right] d s \tag{53}
\end{align*}
$$

$$
\begin{align*}
& \mathrm{f}_{1}^{\prime}(\eta)=\mathrm{k}(\mathrm{~d}-\eta)-2 \beta \mathrm{f}_{0}^{\prime}(\eta)- \\
& -\int_{\mathrm{d}}^{\eta} \frac{1}{\mathrm{~s}}\left[\mathrm{f}_{0}^{\prime}(\mathrm{s})+2 \beta \mathrm{f}_{0}^{\prime 3}(\mathrm{~s})\right] \mathrm{ds} \tag{54}
\end{align*}
$$

where $f_{0}^{\prime}(\eta)$ is the initial approximation which must satisfy the boundary conditions (16):

$$
\begin{equation*}
\mathrm{f}_{0}(\mathrm{l})=0, \mathrm{f}_{0}^{\prime}(\mathrm{d})=0 \tag{55}
\end{equation*}
$$

In the case of VIM, $f_{0}^{\prime}(\eta)$ does not contain unknown parameters. In the case of OVIM, the initial approximation $f_{0}^{\prime}(\eta)$ is chosen such as it depends on a number of unknown constants. For example, if we take into account only the linear part of Eq.(15), we can write

$$
\begin{equation*}
n f^{\prime \prime}+f^{\prime}+k \eta=0, f^{\prime}(d)=0 \tag{56}
\end{equation*}
$$

Therefore, we obtain from Eq.(56) (see Eq,(19)(21)):

$$
\begin{equation*}
f_{0}^{\prime}=\frac{k}{2}\left(\frac{d^{2}}{\eta}-\eta\right) \tag{57}
\end{equation*}
$$

The initial approximation $f_{0}^{\prime}$ which appears in the right side of Eq.(54) is chosen in the form

$$
\begin{align*}
& f_{0}^{\prime}(\eta)=C_{1}\left(\frac{d^{2}}{\eta}-\eta\right)+C_{2}\left(\frac{d^{2}}{\eta}-\eta\right)^{2}+  \tag{58}\\
& +C_{3}\left(\frac{d^{2}}{\eta}-\eta\right)^{3}
\end{align*}
$$

The choose (58) of $f_{0}^{\prime}$ is not unique. This initial approximation can be chosen in the form:

$$
\begin{align*}
& f_{0}^{\prime}(\eta)=C_{1}\left(\frac{d^{2}}{\eta}-\eta\right)+C_{2}\left(\frac{d^{2}}{\eta}-\eta\right)^{3}+  \tag{59}\\
& +C_{3}\left(\frac{d^{2}}{\eta}-\eta\right)^{5}+C_{4}\left(\frac{d^{2}}{\eta}-\eta\right)^{7}
\end{align*}
$$

or

$$
\begin{align*}
& f_{0}^{\prime}(\eta)=C_{1}\left(\frac{d^{2}}{\eta}-\eta\right)+C_{2}\left(\frac{d^{2}}{\eta}-\eta\right)^{2}+  \tag{60}\\
& +C_{3}\left(\frac{d^{2}}{\eta}-\eta\right)^{4}
\end{align*}
$$

and so on.
At this moment, the constants $\mathrm{C}_{1}, \mathrm{C}_{2}$ and $\mathrm{C}_{3}$ are unknown.

Having in view Eq.(54), from Eq.(58) we obtain:

For $\mathrm{n}=0$ into Eq.(53) we have

$$
\begin{array}{ll}
f_{0}^{\prime}(s)+2 \beta f_{0}^{\prime 3}(s)=C_{1}\left(\frac{d^{2}}{s}-s\right)+C_{2}\left(\frac{d^{2}}{s}-s\right)^{2} & +6 \beta C_{2} C_{3}^{2}\left(-\frac{d^{16}}{8 \eta^{8}}+\frac{4 d^{14}}{3 \eta^{6}}-\frac{14 d^{12}}{3 \eta^{4}}+\right. \\
\left(C_{3}+2 \beta C_{1}^{3}\right)\left(\frac{d^{2}}{s}-s\right)^{3}+6 \beta C_{1}^{2} C_{2}\left(\frac{d^{2}}{s}-s\right)^{4}+ & +\frac{28 d^{10}}{\eta^{2}}+70 d^{8} \ln \eta-28 d^{6} \eta^{2}+7 d^{4} \eta^{4}- \\
+2 \beta\left(C_{1} C_{2}^{2}+C_{1}^{2} C_{3}\right)\left(\frac{d^{2}}{s}-s\right)+ & \left.-\frac{4}{3} d^{2} \eta^{6}+\frac{\eta^{8}}{8}-\frac{7}{3} d^{8}-70 d^{8} \ln d\right)+  \tag{62}\\
+2 \beta C_{2}^{3}\left(\frac{d^{2}}{s}-s\right)^{6}+6 \beta\left(C_{1} C_{3}^{2}+\right. & +2 \beta C_{3}^{3}\left(-\frac{d^{18}}{9 \eta^{4}}+\frac{9 d^{16}}{7 \eta^{7}}-\frac{36 d^{14}}{5 \eta^{5}}+\frac{28 d^{12}}{\eta^{3}}-\right. \\
\left.+C_{2}^{2} C_{3}\right)\left(\frac{d^{2}}{s}-s\right)^{7}+6 \beta C_{2} C_{3}^{2}\left(\frac{d^{2}}{s}-s\right)^{8}+ & -\frac{126 d^{10}}{\eta}-126 d^{8} \eta+28 d^{6} \eta^{3}-\frac{36}{5} d^{4} \eta^{5}+ \\
& \left.+\frac{9}{7} d^{2} \eta^{7}-\frac{\eta^{9}}{9}+\frac{64536 d^{9}}{315}\right)
\end{array}
$$

and therefore
$\int_{d}^{\eta} \frac{1}{s}\left[f_{0}^{\prime}(s)+2 \beta f_{0}^{\prime 3}(s)\right] d s=C_{1}\left(2 d-\eta-\frac{d^{2}}{\eta}\right)+$
$+C_{2}\left[d^{2}(1+2 \ln d)-2 d^{2} \eta-\frac{\eta^{2}}{2}-\frac{d^{4}}{2 \eta^{2}}\right]+$
$+\left(C_{3}+2 \beta C_{1}^{3}\right)\left(-\frac{d^{6}}{3 \eta^{3}}+\frac{3 d^{4}}{\eta}+3 d^{2} \eta-\frac{\eta^{3}}{3}-\frac{16 d^{2}}{3}\right)+$

$$
+2 \beta C_{3}^{3}\left(\frac{d^{2}}{s}-\mathrm{s}\right)^{9}
$$

$+6 \beta C_{1}^{2} C_{2}\left(-\frac{d^{8}}{4 \eta^{2}}+\frac{2 d^{6}}{\eta^{2}}+6 d^{4} \ln \eta-2 d^{2} \eta^{2}+\right.$
$\left.+\frac{\eta^{4}}{4}-6 \mathrm{~d}^{4}\right)+2 \beta\left(\mathrm{C}_{1} \mathrm{C}_{2}^{2}+\mathrm{C}_{1}^{2} \mathrm{C}_{3}\right)\left(-\frac{\mathrm{d}^{10}}{5 \eta^{5}}+\frac{5 \mathrm{~d}^{8}}{3 \eta^{3}}-\right.$
$\left.\frac{10 d^{6}}{\eta}-10 d^{4} \eta+\frac{5 d^{2}}{3} \eta^{3}-\frac{\eta^{5}}{5}+\frac{256 d^{5}}{15}\right)+$
$+2 \beta C_{2}^{3}\left(-\frac{d^{12}}{6 \eta^{6}}+\frac{3 d^{10}}{2 \eta^{4}}-\frac{15 d^{8}}{2 \eta^{2}}-20 d^{6} \ln \eta+\right.$
$\left.+\frac{15}{2} d^{4} \eta^{2}-\frac{3}{2} d^{2} \eta^{4}+\frac{\eta^{4}}{6}+20 d^{6}\right)+2 \beta\left(\mathrm{C}_{1} \mathrm{C}_{3}^{2}+\right.$
$\left.+\mathrm{C}_{2}^{2} \mathrm{C}_{3}\right)\left(-\frac{\mathrm{d}^{14}}{7 \eta^{7}}+\frac{7 \mathrm{~d}^{12}}{5 \eta^{5}}-\frac{7 \mathrm{~d}^{10}}{\eta^{3}}+\frac{35 \mathrm{~d}^{8}}{\eta}+35 \mathrm{~d}^{6} \eta-\right.$
$\left.-21 d^{4} \eta^{3}+7 d^{2} \eta^{5}-\frac{\eta^{7}}{7}-\frac{1754 d^{7}}{35}\right)+$

