## Contribution of the Delay Differential Equations to the Complex Economic Macrodynamics

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*Abstract:* In some cases of the economic macrodynamics, delay differential equations (DDEs) may be more suitable to a wide range of economic models. The dynamics of the Kalecki's business cycle model is represented by a linear first-order DDE with constant coefficients, in the capital stock. Such a DDE, with constant or flexible lags, also occurs in the continuous time Solow's vintage capital growth model. This is due to the heterogeneity of goods and assets. DDEs with constant lags may be preferably solved by using Laplace transforms. Numerous techniques are also proposed for the solution of DDEs, like the inverse scattering method, the Jacobian elliptic function method, numerical techniques, the differential transform method, etc. This study retains the Zhou's differential transform method for solving nonlinear DDEs with backward-foreward delays and flexible coefficients. This study also uses a block diagram approach with application to reference economic models, with help of the software *MATHEMATICA 7.0.1* and its specialized packages for signal processing, such as "Control System Professional" and "SchematicSolver".

*Key–Words:* Delay differential equation, Method of steps algorithm, Differential transform method, Laplace transform solution, (x,k)-root plateau.

### **1** Introduction

The elementary theory of DDEs is introduced by solving and representing simple reference examples. *MATHEMATICA* plots will show how the parametrized solutions are generated. The technical aspects also concern the differential transform technique, for practical reasons. Two representative applications refer to economics: the earlier Kalecki's business cycle model and the Solow's vintage capital growth model. The Kalecki's model is a continuous-time system with a fixed delay (Allen [1], Kalecki [20]). The Solow's model is a continuous-time model with a flexible delay in Boucekkine *et al.* [8] and Bambi [4].

# 2 Elementary theory of functional differential equations

### 2.1 Delay differential equations

In 1963, the book of Bellman and Cooke [6] on the differential-difference equations is the first attempt in the study of more complex differential equations, that will better capture the real situations in many scientific domains. Nowadays, numerous contridutions in-

troduce and deepen the functional differential equations, such as with El'sgol' ts and Norkin [10], Hale and Verduyn Lunel [16].

**Definition 1** A delay differential equation (DDE) is a differential equation in which the time derivatives at the current time depends on the solution and possibly its derivatives at previous time. A neutral DDE (NDDE) of differential order n (the highest derivative) and difference order m (the distinct arguments less one) is written

$$y'(t) = F(t, y(t), y(t - \theta_1), \dots, y(t - \theta_n),$$
  
$$y'(t - \sigma_1), \dots, y'(t - \sigma_m)), t \ge t_0$$

given the initial history function

$$y(t) = \phi(t), \ t < t_0.$$

The sets of delays are such that

$$\{\theta_i > 0, i \in \mathbb{N}_n | \theta_1 < \ldots < \theta_n\}$$

and

$$\{\sigma_i > 0, i \in \mathbb{N}_m | \sigma_1 < \ldots < \sigma_m\}$$

The  $\theta$ 's (resp. the  $\sigma$ 's) delays may be constant ( $\theta = C$ ), time dependent ( $\theta(t)$ ) or time and state dependent

 $(\theta(t, y(t)))$ . Many dynamical processes involve time delays in engineering, nonlinear optical device, lasers dynamics, chemical kinetics, physiological systems, population dynamics, economics, etc. The interdisciplinary nature of the DDEs is illustrated by different contributions in Balachandran et al. [3]. These processes lead to models incorporating a dependence on the past history through the state variable. The linear form of a DDE of differential order n and difference order m with constant coefficients takes the form <sup>1</sup>

$$\sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} y^{(j)}(t - \omega_i) = f(t),$$

where  $y^{(j)}(t) \equiv d^j/dt^j y(t)$ .

The subclass of the linear first-order DDE will then be written

$$a_0y'(t) + a_1y'(t-\omega) + b_0y(t) + b_1y(t-\omega) = f(t),$$

where f(t) is assumed to be integrable and of boundary variation. Consider the linear DDE with a forcing term f(t)

$$y'(t) + ay(t) + by(t - \theta) = f(t),$$
 (1)

subject to the initial or boundary condition

$$y(t) = \phi(t)$$
, with  $t \in [-\theta, 0]$ . (2)

Suppose that f is of class  $C^1$  on  $[0,\infty)$  and  $\phi \in C^0$  on  $[0,\theta)$ .

**Theorem 2** (Existence and uniqueness) (Bellman and Cooke [6]). There exists one and only one continuous function for  $t \ge 0$ , which satisfies Eqs.(1-2) for  $t > \theta$ . Moreover, this function is of class  $C^1$  on  $(\theta, \infty)$  and of class  $C^2$  on  $(2\theta, \infty)$ . If  $\phi$  is of class  $C^1$ on  $[-\theta, 0]$ , y' is continuous at  $\theta$ , if and only if

$$\phi'(\theta - 0) + a\phi(\theta) + b\phi(0) = f(\theta).$$
(3)

If  $\phi$  is of class  $C^2$  on  $[-\theta, 0]$ , y'' is continuous at  $2\theta$  if either Eq.(3) holds or else b = 0.

