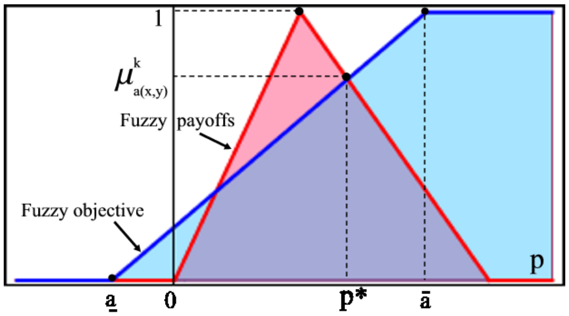

Figure 3: Degree of attainment of a fuzzy goal

Definition 15 *For any pair of strategies* $(\boldsymbol{x}, \boldsymbol{y})$*, an* ***attainment state of the fuzzy goal*** *is represented by the intersection of the fuzzy expected payoff* $\boldsymbol{x}'\widetilde{\boldsymbol{A}}^k\boldsymbol{y}$ *and the fuzzy goal* $\tilde{G}^k_1$*. We have*

$$\mu^k_{a(\boldsymbol{x},\boldsymbol{y})}(p) = \min\left\{\mu^k_{a(\boldsymbol{x}\widetilde{\boldsymbol{A}}^k\boldsymbol{y})}(p), \mu_{\tilde{G}^k_1}(p)\right\},$$

where $p \in D^k_1$ *is a payoff of Player I. The* ***degree of attainment*** *of the* kth *fuzzy goal for Player II is the maximum of the MF, such as*

$$\hat{\mu}^k_{a(\boldsymbol{x},\boldsymbol{y})}(p^*) = \max_p \mu^k_{a(\boldsymbol{x},\boldsymbol{y})}(p).$$

Similarly, the degree of attainment of the fuzzy goal for Player II is

$$\hat{\mu}^l_{b(\boldsymbol{x},\boldsymbol{y})}(p^*) = \max_p\left(\min\left\{\mu^l_{b(\boldsymbol{x}\widetilde{\boldsymbol{B}}^l\boldsymbol{y})}(p), \mu_{\tilde{G}^l_2}(p)\right\}\right).$$

The Figure 3 illustrates the concept.

3.3 Equilibrium solution

An equilibrium solution is defined w.r.t. the degree of attainment of the aggregated fuzzy goal.

Definition 16 *Let* $G = (S^m, S^n, \widetilde{\boldsymbol{A}}^k, \widetilde{\boldsymbol{B}}^l, k, l)$ *be a multiobjective fuzzy bimatrix game, and denote the degrees of attainment of the aggregated fuzzy goal for Players I and II by* $D^1(\boldsymbol{x},\boldsymbol{y})$ *and* $D^2(\boldsymbol{x},\boldsymbol{y})$*, respectively. The* ***equilibrium solution w.r.t. the degree of attainment of the aggregated fuzzy goal*** *is a pair of strategies* $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ *if, for all other strategies, we have*

$$\begin{aligned} D^1(\boldsymbol{x}^*,\boldsymbol{y}^*) &\geq D^1(\boldsymbol{x},\boldsymbol{y}^*), \text{ for all } \boldsymbol{x} \in S^m \\ D^2(\boldsymbol{x}^*,\boldsymbol{y}^*) &\geq D^2(\boldsymbol{x}^*,\boldsymbol{y}), \text{ for all } \boldsymbol{y} \in S^n. \end{aligned}$$

If the fuzzy goals are aggregated by a minimum component method, the classical decision rule by Bellman and Zadeh [2] is used [13]. This aggregation method consists in the intersection of all the fuzzy sets. The Player I's degree of attainment of the aggregated fuzzy goal is defined by

$$D^1(\mathbf{x},\mathbf{y}) = \min_{k\in\mathbb{N}_r} \frac{\mathbf{x}'(\mathbf{A}^k + \Delta^k_{\mathbf{A}})\mathbf{y} - \underline{a}^k}{\bar{a}^k - \underline{a}^k + \mathbf{x}'\Delta^k_{\mathbf{A}}\mathbf{y}}.$$

The Player I's programming problem using the kth payoffs is

$$\left.\begin{array}{r} \max_{\mathbf{x},\sigma} \quad \sigma \\ \text{subject to} \\ \frac{\mathbf{x}'(\mathbf{A}^k+\Delta^k_{\mathbf{A}})\mathbf{y}^* - \underline{a}^k}{\bar{a}^k - \underline{a}^k + \mathbf{x}'\Delta^k_{\mathbf{A}}\mathbf{y}^*} \geq \sigma, \\ \mathbf{e}'_m\mathbf{x} = 1, \\ \mathbf{x} \geq 0. \end{array}\right]$$