Substituting Eq.(62) into Eq.(54) we obtain

$$
\begin{aligned}
& f_{1}^{\prime}(\eta)=C_{1} \frac{d^{2}}{\eta}+\left(C_{1}-k\right) \eta+\left(k-2 C_{1}\right) d+ \\
& +C_{2}\left[\frac{d^{4}}{2 \eta^{2}}+\frac{\eta^{2}}{2}+2 d \ln \eta-d^{2}(1+2 \ln d)\right]- \\
& -2 \beta C_{1}^{3}\left(\frac{d^{2}}{\eta}-\eta\right)^{3}+6 \beta C_{1}^{3} C_{2}\left(\frac{d^{2}}{\eta}-\eta\right)^{4}-
\end{aligned}
$$

$$
-2 \beta\left(C_{1} C_{2}^{2}+C_{1}^{2} C_{3}\right)\left(\frac{d^{2}}{\eta}-\eta\right)^{5}-
$$

$-2 \beta\left(C_{1} C_{2}^{2}+C_{1}^{2} C_{3}\right)\left(\frac{d^{2}}{\eta}-\eta\right)^{5}-$

$$
-2 \beta C_{2}^{3}\left(\frac{d^{2}}{\eta}-\eta\right)^{6}-6 \beta\left(C_{1} C_{3}^{2}+\right.
$$

$-2 \beta C_{2}^{3}\left(\frac{d^{2}}{\eta}-\eta\right)^{6}-6 \beta\left(C_{1} C_{3}^{2}+\right.$

$$
+C_{2}^{2} C_{3}\left(\frac{d^{2}}{\eta}-\eta\right)^{7}+\left(C_{3}+2 \beta C_{1}^{3}\right)\left(\frac{d^{6}}{3 \eta^{3}}-\right.
$$

$+C_{2}^{2} C_{3}\left(\frac{d^{2}}{\eta}-\eta\right)^{7}+\left(C_{3}+2 \beta C_{1}^{3}\right)\left(\frac{d^{6}}{3 \eta^{3}}-\right.$

$$
\left.-\frac{3 d^{4}}{\eta}-3 d^{2} \eta+\frac{\eta^{3}}{3}+\frac{16 d^{2}}{3}\right)+
$$

$\left.-\frac{3 d^{4}}{\eta}-3 d^{2} \eta+\frac{\eta^{3}}{3}+\frac{16 d^{2}}{3}\right)+$

$$
+6 \beta C_{1}^{2} C_{2}\left(\frac{d^{8}}{4 \eta^{2}}-\frac{2 d^{6}}{\eta^{2}}-6 d^{4} \ln \eta+2 d^{2} \eta^{2}-\right.
$$

$$
\left.-\frac{\eta^{4}}{4}+6 \mathrm{~d}^{4}\right)+2 \beta\left(\mathrm{C}_{1} \mathrm{C}_{2}^{2}+\mathrm{C}_{1}^{2} \mathrm{C}_{3}\right)\left(\frac{\mathrm{d}^{10}}{5 \eta^{5}}-\frac{5 \mathrm{~d}^{8}}{3 \eta^{3}}+\right.
$$

$\left.-\frac{\eta^{4}}{4}+6 \mathrm{~d}^{4}\right)+2 \beta\left(\mathrm{C}_{1} \mathrm{C}_{2}^{2}+\mathrm{C}_{1}^{2} \mathrm{C}_{3}\right)\left(\frac{\mathrm{d}^{10}}{5 \eta^{5}}-\frac{5 \mathrm{~d}^{8}}{3 \eta^{3}}+\right.$

$$
\left.+\frac{10 d^{6}}{\eta}+10 d^{4} \eta-\frac{5 d^{2}}{3} \eta^{3}+\frac{\eta^{5}}{5}-\frac{256 d^{5}}{15}\right)+
$$