<sup>1</sup>The general form of a NDDE is given by Bellman and Cooke [6]

$$F\left(t, y(t), y(t-\omega_1), \dots, y(t-\omega_m), \\ y'(t), y'(t-\omega_1), \dots, y'(t-\omega_m), \dots, \\ y^{(n)}(t), y^{(n)}(t-\omega_1), \dots, y^{(n)}(t-\omega_m)\right) = 0$$

*Proof.* See Bellman and Cooke [6], pages 50-51. ■ Basic reference equations are the DDEs of different type: the Frisch-Holme type, the Ikeda type, the Mackey-Glass type, the logistic type and the Lotka-Volterra system of DDEs. The Frisch-Holme type DDE [12] is of the form

$$y'(t) = -ay(t) - by(t - \theta),$$

where *a* and *b* are non-negative constants, and  $\theta$  a given positive delay. This DDE was inspired by the macroeconomic Kalecki model [20]. Frisch and Holme [12] discuss the characteristic solutions. The Ikeda type DDE [19]may be written in the simplified form <sup>2</sup>

$$y'(t) = \sin(t - \theta),$$

where the RHS is a delayed nonlinear feedback. The bifurcation phenomena of the system are discovered by Ikeda and Matsumoto [19] and Sprott [31]. The Mackey-Glass type DDE was introduced to model the physiological control systems (i.e. the production of white blood cells). The DDE takes the form <sup>3</sup>

$$y'(t) = a \frac{y(t-\theta)}{1+y(t-\theta)^n} - by(t),$$

where a, b are positive parameters and  $\theta$  a given delay. The study by Mackey and Glass [28] deals with a firstorder nonlinear DDE. The equation displays dynamic behavior including limit cycle oscillations, a variety of wave forms, aperiodic and chaotic solutions. The logistic function is frequently used in modeling the population growth of persons, animal species or other physiological members. The assumption is that the population grows with the population size, moderated by a competition factor. Indeed, as population grows, its members come into competition for food and other limited resources. Assuming y(y - 1)/2 interactions for a given population of size y, the logistic (non delayed) growth equation (see Shone [30], pages 593-603) is

$$y'(t) = ry(t) - r_1 \frac{y(t)(y(t) - 1)}{2}$$

<sup>2</sup>The Ikeda type DDE was proposed to model an optical bistable resonator system. The generalized form introduces a friction besides a delayed feedback, such as

$$y'(t) = -y(t) + \mu \sin(x(t - t_R) - x_0),$$

where x is the lag of the phase in the electric field,  $\mu$  the laser power intensity injected into the system,  $t_R$  the round trip time of the light in the resonator and  $x_0$  a constant.

<sup>3</sup>The generalization with infinite delay, given by Liz *et al.* [26], is the integro-differential equation

$$y'(t) = -\delta y(t) + \alpha \int_0^\infty \frac{dq(s)}{1 + y^n(t-s)}, \ t > 0, \ y \ge 0,$$

with  $\alpha$ ,  $\delta > 0$  and  $n \in (0, \infty)$ .

or

$$y'(t) = (a - by(t))y(t).$$

Introducing a positive delay, single species models (see Bocharov and Rihan [7], Kuang [25]) and economic time-to-build model may take the form of a DDE, such as for the Hutchinson's equation [18](also referred to as Wright equation [34]). The equation is written

$$y'(t) = r(1 - y(t - \theta)/K)y(t),$$

where the nonegative parameters r and K denote respectively, the intrinsic growth rate and the environment carrying capacity. Also biological problems give rise to Lotka-Volterra systems [27, 33] of logistic delayed DDEs, such as the two-dimensional system

$$\begin{aligned} y_1'(t) &= \left(-a_1 + b_1 y_2(t - \theta_2)\right) y_1(t), \\ y_2'(t) &= \left(-a_2 + b_2 y_1(t - \theta_1)\right) y_2(t), \end{aligned}$$

where  $a_1, b_1, a_2, b_2$  are positive constants, and  $\theta_1, \theta_2$  two delays. Assuming that two species of animal compete for food, the relative growth rate of each species is negatively related with a specific delay to the biomass of other species.

### 2.2 Method of Steps Algorithm

The Bellman's method of step is a numerical integration approach for DDEs. The theoretical aspects of such numerical methods are shown in Bellen and Zennaro [5]. Let a real-valued DDE with constant delay be

$$y'(t) = f(t, y(t), y(t-\theta)), \ t > \theta, \ \theta > 0,$$

where f is continuous in all arguments. The method of steps algorithm (MSA) consists in extending foreward an initial solution in the direction of increasing t (see Halanay [15]. The equation is solved for each meshpoint  $\{0, \theta, 2\theta, \ldots, k\theta, \ldots\}$ . Suppose that  $y't) = y_0(t)$  is given on  $[-\theta, 0]$ . The computation process consists of the following steps:

**Ist step:** a solution  $y(t) = y_1(t)$  is determined on  $[0, \theta]$  by solving, analytically or numerically, an ordinary differential equation (ODE) with the initial condition y(0). We have

$$y'(t) = f(t, y(t), y_0(t - \theta)), t \in [0, \theta]$$
  

$$y(0) = y_0(\theta), t \in [-\theta, 0].$$

**2nd step** : a solution  $y(t) = y_2(t)$  is determined on

 $[\theta, 2\theta]$  by solving an ODE with the initial condition  $y(\theta)$ . We have

$$y'(t) = f(t, y(t), y_1(t - \theta)), t \in [\theta, 2\theta]$$
  

$$y(\theta) = y_1(\theta), t \in [0, \theta].$$
  
...

**kth step** : a solution  $y(t) = y_k(t)$  is determined on  $[(k-1)\theta, k\theta]$  by solving an ODE with the initial condition  $y((k-1)\theta)$ . We have

$$y'(t) = f(t, y(t), y_{k-1}(t-\theta)), t \in [(k-1)\theta, k\theta],$$

with

$$y((k-1)\theta) = y_{k-1}(\theta), \ t \in [(k-2)\theta, (k-1)\theta].$$

### 2.2.1 Dynamics of a Frisch-Holme type DDE

A Frisch-Holme type DDE  $[12]^4$  is defined by

$$y'(t) = -ay(t) - by(t - \theta)$$

With the parameter values  $a = 0, b = -1, \theta = 1$ , the DDE is now

$$y'(t) = y(t-1).$$

Let the initial function be simply set to  $y_0(t) = 1$ , at period 0. The ODE to solve at next period will be  $y'(t) = y_0(t-1), y(1) = y_0(1)$ , at period 1, and that of the next period will be  $y'(t) = y_1(t-1), y(2) =$  $y_1(2)$ , a.s.o.