The Player II's programming problem for Player II using the lth payoffs is

$$\left.\begin{array}{r} \max_{\mathbf{y},\delta} \quad \delta \\ \text{subject to} \\ \frac{\mathbf{x}'^*(\mathbf{B}^l+\Delta^l_{\mathbf{B}})\mathbf{y} - \underline{b}^l}{\bar{b}^l - \underline{b}^l + \mathbf{x}'^*\Delta^l_{\mathbf{B}}\mathbf{y}} \geq \delta, \\ \mathbf{e}'_n\mathbf{y} = 1, \\ \mathbf{y} \geq 0. \end{array}\right]$$

Applying the Kuhn-Tucker necessary and sufficient conditions, we have the equivalence Theorem 17.

Theorem 17 (Equivalence Theorem) *Let* $G = (S^m, S^n, \widetilde{\boldsymbol{A}}^k, \widetilde{\boldsymbol{B}}^l)$ *be a multiobjective fuzzy bimatrix game, a necessary and sufficient condition that* $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ *be an equilibrium point is the solution of*

[13] One another method for aggregating multiple fuzzy goals is weighting the coefficients.

the nonlinear programming problem

$$
\begin{aligned}
\max_{\boldsymbol{x},\boldsymbol{y},\psi,\xi,\sigma,\delta,\Lambda,\Theta} \quad & \Bigg\{ \sum_{k=1}^{r} \lambda_k \Bigg[\frac{\underline{a}^k (2\boldsymbol{x}'\Delta_{\boldsymbol{A}}^k \boldsymbol{y} + \bar{a}^k - \underline{a}^k)}{(\bar{a}^k - \underline{a}^k + \boldsymbol{x}'\Delta_{\boldsymbol{A}}^k \boldsymbol{y})^2} \\
& - \frac{\boldsymbol{x}'\Delta_{\boldsymbol{A}}^k \boldsymbol{y} \times \boldsymbol{x}'(\boldsymbol{A}^k + \Delta_{\boldsymbol{A}}^k)\boldsymbol{y}}{(\bar{a}^k - \underline{a}^k + \boldsymbol{x}'\Delta_{\boldsymbol{A}}^k \boldsymbol{y})^2} \Bigg] + \sigma - \psi \\
& + \sum_{l=1}^{s} \theta_l \Bigg[\frac{\underline{b}^l (2\boldsymbol{x}'\Delta_{\boldsymbol{B}}^l \boldsymbol{y} + \bar{b}^l - \underline{b}^l)}{(\bar{b}^l - \underline{b}^l + \boldsymbol{x}'\Delta_{\boldsymbol{B}}^l \boldsymbol{y})^2} \\
& - \frac{\boldsymbol{x}'\Delta_{\boldsymbol{B}}^l \boldsymbol{y} \times \boldsymbol{x}'(\boldsymbol{B}^k + \Delta_{\boldsymbol{B}}^l)\boldsymbol{y}}{(\bar{b}^l - \underline{b}^l + \boldsymbol{x}'\Delta_{\boldsymbol{B}}^l \boldsymbol{y})^2} \Bigg] + \delta - \xi \Bigg\} \\
& \textit{subject to} \\
& \sum_{k=1}^{r} \lambda_k \Bigg[\frac{(\bar{a}^k - \underline{a}^k + \boldsymbol{x}'\Delta_{\boldsymbol{A}}^k \boldsymbol{y})\boldsymbol{A}_1^k \boldsymbol{y}}{(\bar{a}^k - \underline{a}^k + \boldsymbol{x}'\Delta_{\boldsymbol{A}}^k \boldsymbol{y})^2} \\
& + \frac{(\bar{a}^k - \boldsymbol{x}'\boldsymbol{A}^k \boldsymbol{y})(\Delta_{\boldsymbol{A}}^k)_1 \boldsymbol{y}}{(\bar{a}^k - \underline{a}^k + \boldsymbol{x}'\Delta_{\boldsymbol{A}}^k \boldsymbol{y})^2} \Bigg] - \psi \leq 0, \\
& \sum_{k=1}^{r} \lambda_k \Bigg[\frac{(\bar{a}^k - \underline{a}^k + \boldsymbol{x}'\Delta_{\boldsymbol{A}}^k \boldsymbol{y})\boldsymbol{A}_2^k \boldsymbol{y}}{(\bar{a}^k - \underline{a}^k + \boldsymbol{x}'\Delta_{\boldsymbol{A}}^k \boldsymbol{y})^2} \\