$\left.+\frac{10 d^{6}}{\eta}+10 d^{4} \eta-\frac{5 d^{2}}{3} \eta^{3}+\frac{\eta^{5}}{5}-\frac{256 d^{5}}{15}\right)+$

$$
+2 \beta C_{2}^{3}\left(\frac{d^{12}}{6 \eta^{6}}-\frac{3 d^{10}}{2 \eta^{4}}+\frac{15 d^{8}}{2 \eta^{2}}+20 d^{6} \ln \eta-\right.
$$

$+2 \beta C_{2}^{3}\left(\frac{d^{12}}{6 \eta^{6}}-\frac{3 d^{10}}{2 \eta^{4}}+\frac{15 d^{8}}{2 \eta^{2}}+20 d^{6} \ln \eta-\right.$

$$
\begin{aligned}
& \left.-\frac{15}{2} d^{4} \eta^{2}+\frac{3}{2} d^{2} \eta^{4}-\frac{\eta^{4}}{6}-20 d^{6}\right)+2 \beta\left(\mathrm{C}_{1} \mathrm{C}_{3}^{2}+\right. \\
& \left.+\mathrm{C}_{2}^{2} \mathrm{C}_{3}\right)\left(\frac{\mathrm{d}^{14}}{7 \eta^{7}}-\frac{7 d^{12}}{5 \eta^{5}}+\frac{7 \mathrm{~d}^{10}}{\eta^{3}}-\frac{35 \mathrm{~d}^{8}}{\eta}-35 \mathrm{~d}^{6} \eta+(63)\right. \\
& \left.+21 d^{4} \eta^{3}-7 d^{2} \eta^{5}+\frac{\eta^{7}}{7}+\frac{1754 d^{7}}{35}\right)+\ldots \ldots . .
\end{aligned}
$$

The residual functional J from Eq. (8) is

$$
\begin{align*}
& J=\int_{1}^{d}\left\{\eta f_{1}^{\prime \prime}(\eta)+f_{1}^{\prime}(\eta)+k \eta+2 \beta\left[f_{1}^{\prime 3}(\eta)+\right.\right.  \tag{64}\\
& \left.\left.+3 \eta f_{1}^{\prime 2}(\eta) f_{1}^{\prime \prime}(\eta)\right]\right\}^{2} d \eta
\end{align*}
$$

The constants $\mathrm{C}_{1}, \mathrm{C}_{2}$ and $\mathrm{C}_{3}$ are determined from Eqs.(9) and thus from the conditions

$$
\begin{equation*}
\frac{\partial \mathbf{J}}{\partial \mathrm{C}_{1}}=\frac{\partial \mathbf{J}}{\partial \mathrm{C}_{2}}=\frac{\partial \mathbf{J}}{\partial \mathrm{C}_{3}}=0 \tag{65}
\end{equation*}
$$

## 6 Numerical examples

The explicit analytic expression given by Eq.(63) contains the parameters $\mathrm{C}_{1}, \mathrm{C}_{2}$ and $\mathrm{C}_{3}$ which give the convergence region and rate of approximation for the OVIM. In order to prove the efficiency of the OVIM we consider different cases for some values of the parameters $\mathrm{k}, \beta$ and d .

### 6.1. The case $k=1, \beta=1$

Form the system (65) we obtain
$\mathrm{C}_{1}=0.5, \mathrm{C}_{2}=0.00001127, \mathrm{C}_{3}=0.74789281$
In tables 3 and 4 is presented a comparison between the present solution obtained from formula (63) and exact solution of Eq.(15) for $\mathrm{d}=1.02$ and $\mathrm{d}=1.04$ respectively. It can be seen that the solution obtained by the present method is identical with that given by the exact solution, demonstrating a very good accuracy.

| $\eta$ | $\overline{\mathrm{f}}^{\prime}$ given by Eq.(63) | $\mathrm{f}^{\prime}$ exact |
| :--- | :--- | :--- |
| 1 | 0.020183555 | 0.020183555 |
| 1.005 | 0.015105047 | 0.015105047 |
| 1.007 | 0.013079437 | 0.013079437 |
| 1.01 | 0.010047475 | 0.010047475 |
| 1.0105 | 0.009542918 | 0.009542918 |
| 1.0108 | 0.009240289 | 0.009240289 |