Suppose that y(t) = 1 for  $t \in (0, 1]$ . If y'(t) = y(t-1) is to hold for t > 1, the values of y'(t) for  $t \in (1, 2)$  are determined. Since y(t) is required to be continuous at t = 1, these values determine y(t) for  $t \in (1, 2)$ , we have

$$y(t) = 1 + (t - 1) = t, t \in [1, 2].$$

Extending the definition of y(t) from one interval to the next, we find the expression <sup>5</sup>

$$y(t) = \sum_{k=0}^{n} \frac{(t-k)^k}{k!}, \ t \in [n, n+1], \ n \in \mathbb{N}_0.$$

<sup>4</sup>Kalecki [20] early retained a similar DDE for the macroeconomic system

$$y'(t) = \frac{m}{\theta}y(t) - \frac{m+n\theta}{\theta}y(t-\theta),$$

for which a main cyclical solution is obtained for the parameter values: m = .95, n = .121 and  $\theta = .6$  (7 months). In the Kalecki's model, y(t) denotes a deviation of investment from the constant demand for restoration of equipement. The parameters m and n are coefficients, providing from a relation between the proportion of investment and the expected net yield.

<sup>5</sup>In Appendix A, the general solution is obtained by applying Laplace transforms to a similar DDE with different values for *b*. The Tinbergen's shipbuilding cycle model, in Appendix C, also uses this type of DDE for a value of the delay which differs from unity.

Let two simple real-valued DDEs with constant unit delay be DDE1 for y'(t) = y(t-1) and DDE2 for y'(t) = -y(t-1) with history  $y_0(t) = 1$  on [-1,0]for both equations. The first three steps are presented in Figure 1.

interval	equation DDE1	equation DDE2
[-1,0]	1	1
[0,1]	1+t	1-t
[1,2]	$\frac{1}{2}(3+t^2)$	$\frac{1}{2}(3-4t+t^2)$
[2,3]	$\frac{1}{6}(1+12t-3t^2+t^3)$	$\frac{1}{6}(17-24t+9t^2-t^3)$

Figure 1: Solution of equations DDE1 and DDE2 for the first three steps

For each case, the solutions are shown in Figure 2. The cartesian plots show the integral solution in y(t), the velocity y'(t) and acceleration y''(t). The phase plots in (y(t-1), y(t)) plane show different dynamics of the two simple models. Figure 3 allows the evalu-



Figure 2: Solutions for y'(t) = y(t-1) (DDE1) and y'(t) = -y(t-1) (DDE2)

ation of the Frisch-Holme dynamics, with parameters a and b, within some ranges. The solutions are explored for different values of the coefficients a and b, of the delay  $\theta$ , and for different history functions, through the control of sliders. An additional discrete slider allows the choice between the cartesian and the parametric plot. Theoretical studies show that delayed



Figure 3: Frisch-Holme dynamics

feedback mechanisms or multiple feedback loops may generate complex periodic, aperiodic or chaotic dynamics (in Glass *et al.* [13]).

### 2.2.2 Stability of the Frisch-Holme DDE

The stability of all solutions is achieved asymptotically, provided that all the characteristic roots lie in the negative complex left half-space (LHP) (in Bellman and Cooke [6], Kolmanovskii and Nosov [22], Hairer *et al.* [14], Kolmanovskii and Myskis [23, 24]). Searching for solutions of the form

$$y(t) = ce^{\rho t}$$
, with  $\rho = \alpha + j\beta$ ,  $j = \sqrt{-1}$  (4)

leads to the characteristic equation

$$D(\rho) \equiv \rho + a + be^{-\rho\theta} = 0, \tag{5}$$

which possesses many solutions for  $b \neq 0$ . We are searching for (a, b)-values for which  $D(\rho) = 0$ . A zero root is obtained for a + b = 0. The purely imaginary root is  $\rho = j\omega$ , for  $\omega$  real. Inserting the complex  $\rho$  in Eq.(4) into Eq.(5), and separating the real and imaginary parts, we obtain a parametric function in  $\omega$  for a and b, and a fixed delay  $\theta$ . The parametric equations are defined by

$$a = -\omega \cot \omega \theta, \tag{6}$$

$$b = \omega \sec \omega \theta. \tag{7}$$

The spectrum of roots of  $D(\rho)$  in the complex plane is plotted <sup>6</sup> in the LHS Figure 4 for (a, b) = (-.5, 1). All the the roots stay in the LHP. The RHS Figure 4 shows a unique stability region in the (a,b)-plane. At the point C(-.5, 1), in the stability region  $\Gamma_0$ , all characteristic roots of the transcendental function are negative (see the LHS Figure 4).

**Definition 3** (*D*-Subdivision). Given a characteristic equation of a DDE with constant coefficients, a D-subdivision is a partition of the coefficients (a, b)plane into regions by hypersurfaces, the point of which corresponds to quasi-polynomials having at least one zero on the imaginary axis.