& + \frac{(\bar{a}^k - \boldsymbol{x}'\boldsymbol{A}^k \boldsymbol{y})(\Delta_{\boldsymbol{A}}^k)_2 \boldsymbol{y}}{(\bar{a}^k - \underline{a}^k + \boldsymbol{x}'\Delta_{\boldsymbol{A}}^k \boldsymbol{y})^2} \Bigg] - \psi \leq 0, \\
& \sum_{l=1}^{s} \theta_l \Bigg[\frac{(\bar{b}^l - \underline{b}^l + \boldsymbol{x}'\Delta_{\boldsymbol{B}}^l \boldsymbol{y})(\boldsymbol{B}_1^l)'\boldsymbol{x}}{(\bar{b}^l - \underline{b}^l + \boldsymbol{x}'\Delta_{\boldsymbol{B}}^l \boldsymbol{y})^2} \\
& + \frac{(\bar{b}^l - \boldsymbol{x}'\boldsymbol{B}^l \boldsymbol{y})(\Delta_{\boldsymbol{B}}^l)_1' \boldsymbol{x}}{(\bar{b}^l - \underline{b}^l + \boldsymbol{x}'\Delta_{\boldsymbol{B}}^l \boldsymbol{y})^2} \Bigg] - \xi \leq 0, \\
& \sum_{l=1}^{s} \theta_l \Bigg[\frac{(\bar{b}^l - \underline{b}^l + \boldsymbol{x}'\Delta_{\boldsymbol{B}}^l \boldsymbol{y})(\boldsymbol{B}_2^l)'\boldsymbol{x}}{(\bar{b}^l - \underline{b}^l + \boldsymbol{x}'\Delta_{\boldsymbol{B}}^l \boldsymbol{y})^2} \\
& + \frac{(\bar{b}^l + \boldsymbol{x}'\boldsymbol{B}^l \boldsymbol{y})(\Delta_{\boldsymbol{B}}^l)_2' \boldsymbol{x}}{(\bar{b}^l - \underline{b}^l + \boldsymbol{x}'\Delta_{\boldsymbol{B}}^l \boldsymbol{y})^2} \Bigg] - \xi \leq 0, \\
& \sum_{l=1}^{s} \theta_l \Bigg[\frac{(\bar{b}^l - \underline{b}^l + \boldsymbol{x}'\Delta_{\boldsymbol{B}}^l \boldsymbol{y})(\boldsymbol{B}_3^l)'\boldsymbol{x}}{(\bar{b}^l - \underline{b}^l + \boldsymbol{x}'\Delta_{\boldsymbol{B}}^l \boldsymbol{y})^2} \\
& + \frac{(\bar{b}^l + \boldsymbol{x}'\boldsymbol{B}^l \boldsymbol{y})(\Delta_{\boldsymbol{B}}^l)_3' \boldsymbol{x}}{(\bar{b}^l - \underline{b}^l + \boldsymbol{x}'\Delta_{\boldsymbol{B}}^l \boldsymbol{y})^2} \Bigg] - \xi \leq 0, \\
& \frac{\boldsymbol{x}'(\boldsymbol{A}^k + \Delta_{\boldsymbol{A}}^k)\boldsymbol{y} - \underline{a}^k}{\bar{a}^k - \underline{a}^k + \boldsymbol{x}'\Delta_{\boldsymbol{A}}^k \boldsymbol{y}} - \sigma \geq 0, \ k \in \mathbb{N}_r \\
& \frac{\boldsymbol{x}'(\boldsymbol{B}^l + \Delta_{\boldsymbol{B}}^l)\boldsymbol{y} - \underline{b}^l}{\bar{b}^l - \underline{b}^l + \boldsymbol{x}'\Delta_{\boldsymbol{B}}^l \boldsymbol{y}} - \delta \geq 0, \ l \in \mathbb{N}_s \\
& \boldsymbol{e}_m'\boldsymbol{x} = 1, \\
& \boldsymbol{e}_n'\boldsymbol{y} = 1, \\
& \boldsymbol{x} \geq 0, \ \boldsymbol{y} \geq 0, \ \Lambda \geq 0, \ \Theta \geq 0,
\end{aligned}
$$

where ψ, ξ are scalars and $\Lambda' = (\lambda_k)_{1\times 3}$, $\Theta' = (\theta_l)_{1\times 3}$, scalar entries. The vector $\boldsymbol{A}_i^k$, $i = 1, 2$ denotes the ith *row of the matrix $\boldsymbol{A}^k$ and similarly for the transposed matrix $(\boldsymbol{B}^l)_j'$, $j = 1, 2, 3$.*