Table 3: Comparison between the present solution (63) and the exact solution for $k=\beta=1, d=1.02$

| $\eta$ | $\overline{\mathrm{f}}$ 'given by (63) | f'exact |
| :--- | :--- | :--- |
| 1 | 0.040665504 | 0.040665504 |
| 1.008 | 0.032439661 | 0.032439661 |
| 1.016 | 0.024254926 | 0.024254926 |
| 1.023 | 0.017131195 | 0.017131195 |
| 1.03 | 0.010046515 | 0.010046515 |
| 1.038 | 0.00200191 | 0.00200191 |

Table 4: Comparison between the present solution (63) and the exact solution for $\mathrm{k}=\beta=1, \mathrm{~d}=1.04$

Figures 1 and 2 present a comparison between the present solution given by Eq.(63) and the exact solution of Eq.(15) for $\mathrm{d}=1.02$ and $\mathrm{d}=1.04$, respectively.


Fig. 1 Comparison between the present solution (63) and the numerical solution for $k=1, \beta=1, d=1.02$


Fig. 2 Comparison between the present solution (63) and the numerical solution for $\mathrm{k}=1, \beta=1, \mathrm{~d}=1.04$

### 6.2. The case $k=1, \boldsymbol{\beta}=1.5$

In this case we obtain
$\mathrm{C}_{1}=0.5, \mathrm{C}_{2}=0.000012321, \mathrm{C}_{3}=0.74791023$ (64)
In tables 5 and 6 is presented a comparison between the present solution (63) and the exact solution for $\mathrm{d}=1.02$ and $\mathrm{d}=1.04$, respectively

| $\eta$ | $\overline{\mathrm{f}}^{\prime}$ given by Eq.(63) | $\mathrm{f}^{\prime}$ exact |
| :--- | :--- | :--- |
| 1 | 0.020175363 | 0.020175363 |
| 1.005 | 0.015101608 | 0.015101607 |
| 1.007 | 0.013077203 | 0.013077203 |
| 1.01 | 0.010046462 | 0.010046462 |
| 1.0105 | 0.009542049 | 0.009542049 |
| 1.0108 | 0.009239501 | 0.009239501 |

Table 5: Comparison between the present solution (63) and the exact solution for $k=1, \beta=1.2, d=1.02$

| $\eta$ | $\overline{\mathrm{f}}^{\prime}$ given by Eq.(63) | $\mathrm{f}^{\prime}$ exact |
| :--- | :--- | :--- |
| 1 | 0.040599239 | 0.040599241 |
| 1.008 | 0.032405844 | 0.032405844 |
| 1.016 | 0.024240732 | 0.024240731 |
| 1.023 | 0.017126181 | 0.017126181 |
| 1.03 | 0.010045502 | 0.010045502 |
| 1.038 | 0.002001902 | 0.002001901 |

Table 6: Comparison between the present solution (63) and the exact solution for $k=1, \beta=1.2, d=1.04$

It is easy to verify the accuracy of the obtained solutions if we graphically compare these analytical solutions with the exact ones.


Fig. 3 Comparison between the present solution (63) and the numerical solution for $\mathrm{k}=1, \beta=1.2, \mathrm{~d}=1.02$


Fig. 4 Comparison between the present solution (63) and the numerical solution for $\mathrm{k}=1, \beta=1.2, \mathrm{~d}=1.04$

## 8 Conclusions

In his paper, a new technique was proposed to solve the nonlinear problem of thin film flow of a fourth grade fluid down a vertical cylinder.

This procedure is very effective and has a distinct advantage over usual approximation methods in that it proves to be valid not only for weakly nonlinear equations, but also for highly complex nonlinear ones.

Convergence and errors are remarkable and this method provides a convenient way to control the convergence of approximate solution. This is realized using the auxiliary functions $\alpha\left(\eta, \mathrm{C}_{\mathrm{i}}\right)$ and $\lambda\left(\eta, \mathrm{C}_{\mathrm{j}}\right)$ used for adjusting and controlling the convergence of solution. These coefficients are determined by minimizing the residual square errors which is a very rigorous and effective procedure.

It is observed that we need only one iteration to obtain a remarkable accuracy. The results obtained through the proposed method reveal very good agreement with the exact results.

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