The boundaries of the D-Subdivision are plotted in the RHS Figure 4. One boundary is the line a + b = 0 for a real zero root. The other boundaries are defined by the parametric functions Eqs.(6-7) in  $\omega$  for a given delay  $\theta$ . A boundary is drawn for any of the open intervals  $(0, \frac{\pi}{\theta}), (\frac{\pi}{\theta}, \frac{2\pi}{\theta}), (\frac{2\pi}{\theta}, \frac{3\pi}{\theta}), \ldots$  The solutions are either monotonic or oscillatory. The D-Subdivision method determines the number of roots having posisitive real part (p-zeros) in accordance with the coefficients a and b. For continuous variation of the coefficients, the number of the p-zeros may change when passing accross the boundary of a region of the D-Subdivision. To every region  $\Gamma_k$ , it is possible to assign a number k of p-zeros, as in the RHS Figure 4. The regions  $\Gamma_0$  (if they exist) are the regions of asymptotic stability of solutions. There is one such shaded stability region in Figure 4. The extension to a linear (linearized) system of DDEs introduces the matrix equation (Engelborghs and Roose [11])

$$\mathbf{y}'(t) = \mathbf{A}_0 \mathbf{y}(t) + \sum_{i=1}^m \mathbf{A}_i \mathbf{y}(t-\theta_i), \ \mathbf{A}_i \in \mathbb{R}^{n \times m}, \ i \in \mathbb{N}_m,$$

whose characteristic equation reads

$$\det(\mathbf{D}(\rho)) = \rho \mathbf{I} - \mathbf{A}_0 - \sum_{i=1}^m \mathbf{A}_i e^{-\rho \theta_i} = 0.$$

### 2.3 Differential Transform Method

The approximate solution of a DDE by using the Zhou's differential transform method (DTM) was ex-

$$\begin{cases} a + \alpha + be^{-\alpha} \cos \beta &= 0, \\ \beta - be^{-\alpha} \sin \beta &= 0. \end{cases}$$

The roots can be computed by using a Newton-Raphson method, with adequate starting values.



Figure 4: Asymptotic stability of solutions with D-Subdivision.

tended by Arikoglu and Ozkol [2]. The DTM approach is used to solve linear and nonlinear DDEs with variable coefficients.

### 2.3.1 Introduction to the method

The transformation of the *k*th derivative of a function f(t) is

$$F(k) = \frac{1}{k!} f^{(k)}(t)|_{t=t_0}$$

and the inverse transformation is defined by

$$f(t) = \sum_{k=0}^{\infty} F(k)(t-t_0)^k.$$

The transformation of usual functions is presented in Appendix B. Let a DDE take the general form

$$f\left(g^{(n_1)}(t+\theta_1), g^{(n_2)}(t+\theta_2), \dots, g^{(n_p)}(t+\theta_p)\right) = 0,$$
(8)

with the boundary conditions (BCs)

$$g^{(a_i)}(t)|_{t=b_i} = c_i, \ i \in \mathbb{N}_m.$$

The differential transformation  $\mathcal{D}[.]$  of the BCs is given by

$$\sum_{k=0}^{T} \frac{k!}{(k-a_i)!} G(k) (b_i - t_0)^{k-a_i} = c_i, \ i \in \mathbb{N}_m.$$

**Theorem 4** (Arikoglu and Ozhol [2]). Let  $G(k) = \mathcal{D}[g(t)]$ , the solution of Eq.(8) depends on the solution of the unknown coefficients  $G(0), G(1), \ldots, G(T)$ . We have

$$F_k\left(G(0), G(1), \dots, G(T)\right) = 0, \ k \in \mathbb{N}_{0, T-m}.$$
(9)

<sup>&</sup>lt;sup>6</sup>The characteristic equation  $D(\rho) = 0$  with  $\rho = \alpha + \beta j$ ,  $j = \sqrt{-1}$  is numerically solved for (a, b) = (-.5, 1). The set of equations to be solved is

Eq.(9) consists in m equations, the remaining T-m+1 standing from the transformation of Eq.(8).

*Proof.* See Arikoglu and Ozkhol [2]. ■

### 2.3.2 Application to the Kalecki's system

The continuous-time Kalecki's business cycle system [20] is

$$Y(t) = C(t) + I(t),$$
 (10)

$$C(t) = cY(t) - u(t), \ c \in (0,1),$$
(11)

$$I(t) = \frac{1}{\theta} \int_{t-\theta}^{t} B(\tau) d\tau, \qquad (12)$$

$$B(t) = \lambda (vY(t) - K(t)), \ \lambda \in (0, 1), \ (13)$$
  

$$K'(t) = B(t - \theta).$$
(14)

The aggregate demand in Eq.(10) of consumption C and investment outlays I equals the total revenue Y. In Eq.(11), consumption is proportional to the total revenue and is influenced by a stabilization policy u(t). In Eq.(12), the investment orders depend on the past investment decisions B with a fixed delay of  $\theta$ . The determination of the investment decisions, in Eq.(13), is proportional to the gap between the desired level of equipment vY(t) and the existing capital stock K. According to Eq.(14), a fixed delay separates the deliveries of capital goods from the orders. After some algebraic manipulations <sup>7</sup>, the system may be reduced to a linear first-order DDE with constant coefficients, in the variable K(t)

$$K'(t) = aK(t) - bK(t - \theta), \ K(0) = 1,$$
 (15)

where  $a = \frac{\lambda v}{\theta(1-c)}$  and  $b = \lambda \left(1 + \frac{v}{\theta(1-c)}\right)$ . Using the Table B.1, we obtain the transformation of Eq.(15)

$$(k+1)\bar{K}(k+1) - a\bar{K}(k) + b\sum_{h_1=k}^T \binom{h_1}{k} (-1)^{h_1-k} - \bar{K}(h_1) = 0.$$

The unique BC is also transformed to  $\overline{K}(0) = 1$ . Taking for numerical values  $\lambda = 2/5, \ \theta = 1, \ c =$ 3/4, v = 1/2, the coefficients are a = .8 and b = 1.2. Choosing T = 4, with the BC and k = 0, 1, 2, 3 and taking for numerical values, a linear system in the

$$I(t) = \theta^{-1} \int_{t}^{t+\theta} K'(x) dx = \theta^{-1} \big( K(t+\theta) - K(t) \big),$$
  
ere  $x = \tau + \theta.$ 

variables  $\bar{K}(1), \bar{K}(2), \bar{K}(3), \bar{K}(4)$  is solved. We have the system

$$\begin{aligned} 2\bar{K}(1) &- 1.2\bar{K}(2) + 1.2\bar{K}(3) - 1.2\bar{K}(4) &= .4, \\ .4\bar{K}(1) &- .4\bar{K}(2) + 3.6\bar{K}(3) - 4.8\bar{K}(4) &= 0, \\ .4\bar{K}(2) &- .6\bar{K}(3) + 7.2\bar{K}(4) &= 0, \\ .4\bar{K}(2) &- .8\bar{K}(4) &= 0. \end{aligned}$$