Proof: see Nishizaki and Sakawa [17], pages 110–114. ■

3.4 Numerical example

In the following two players example [14][15], Players I and II have respectively two and three pure strategies and three different objectives. The goals of the two players are fuzzy. The payoffs are triangular fuzzy numbers. The LR-representation of the payoffs are the tensors $\widetilde{\mathbf{A}}^k \in \mathbb{R}^{2\times 3\times 3}$, $k \in \mathbb{N}_3$ and $\widetilde{\mathbf{B}} \in \mathbb{R}^{2\times 3\times 3}$, $l \in \mathbb{N}_3$ for Players I and II respectively, are

$$\widetilde{\mathbf{A}}_{LR}^1 = \begin{pmatrix} (1,.5,1) & (4,1,1) & (3,.5,1.5) \\ (2,1,1) & (4,.5,.5) & (1,1,1) \end{pmatrix},$$

$$\widetilde{\mathbf{A}}_{LR}^2 = \begin{pmatrix} (4,.5,1) & (3,1,1) & (2,1,.5) \\ (1,1,1) & (5,1,.5) & (1,.5,1) \end{pmatrix},$$

$$\widetilde{\mathbf{A}}_{LR}^3 = \begin{pmatrix} (2,1,1.5) & (0,0,1.5) & (1,.5,1) \\ (4,1.5,1.5) & (1,.5,.5) & (3,1,.5) \end{pmatrix}$$

and

$$\widetilde{\mathbf{B}}_{LR}^1 = \begin{pmatrix} (0,0,1) & (2,1.5,1) & (2,1,1) \\ (5,.5,1) & (5,1,1) & (1,.5,.5) \end{pmatrix},$$

$$\widetilde{\mathbf{B}}_{LR}^2 = \begin{pmatrix} (4,.5,1) & (2,1,1.5) & (5,1,.5) \\ (0,0,1) & (5,.5,.5) & (4,1.5,1) \end{pmatrix},$$

$$\widetilde{\mathbf{B}}_{LR}^3 = \begin{pmatrix} (2,1,1.5) & (1,.5,1) & (4,1,1.5) \\ (1,.5,.5) & (0,0,1.5) & (1,1,1) \end{pmatrix}.$$

The right spread matrices for Player I are

$$\Delta_{\mathbf{A}}^1 = \begin{pmatrix} 1 & 1 & 1.5 \\ 1 & .5 & 1 \end{pmatrix}, \Delta_{\mathbf{A}}^2 = \begin{pmatrix} 1 & 1 & .5 \\ 1 & .5 & 1 \end{pmatrix},$$

$$\Delta_{\mathbf{A}}^3 = \begin{pmatrix} 1.5 & 1.5 & 1 \\ 1.5 & .5 & .5 \end{pmatrix}.$$

The right spread matrices for Player II are

$$\Delta_{\mathbf{B}}^1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & .5 \end{pmatrix}, \Delta_{\mathbf{B}}^2 = \begin{pmatrix} 1 & 1.5 & .5 \\ 1 & .5 & 1 \end{pmatrix},$$

$$\Delta_{\mathbf{B}}^3 = \begin{pmatrix} 1.5 & 1 & 1.5 \\ .5 & 1.5 & 1 \end{pmatrix}.$$

The optimal solutions [16] of Player I are $x_1^* = .6438$ and $x_2^* = .3562$ w.r.t. a degree of attainment of the

[14] This numerical application is an extension of the Chen's example [6].

[15] The contribution by Keller [9] introduces to the fuzzy optimization techniques, using the software *MATHEMATICA*. Simple classic economic examples are analysed.

[16] The numerical solutions have been obtained using the primitive 'Minimize' of *MATHEMATICA* for a timing of 7 minutes 46 for an Intel(R) Corel(TM)2 CPU6400@2.13 GHz.

goal [17] of 58.5 per cent. The optimal solutions of Player II are $y_1^* = .5226$, $y_2^* = .3149$ and $y_3^* = .1625$ w.r.t. a degree of attainment of the goal of 52.5 per cent.

4 Conclusion

The crisp bimatrix games have an equivalent QP problem for finding Nash equilibrium solutions. The single objective fuzzy bimatrix game have an equivalent nonlinear programming problem. The multiple objective bimatrix games have an extended nonlinear programming problem. All these problems may be solve by different ways, by using algorithms and optimization techniques (Lemke-Howson's algorithm, multipliers in Varian [21], Van de Panne's two phase method [20], symmetric Zimmermann's approach [25, 26]), genetic algorithm in Wang *et al.* [24], the relaxation procedure for min-max problems subject to separate constraints (Shimizu and Aiyoshi [19]).

A Karush- Kuhn- Tucker (KKT) Optimality Conditions [3]

Let a nonlinear programming problem be (see Boyd and Vandenberghe [3])

$$\left.\begin{array}{r}\min_{\mathbf{x}}\ f_0(\mathbf{x})\\ \text{subject to}\\ f_i(\mathbf{x}) \le 0,\ i \in \mathbb{N}_m\\ h_j(\mathbf{x}) = 0,\ j \in \mathbb{N}_p\end{array}\right]$$

The optimization variables are $\mathbf{x} \in \mathbb{R}^n$, the objective function is $f_0 : \mathbb{R}^n \mapsto \mathbb{R}$, the m inequality constraints are $f_i(\mathbf{x}) \le 0,\ i \in \mathbb{N}_m$, and the p equality constraints are $h_j(\mathbf{x}) = 0,\ j \in \mathbb{N}_p$. All the functions $f_0, f_1, \ldots, f_m, h_1, h_2, \ldots, h_p$ are differentiable. The domain of the optimization problem is defined by

$$\mathcal{D} = \bigcap_{i=0}^{m} \textbf{dom}\ f_i \bigcap \bigcap_{j=0}^{p} \textbf{dom}\ h_j.$$

Associating the m-dimensional multiplier λ and the p-dimensional multiplier ν, we have the lagrangian

$$\mathcal{L}(\mathbf{x}, \lambda, \nu) = f_0(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i f_i(\mathbf{x}) + \sum_{j=1}^{p} \nu_j h_j(\mathbf{x}).$$

Since $\mathbf{x}^*$ minimizes $\mathcal{L}(\mathbf{x}, \lambda^*, \nu^*)$ over $\mathbf{x}$, it follows that

$$\nabla f_0(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(\mathbf{x}^*) + \sum_{j=1}^{p} \nu_j^* \nabla h_j(\mathbf{x}) = 0.$$

Thus we have the KKT conditions at the primal point x^* and dual points (λ^*, ν^*)

$$\left.\begin{array}{r} f_i(\mathbf{x}^*) \le 0,\ i \in \mathbb{N}_m\\ h_j(\mathbf{x}^*) = 0, j \in \mathbb{N}_p\\ \lambda_i^* \ge 0,\ i \in \mathbb{N}_m\\ \nabla f_0(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(\mathbf{x}^*) + \sum_{j=1}^{p} \nu_j^* \nabla h_j(\mathbf{x}) = 0,\end{array}\right]$$

Boyd and Vandenberghe [3](page 244) present the minimization quadratic problem

$$\left.\begin{array}{r}\min_{\mathbf{x}}\ \frac{1}{2}\mathbf{x}'\mathbf{P}\mathbf{x} + \mathbf{q}'\mathbf{x} + r\\ \text{subject to}\\ \mathbf{A}\mathbf{x} = \mathbf{b},\end{array}\right]$$

where we have $\mathbf{P} \in \mathbb{R}^{n\times n}$ a symmetric positive semi-definite matrix, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{A} \in \mathbb{R}^{m\times n}$. Let the multipliers be the vector ν, the KKT conditions are

$$\begin{array}{rcl}\mathbf{A}\mathbf{x}^* &=& \mathbf{b}\\ \mathbf{P}\mathbf{x}^* + \mathbf{q} - \mathbf{A}'\nu^* &=& 0.\end{array}$$

Thereafter, the optimal primal and dual variables are obtained by solving this set of $m+n$ equations in the $m+n$ variables $\mathbf{x}^*$ and ν^*.

B Proof of the Equivalence Theorem

The objectives of the Players I and II are achieved by solving the two programming problems, respectively

$$\left.\begin{array}{r}\max_{\mathbf{x}}\ \mathbf{x}'\mathbf{A}\mathbf{y}^*\\ \text{subject to}\\ \mathbf{e}_m'\mathbf{x} = 1,\\ \mathbf{x} \ge 0\end{array}\right]$$

and

$$\left.\begin{array}{r}\max_{\mathbf{y}}\ \mathbf{x}'^*\mathbf{B}\mathbf{y}\\ \text{subject to}\\ \mathbf{e}_n'\mathbf{y} = 1,\\ \mathbf{y} \ge 0\end{array}\right]$$

The equilibrium solution can be obtained by solving (see Chen [6], Mangasarian and Stone [13])

$$\left.\begin{array}{r}\max_{\mathbf{x},\mathbf{y}}\ \mathbf{x}'\mathbf{A}\mathbf{y}^* + \mathbf{x}'^*\mathbf{B}\mathbf{y}\\ \text{subject to}\\ \mathbf{e}_m'\mathbf{x} = 1,\\ \mathbf{e}_n'\mathbf{y} = 1,\\ \mathbf{x} \ge 0,\ \mathbf{y} \ge 0,\end{array}\right]$$

[17] We have

$$D^{1*} = \min_k \left\{ \frac{\mathbf{x}^*(\mathbf{A}^k + \Delta_{\mathbf{A}}^k)\mathbf{y}^* - \underline{a}^k}{\bar{a}^k - \underline{a}^k + \mathbf{x}^*\Delta_{\mathbf{A}}^k\mathbf{y}^*}, k \in \mathbb{N}_3 \right\} = .5840$$

.