We obtain the solution

- . .

$$\bar{K}(1) = -.56, \bar{K}(2) = -.4, \bar{K}(3) = .0533,$$
  
 $\bar{K}(4) = .0266.$ 

Using the inverse transformations from Table B.1, the series solution for the problem is divergent with

$$K(t) = 1 - 1.56t - .4t^2 + .0533t^3 + .0266t^4 + \mathcal{O}(t^5).$$

#### 3 Generalized DDEs for economic systems

### 3.1 Kalecki's business cycle model with discrete delay

The Laplace transformed variables X(t) are written  $\mathcal{L}[X(t)] = \overline{X}(s)$ . The Laplace transform uses notably the following property that<sup>8</sup>.

$$\mathcal{L}[f(t+\theta)] = e^{s\theta} \mathcal{L}[f(t)].$$

The Laplace transform of the system Eqs.(10-14) is

$$\begin{split} \bar{Y}(s) &= \frac{\bar{I}(s)}{1-c} - \frac{\bar{U}(s)}{1-c}, \\ \bar{I}(s) &= \frac{1}{\theta} (e^{\theta s} - 1) \bar{K}(s), \\ \bar{B} &= \lambda v \bar{Y}(s) - \lambda \bar{K}(s), \\ s \bar{K}(s) &= e^{-\theta s} \bar{B}(s), \end{split}$$

where the initial conditions are zero and where  $\overline{U}(s) = \mathcal{L}[u(t)]$  denotes the input of the system. Solving the system w.r.t.  $\bar{K}(s)$ , the transfer function is

$$\frac{K(s)}{\bar{U}(s)} = \frac{\phi(s,\theta)\lambda v}{(1-c)\left(s+\phi(s,\theta)\lambda - \frac{(1-\phi(s,\theta))\lambda v}{\theta(1-c)}\right)},$$

<sup>8</sup>In fact, we have

$$\begin{split} \mathcal{L}[f(t+\theta)] &= \int_0^\infty e^{-st} f(t+\theta) dt \\ \text{or} \\ &e^{st} \int_0^\infty e^{-st_1} f(t_1) dt_1 = e^{s\theta} \mathcal{L}[f(t)], \\ \text{where } t_1 &= t+\theta. \text{ QED} \blacksquare \end{split}$$

wh

or

<sup>&</sup>lt;sup>7</sup>The integral Eq.(12) is also written as

where  $\phi(s,\theta) = e^{-\theta s}$  is approximated by the Taylor's series  $1 - \theta s + \frac{\theta^2 s^2}{2} + \mathcal{O}(s^3)$ . Taking the parameter values  $\lambda = 2/5, \ \theta = 1, \ c = 3/4, \ v = 1/2$  and a unit delay  $\theta = 1$ , the transfer function is

$$H(s) = 2\frac{\frac{s^2}{2} - s + 1}{\frac{3s^2}{2} - \frac{s}{2} + 1}.$$

The constant is 2 (about 6 dB), the zeros are the complex conjugates  $1\pm j$ , so as the poles  $\frac{1}{6}(1\pm j\sqrt{23})$ . Using the *MATHEMATICA* package "SchematicSolver", the system may be represented by the block-diagram of the Figure 5. The system has two positive feedback loops, for the consumption and the investment block G(s).



Figure 5: Block-diagram of the Kalecki's model

The Bode diagrams in Figure 6 of the transfer function show the response of the system to a sine signal. The impact of a shorter/longer time delay is illustrated in Figure 7 $^{9}$ .

# **3.2** Solow's vintage capital growth model with variable delay

The purpose of a growth model with heterogeneous productive capital, is to determine the optimal age structure of machines by taking adequate investment decisions.

### **3.2.1** Equations of the system

The Solow's vintage capital growth system consists of four equations. The first two equations describe the



Figure 6: Bode diagrams for different delays

<sup>&</sup>lt;sup>9</sup>The block-diagram approach has been applied by Keller [21] to other macroeconomic models, such as with the linear Phillips' model and the nonlinear Goodwin model, to the study of the stabilization effects of economic policies in an uncertain environment.



Figure 7: Impact of time delay on simulation results

vintage technology <sup>10</sup> and the market good equilibrium. The next two equations concern the labor market (labor demand and equilibrium). We have the system<sup>11</sup>

$$y(t) = \int_{t-T(t)}^{t} i(\tau) d\tau, \qquad (16)$$

$$l(t) = \int_{t-T(t)}^{t} i(\tau) e^{-a(\tau)\tau} d\tau, \qquad (17)$$

$$i(t) = (1-c)y(t),$$
 (18)

$$l(t) = 1.$$
 (19)

Eq.(16) represents an AK technology (A = 1) in the vintage productive capital, where all existing machines are supposed to be in use. The variables are defined by: production y, investment i, age T of the oldest machine, and  $\tau$  a generation of machines. Eq.(17) is the labor demand l(t). Each machine at t requires  $e^{-a(t)t}$  workers, and new machines are more productive, since the element a(t)t with (a(t)t)' > 0 denotes the technological progress. According to the equilibrium condition Eq.(18) of the good market, a fixed proportion 1 - c of income is saved and totally invested. Since the labor supply is assumed to be constant over time, the equilibrium condition on the labor market is defined by Eq.(19).

### 3.2.2 System of DDEs with flexible delays

The differentiation of the system Eqs.(16 - 19) leads to a system of two DDEs in y'(t) and T'(t) with flexible delays <sup>12</sup>. We have

$$y'(t) = (1-c)y(t)(1-\Psi(t)),$$
  

$$T'(t) = 1 - \frac{y(t)}{y(t-T(t))}\Psi(t),$$

where

$$\Psi(t) = e^{-a(t)t} / e^{-a\left(t - T(t)\right) \times \left(t - T(t)\right)}$$

The function  $\Psi(t)$  is a ratio between the labor requirement of the new machines at t to the replaced ones at t - T(t).

$$y'(t) = (1-c) \left( y(t) - y (t - T(t)) \times (1 - T'(t)) \right).$$

The differentiation of Eq.(17) leads clearly to l'(t) = 0, which is solved w.r.t. T'(t).

<sup>&</sup>lt;sup>10</sup>The productive capital stock is made up of different homogeneous vintages (Allen [1]). Each vintage consists of a comparable set of machines, in use at t but installed at t - T.

<sup>&</sup>lt;sup>11</sup>This presentation is inspired from Boucekkine *et al.*[8].

 $<sup>^{12}</sup>$ Differentiating Eq.(16), we obtain

#### 3.2.3 Modified Method of Steps Algorithm

The MSA cannot be extended directly to the timevarying state-dependent DDE

$$y'(t) = f\bigg(t, y(t), y\big(t - \theta(t, y(t))\bigg),$$

where  $\theta(t, y(t))$  is the lag function of time t and state variable y(t). In fact, the successive meshpoints  $\{0, \sigma_1, \sigma_2, \ldots, \sigma_k, \ldots\}$  will differ from each other and are unknown. At each step k, the meshpoint  $\sigma_{k+1}$  must be determined, given the computed solution  $y(t) = y_{i+1}(t)$  on  $[\sigma_i, \sigma_{i+1}]$  for  $i = 0, \ldots, k$ . The equation

$$\sigma_{k+1} - \theta(\sigma_{k+1}, y(\sigma_{k+1})) = \sigma_i$$

is solved for the meshpoint  $\sigma_i$  corresponding to the smallest  $\sigma_{k+1} > \sigma_k$ . One illustration of the process is given by Boucekkine *et al.* [8]. Suppose that the generalized DDE is given by

$$y'(t) = y(t - \theta(t)),$$

where the lag function is defined by  $\theta(t) = t + \sin t$ and the initial function  $y_0(t) = 1$ .

At the **step 1**, for i = 0,  $i \in \{0, 1\}$  we have to solve  $y'(t) = y_0(t) = 1$  on  $[0, \sigma_1]$ . We find the solution  $y(t) = y_1(t) = 1 + t$ . At this step, for i = 1, we have to solve  $y'(t) = y_1(-\sin t)$ , which gives contradictory results on  $[0, 2\pi]$ . Indeed, the solution of  $y'(t) = -\sin t + 1$ , y(0) = 1 is  $t + \cos t$  on  $[0, 2\pi]$ , whereas the solution of  $y'(t) = -\sin t + 1$ ,  $y(\pi) = 1 + \pi$  is  $2 + t + \cos t$  on  $[\pi, 2\pi]$ . Since, the smallest value may be chosen in practice, we retain  $\sigma_1 = \pi$ . The modified algorithm then introduces two substeps for each step of the MSA (Boucekkine *et al.*[8]):

(i) Replacing the solution  $y_i(t)$  on  $[\sigma_{i-1}, \sigma_i]$  for  $i = 0, \ldots, k$  in the DDE, the resulting ODE is solved and gives  $y_{k+1}(t)$ .

(*ii*) The largest meshpoint value  $\sigma_{k+1}$  is such that  $y'_{k+1}(t)$  is consistent with the DDE over  $[\sigma_k, \sigma_{k+1}]$ . The solution is  $y(t) = y_{k+1}(t)$  on  $[\sigma_k, \sigma_{k+1}]$ . The numerical code used by Boucekkine *et al.* [8] is based on the 5th-order Runge-Kutta method.

## A Laplace transform solution of equation y'(t) = -by(t-1)

The Laplace transform method is used to solve a linear first-order DDE:

$$y'(t) + by(t-1) = 0, t > 0,$$
 (A.1)

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which BCs are  $y(t) = 0, t \in [-1, 0]$ , and where b is a constant. We know that  ${}^{13} \mathcal{L}[y(t)] = \int_0^\infty y(t_1)e^{-st_1}dt_1 = Y(s), s \in \mathbb{C}$ , and  $\mathcal{L}[y'(t)] = sY(s) - y_0$ . Multiplying Eq.(A.1) by  $e^{-st}$ , and integrating from 1 to infinity, we have also

$$\int_{1}^{\infty} y'(t)e^{-st}dt + b\int_{1}^{\infty} y(t-1)e^{-st}dt = 0.$$
 (A.2)

Let us examine the two integrals of Eq.(A.2). Integrating by parts the first integral and assuming  $y(t)e^{-st} \rightarrow 0$  as  $t \rightarrow \infty$ , we obtain

$$\int_{1}^{\infty} y'(t)e^{-st}dt = -y(1)e^{-s} + s\int_{1}^{\infty} y(t)e^{-st}dt.$$
(A.3)

Using a change of variable for the second integral yields

$$b \int_{1}^{\infty} y(t-1)e^{-st}dt = b \int_{-1}^{\infty} y(t_1)e^{-s(t_1+1)}dt_1,$$
  
$$= by_0e^{-s} \int_{-1}^{0} e^{-st}dt + be^{-s} \int_{-1}^{\infty} y(t_1)e^{-st_1}dt_1,$$
  
$$= by_0e^{-s} \left[ -\frac{e^{st}}{s} \right]_{-1}^{0} + be^{-s}Y(s),$$
  
$$= \frac{by_0(1-e^{-s})}{s} + be^{-s}Y(s). \quad (A.4)$$

Placing Eqs.(A.3) and (A.4) into Eq.(A.2) yields

$$sY(s) - y_0 + \frac{by_0(1 - e^{-s})}{s} + be^{-s}Y(s) = 0.$$
 (A.5)