Let $p = \max_{\mathbf{x}} \mathbf{x}'\mathbf{A}\mathbf{y}^*$ and $q = \max_{\mathbf{y}} \mathbf{x}'^*\mathbf{B}\mathbf{y}$. The following inequalities are also true $p \geq \mathbf{x}'\mathbf{A}\mathbf{y}^* \geq \mathbf{x}'\mathbf{A}\mathbf{y}$, for all $\mathbf{x} \geq 0$. So, we have the simplification $p\mathbf{e}_m \geq \mathbf{A}\mathbf{y}$. We also have the inequalities $q \geq \mathbf{x}'^*\mathbf{B}\mathbf{y} \geq \mathbf{x}'\mathbf{B}\mathbf{y}$, for all $\mathbf{y} \geq 0$. So, we have the simplification $q\mathbf{e}_n' \geq \mathbf{B}'\mathbf{x}$. The QP problem is

$$\left.\begin{array}{r}\min_{\mathbf{x},\mathbf{y},p,q}\ (p - \mathbf{x}'\mathbf{A}\mathbf{y}) + (q - \mathbf{x}'\mathbf{B}\mathbf{y}) \\ \text{subject to} \\ \mathbf{B}'\mathbf{x} \leq q\mathbf{e}_n', \\ \mathbf{A}\mathbf{y} \leq p\mathbf{e}_m, \\ \mathbf{e}_m'\mathbf{x} = 1, \\ \mathbf{e}_n'\mathbf{y} = 1, \\ \mathbf{x} \geq 0,\ \mathbf{y} \geq 0. \end{array}\right]$$

Then, the QP problem of the equivalence Theorem 2 is deduced ■

C Fuzzy decision sets

C.1 Bellman-Zadeh fuzzy decision rules

According to the Bellman-Zadeh symmetry principle, a fuzzy decision set is achieved by using an appropriate aggregation of the fuzzy sets.

Definition 18 *Let X be a set of possible actions, $\{\tilde{G}_j\ (j \in \mathbb{N}_n\}$ a set of fuzzy objectives, and $\{\tilde{C}_i\ (i \in \mathbb{N}_m\}$ the decision set is defined by*

$$\tilde{D} = \Big(\bigcap_{j=1}^{n} \tilde{G}_j\Big) \bigcap \Big(\bigcap_{i=1}^{m} \tilde{C}_i\Big),$$

with MF $\mu^1_{\tilde{D}} : X \mapsto [0,1]$ given by

$$\mu^1_{\tilde{D}}(x) = \Big(\bigwedge_{j=1}^{n} \mu_{\tilde{G}_j}(x)\Big) \bigwedge \Big(\bigwedge_{i=1}^{m} \mu_{\tilde{C}_i}(x)\Big).$$

The MFs of the aggregate fuzzy goal can be expressed as

$$\mu(x,y) = \min_{k \in \mathbb{N}_r} \Big\{ \mu_k(\mathbf{x}'\mathbf{A}^k\mathbf{y}) \Big\}.$$

Hence, we have with linear MFs

$$\mu(x,y) = \min_{k \in \mathbb{N}_r} \Big\{ \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{a^k_{ij}}{\bar{a}^k - \underline{a}^k} x_i y_j - \frac{\underline{a}^k}{\bar{a}^k - \underline{a}^k} \Big\}.$$

Considering the unequal importance of the fuzzy goals and constraints, Bellman and Zadeh also suggest another decision rule. This rule is defined the following convex combination of the fuzzy objective functions and constraints

$$\mu^2_D = \sum_{i=1}^{r} \alpha_i \mu_{G_i}(x) + \sum_{j=1}^{m} \beta_i \mu_{C_j}(x),$$

where all the nonnegative weighting coefficients α_i and β_j sum to one.