Solving Eq.(A.5) for Y(s) and assuming that  $s - e^{-s} \neq 0$ , we get

$$Y(s) = \frac{y_0}{s} - \frac{by_0}{s(s+be^{-s})}.$$
 (A.6)

**Theorem 5** (*Pinney* [29]). Let f(t) be integrable over every finite interval such that

(i)  $\int_0^\infty f(t)e^{-st}$  converges absolutely on the real line  $\operatorname{Re} s = c$ (ii) f(t) is of bounded variation in the neighborhood of t, then

$$\oint_{(c)} F(s)e^{st}ds = \frac{1}{2} \bigg( f(t+0) - f(t-0) \bigg),$$

<sup>13</sup>The inverse Laplace transform is given for a suitable c by

$$\mathcal{L}^{-1}[Y(s)] = (2\pi j)^{-1} \int_{c-j\infty}^{c+j\infty} Y(s) e^{st} ds = y(t).$$

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where the LHS is a contour integral, taken along the line from c - jT to c + jT in the complex plane<sup>14</sup>.

From Eq.(A.6), Y(s) may also be expressed as

$$Y(s) = \frac{y_0}{s} - \frac{by_0}{s^2(1+b\frac{e^{-s}}{s})},$$
  
=  $y_0 \left(\frac{1}{s} - \sum_{p=0}^{\infty} (-1)^p b^{p+1} e^{-ps} s^{-p-2}\right).$ 

Applying the theorem 5, we have

$$y(t) = \oint_{(c)} y_0 \left( \frac{e^{st}}{s} - \sum_{p=0}^{\infty} (-1)^p b^{p+1} e^{s(t-p)} / e^{p+2} \right) ds,$$
  
=  $y_0 \left( \oint_{(c)} \frac{e^{st}}{s} ds - \sum_{p=0}^{\infty} (-1)^p b^{p+1} \oint_{(c)} e^{s(t-p)} / s^{p+2} ds \right)$ 

Giving that (see Pinney [29], page 8)

$$\oint_{(c)} \frac{e^{hk}}{k^{n+1}} = \begin{cases} \frac{h^n}{n!}, & \operatorname{Re} h > 0, \\ 0, & \operatorname{Re} h < 0, \end{cases}$$

we obtain the solution

$$y(t) = y_0 \left( 1 - \sum_{p=0}^{\lfloor t \rfloor} (-1)^p b^{p+1} \frac{(t-p)^{p+1}}{(p+1)!} \right).$$

where [t] denotes the largest integer less or equal to t.

# **B** Differential Transform Method Evaluation)

Let a nonlinear DDE with backward-foreward delays, and variable coefficients  $^{15}$ 

$$y''(t) - e^{-t}y'(t-1)y(t+1) = 0,$$
 (B.1)

where the BCs are y(0) = y'(0) = 1,  $t \in [0, 1]$ . The exact solution is clearly  $e^t$ . Using the table B.1 <sup>16</sup>, knowing that  $\mathcal{D}[e^{-x}] = (-1)^k/k!$ , we obtain the

$$\oint_{(c)} F(s)e^{st}ds \equiv \lim_{T \to \infty} \frac{1}{2\pi j} \int_{c-jT}^{c+jT} F(s)e^{st}ds.$$

<sup>16</sup>The Table B.1 of usual differential transforms is derived from the theorems by Arikoglu and Ozhol [2].

transformation of Eq.(B.1)

$$(k+1)(k+2)Y(k+2) - \sum_{k_2=0}^{k} \sum_{k_1=0}^{k_2} \sum_{h_1=k_1+1}^{T} \sum_{h_2=k_2-k_1}^{T} (k_1+1) \binom{h_1}{k_1+1} \times \binom{h_2}{k_2-k_1} (-1)^{h_1-k_1-1}Y(h_1)Y(h_2) \times \frac{(-1)^{k-k_2}}{(k-k_2)!} = 0.$$

The BCs are also transformed to Y(0) = Y(1) = 1. Choosing T = 8, with the BCs and k = 0, 1, 2, a nonlinear system in  $Y(1), Y(2), \ldots, Y(8)$  is solved <sup>17</sup>. Using the inverse transformations from Table B.1, the series solution for the problem is obtained

$$y(t) = 1 + t + .499973t^{2} + .166615t^{3} + .041599t^{4}$$
  
+.008289t^{5} + .001373t^{6} + .000187t^{7} + .000016t^{8} + \mathcal{O}(t^{9}).

The errors increase significantly with time in Figure B.1.



Figure B.1: DTM errors

<sup>&</sup>lt;sup>14</sup>The contour integral is represented by

<sup>&</sup>lt;sup>15</sup>This example is drawn from Arikoglu and Ozhol [2] with adapted notations, and using the package *MATHEMATICA*.