C.2 Product fuzzy decision set

The product fuzzy decision is an alternative decision set, defined by

$$\mu^3_D = \Big(\prod_{i=1}^{r} \alpha_i \mu_{G_i}(x)\Big) \times \Big(\prod_{j=1}^{m} \beta_i \mu_{C_j}(x)\Big),$$

C.3 Comparison of the fuzzy decision rules

The three types of decision sets are related by the inequalities

$$\mu^3_D(x) \leq \mu^1_D(x) \leq \mu^2_D(x).$$

The following example is taken from Sakawa [18] with one objective and one constraint. According to the fuzzy goal "x should be much larger than 10", and according to the fuzzy constraint "x should be substantially less or equal than 30". The MFs of the fuzzy objective and the fuzzy constraint are respectively defined by

$$\mu_G(x) = \begin{cases} 0, & x \leq 10 \\ 1 - \frac{1}{1+\left(\frac{x-10}{10}\right)^2}, & x > 10 \end{cases}$$

and

$$\mu_C(x) = \begin{cases} 0, & x \leq 30 \\ \frac{1}{1+\frac{x}{x-30}}, & x < 30 \end{cases}$$

The Figure 4 compares the fuzzy rules. In this examples, the maximum decisions are obtained for $(x^*_1, \mu^*_1) = (11.7549, .7549)$, $(x^*_2, \mu^*_2) = (11.3841, .8500)$ and $(x^*_3, \mu^*_3) = (11.4811, .6520)$.

D Fuzzy quadratic programming

The symmetric approach by Zimmermann [25] may be used for solving fuzzy programming problems. For this approach, membership functions are defined, by using a given aspiration level of the decision maker for the objective, and accepted tolerances for the objective and the constraint functions. An equivalent crisp QP problem is obtained with a quadratic constraint. This particular QP problem can be solved by using van de Panne 's two-phase method [20]

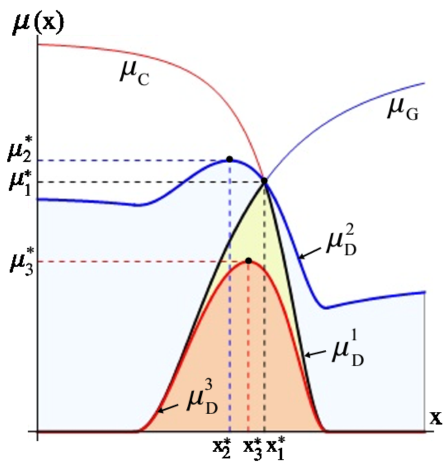

Figure 4: Fuzzy decision rules: intersection, convex and product fuzzy decision

D.1 Fuzzy QP problem

The fuzzy QP problem may be defined by a convex quadratic objective function together with a bounded feasible region such as in Bector and Chandra [1]

$$\left.\begin{array}{r} \widetilde{\min}_{\mathbf{x}}\ \mathbf{c}'\mathbf{x}+\frac{1}{2}\mathbf{x}'\mathbf{Q}\mathbf{x} \\ \text{subject to} \\ \mathbf{A}_i\mathbf{x} \lesssim b_i,\ i\in\mathbb{N}_m \\ \mathbf{x}\geq 0, \end{array}\right]$$

where $\mathbf{c},\ \mathbf{x}\in\mathbb{R}^n,\ \mathbf{b}\in\mathbb{R}^m,\ \mathbf{A}\in\mathbb{R}^{m\times n},\ \mathbf{Q}\in\mathbb{R}^{n\times n}$. The vector $\mathbf{A}_i$ denotes the *i*th row of matrix $\mathbf{A}$. The symmetric matrix $\mathbf{Q}$ is supposed to be positive semi definite.

D.2 Symmetric fuzzy QP problem

According to Zimmermann [25, 26], the symmetric version of the fuzzy QP problem is

$$\left.\begin{array}{r} \text{Find } \mathbf{x} \\ \text{such that} \\ \mathbf{c}'\mathbf{x}+\frac{1}{2}\mathbf{x}'\mathbf{Q}\mathbf{x} \gtrsim z_0, \\ \mathbf{A}_i\mathbf{x} \lesssim b_i,\ i\in\mathbb{N}_m \\ \mathbf{x}\geq 0, \end{array}\right]$$

where $z_0\in\mathbb{R}$ is the aspiration level of the DM and $p_0,\ p_i,\ i\in\mathbb{N}_m$ the tolerances for the objective and the set constraints, respectively. The membership function for the objective is defined by

$$\mu_0(z)=\left\{\begin{array}{l} 1,\ \ z<z_0, \\ \frac{(z_0+p_0)-z}{p_0},\ z\in[z_0,\leq z_0+p_0] \\ 0,\ z\geq z_0+p_0, \end{array}\right.$$

The membership function for the *i*th ($i\in\mathbb{N}_m$) constraint is also defined by

$$\mu_i(\mathbf{A}_i\mathbf{x})=\left\{\begin{array}{l} 1,\ \ \mathbf{A}_i\mathbf{x}<b_i, \\ \frac{(b_i+p_i)-\mathbf{A}_i\mathbf{x}}{p_i},\ \mathbf{A}_i\mathbf{x}\in[b_i,b_i+p_i] \\ 0,\ \mathbf{A}_i\mathbf{x}\geq b_i+p_i. \end{array}\right.$$

An optimal solution is obtained by solving the crisp equivalent QP problem

$$\left.\begin{array}{r} \text{Find } \alpha \\ \text{such that} \\ \mathbf{c}'\mathbf{x}+\frac{1}{2}\mathbf{x}'\mathbf{Q}\mathbf{x}+\alpha p_0\leq z_0+p_0, \\ \mathbf{A}_i\mathbf{x}+\alpha p_i\leq b_i+p_i,\ i\in\mathbb{N}_m \\ \mathbf{x}\geq 0,\ \alpha\in[0,1]. \end{array}\right]$$

D.3 Multiplier method

Let a nonlinear programming problem be defined as in Varian [21]

$$\min_{\mathbf{x}} f(\mathbf{x}) \text{ subject to } \mathbf{g}(\mathbf{x})=0 \text{ and } \mathbf{h}(\mathbf{x})\leq 0,$$

where $\mathbf{g}$, $\mathbf{h}$ are nonlinear vectorial functions and $\mathbf{x}$ a vector of variables. The multiplier method is based on the Uzawa algorithm, which is a dual gradient ascent algorithm. [18] The principle of the method may be described by the three steps:
i) predict the multipliers $\mathbf{p}^{(k)}$ and $\mathbf{q}^{(k)}$ that are associated with the constraints $\mathbf{g}(\mathbf{x})=0$ and $\mathbf{h}(\mathbf{x})\leq 0$,
ii) then, minimize $f(\mathbf{x})+\mathbf{p}^{(k)}\mathbf{g}(\mathbf{x})+\mathbf{q}^{(k)}\mathbf{h}(\mathbf{x})$,
iii) then, update until to convergence as $\mathbf{p}^{(k+1)}=\mathbf{p}^{(k)}+c_1\mathbf{g}(\mathbf{x}^{(k)})$ and $\mathbf{q}^{(k+1)}=\mathbf{q}^{(k)}+c_2\max\{0,\mathbf{h}(\mathbf{x}^{(k)})\}$, where the numbers $c_i, i=1,2$ are positive.

D.4 Numerical example

The following numerical example is taken from Bector and Chandra [1], pages 77-78. The fuzzy symmet-

[18] The multiplier method (also called augmented Lagrangian method) package of the software *MATHEMATICA* (see Varian [21]) uses the primitive $MultiplierMethod[f,\mathbf{g},\mathbf{h},\mathbf{x},\mathbf{x0},DualParameter\rightarrow True]$. This primitive is finding a local solution to a minimization problem where f is the criterion to be minimized, $\mathbf{g}$ a list (possibly empty) of equality constraints, $\mathbf{h}$ a list (possibly empty) of inequality constraints of the form $\mathbf{h}(\mathbf{x})\leq 0$, $\mathbf{x}$ the list of variables and $\mathbf{x0}$ the initial conditions for $\mathbf{x}$. It returns the list of results $\left\{f^*,\{x1\rightarrow x_1^*,\ldots\}\right\}$. The option DualParameter is providing information on feasibility and Lagrange and/or KKT multipliers.

ric QP problem is

$$\left.\begin{array}{r}\text{Find } (x_1, x_2)\\ \text{such that}\\ 2x_1 + x_2 + 4x_1^2 + 4x_1x_2 + 2x_2^2 \lesssim 51.88,\\ 4x_1 + 5x_2 \gtrsim 20,\\ 5x_1 + 4x_2 \gtrsim 20,\\ x_1 + x_2 \lesssim 30,\\ x_1,\ x_2 \geq 0.\end{array}\right]$$

Let the tolerances be $p_0 = 2.12,\ p_1 = 2,\ p_2 = 1,\ p_3 = 3$, the equivalent crisp QP problem is

$$\left.\begin{array}{r}\max \alpha\\ \text{subject to}\\ 2x_1 + x_2 + 4x_1^2 + 4x_1x_2 + 2x_2^2 + 2.12\alpha \leq 54,\\ 4x_1 + 5x_2 - 2\alpha \geq 18,\\ 5x_1 + 4x_2 - \alpha \geq 19,\\ x_1 + x_2 + 3\alpha \leq 33,\\ x_1,\ x_2 \geq 0,\\ \alpha \in [0, 1].\end{array}\right]$$

The optimum solution of the QP problem, given by the multiplier method is $x_1^* = .9918,\ x_2^* = 3.7253,\ \alpha^* = .8599$. This result tells that the solution is obtained with a satisfaction level of 86 per cent.

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