<sup>&</sup>lt;sup>17</sup>Solving the nonlinear system by using the primitives of *MATHEMATICA*, 128 solutions are obtained. The 20 real solutions are shown in the LHS Figure B.1.

f(t)	$F(k) = \mathcal{D}[f(t)]$
$g(t) \pm h(t)$	$G(k) \pm H(k)$
$c \ g(t)$	$c G(k_1)$
$t^n$	$\delta(k-n) = \begin{cases} 1, & k=n\\ 0, & k \neq n \end{cases}$
g(t) h(t)	$\sum_{k_1=0}^{k} G(k_1) H(k-k_1)$
$g^{(n)}(t)$	$\frac{(k+n)!}{k!}G(k+n)$
$p(t) g^{(n)}(t)$	$\sum_{h_1=0}^{k} P(k_1) \frac{(k-k_1+n)!}{(k-k_1)!} \times G(k-k_1+n)$
g(t+a)	$\lim_{T \to \infty} \sum_{h_1=k}^T \begin{pmatrix} h_1 \\ k \end{pmatrix}$ $\times a^{h_1-k} G(h_1)$
$g^{(n)}(t+a)$	$\lim_{T \to \infty} \frac{(k+n)!}{k!} \sum_{h_1=k+n}^{T} \binom{h_1}{k+n} a^{h_1-k-n} G(h_1)$
$p(t) g^{(n)}(t+a)$	$ \lim_{T \to \infty} \sum_{k_1=0}^{k} \sum_{h_1=k-k_1+n}^{T} \frac{(k-k_1+n)!}{(k-k_1)!} \binom{h_1}{k-k_1+n} \times a^{h_1-k+k_1-n} P(k_1) G(h_1) $

Table B.1: Differential transforms  $F(k) = \mathcal{D}[f(t)]$ 

### C Tinbergen's shipbuilding cycle

The Tinbergen's equation [32] is of the form  $^{18}$ 

$$y'(t) = -by(t - \theta), \ b > 0, t > \theta.$$
 (C.1)

It is also assumed that y(t) = h(t),  $t \in [0, \theta)$ , where h(t) is some given function. In this application to the shipbuilding industry, y denotes the deviation of the tonnage from a mean value and  $\theta$  a given constant construction time. In this equation, Tinbergen assumes the rate of new shipbuilding to be proportional to a delayed tonnage deviation, with a one to two years delay  $\theta$  and a ranged intensity reaction  $b \in [\frac{1}{2}, 1]$ . An endogenous cycle is found for the shipbuilding industry, with a period of 7 years 6 months for  $\theta = 1$  and 8 years 9 months for  $\theta = 2$ .

### C.1 Characteristic equation

Let the form of the unknown function be  $y(t) = e^{\rho t}$ , the characteristic equation from Eq.(C.1) is

$$D(\rho) \equiv \rho + be^{-\rho\theta} = 0, \ \rho \in \mathbb{C},$$
 (C.2)

where  $\rho = \beta + \alpha j$ ,  $j = \sqrt{-1}$ . Inserting  $\rho$  into Eq.(C.1) and separating the real and imaginary parts, we get the system

$$\cos u = \frac{-v}{\theta b} e^{v},$$
$$\frac{\sin u}{u} = \frac{1}{\theta b} e^{v},$$

where  $u \equiv \alpha \theta$  and  $v \equiv \beta \theta$ . Eliminating v, we obtain an even function f(u) in which the structural coefficients  $\theta$ , b are not explicit. We have

$$f(u) = \frac{u}{\tan u} + \ln \frac{\sin u}{u} = C, \qquad (C.3)$$

where  $C \equiv -\ln(\theta b)$ . A further analysis of the characteristic equation is given by Pinney [29] by means of the (x, k)-root plateau in the parameter space<sup>19</sup>. The properties of the characteristic equation are summarized in Figure C.1(see also Hayes [17])

<sup>18</sup>A nonlinear DDE version is given by Pinney [29]

$$y'(t) = -by(t-\theta) - \varepsilon y^{3}(t-1), \ \varepsilon, b > 0.$$

<sup>19</sup>According to this concept, the parameters may be chosen in order to achieve some desired properties for the system. Let the complex number be  $\rho = x + jy$ , the (x,k)-root plateau represents the sets of parameter values for which the caracteristic equation has k pseudo roots greater than x. The equations of the (x, k)-root plateau on the b-line are

$$\operatorname{Re} D(\rho) = x + be^{-x} \cos y,$$
  

$$\operatorname{Im} D(\rho) = y - be^{-x} \sin y.$$

existence of roots	kind of roots	range of b
no root	pseudo-positive	$(0, \frac{\pi}{2})$
one root	pseudo-positive	$(-\frac{3\pi}{2},0)$
2n roots	pseudo-positive	$((2n-\frac{3}{2})\pi,(2n+\frac{1}{2})\pi), n \in \mathbb{N}$
2n+1 roots	pseudo-positive	$(-(2n+\frac{3}{2})\pi,-(2n-\frac{1}{2})\pi), n \in \mathbb{N}$

Figure C.1: Characteristic equation properties

### C.2 Existence of exponential components

Let  $z \equiv \rho \theta$ , Eq.(C.2) may be expressed as

$$-\frac{z}{\theta b} = e^{-z}.$$
 (C.4)

The two parts of Eq.(C.2) are plotted <sup>20</sup> in Figure C.2. The condition for tangency of the two curves  $1/(\theta b) = e^{-z}$  is  $z = \ln(\theta b)$ . Inserting in Eq.(C.3), we get C = 1. The solution of the DDE Eq.(C.1) is a pure exponential <sup>21</sup> of the type

$$y(t) = (C_1 + C_2 t)e^{\rho t}.$$

For C > 1, the solutions are composed of two expo-



Figure C.2: Exponential behavior

nentials in the period ranges

$$T\in \left(\frac{\theta}{k},\frac{\theta}{k-\frac{1}{2}}\right),\;k\in\mathbb{N}$$

<sup>20</sup>Figure C.2 shows a state of a dynamic interactive *MATHE-MATICA* plotting, with automatic sliders and controls.

### C.3 Existence of cyclical components

A cycle corresponds to each real solution of Eq.(C.4) when C < 1. The two sides of this equation are represented in Figure C.3. Real branches of f(u) decrease monotically in all the intervals  $[k2\pi, (2k+1)\pi], k \in \mathbb{N}_0$ . According to  $u = \alpha \theta$  and  $\alpha = \frac{2\pi}{T}$ , the corresponding period ranges are

$$T \in \left(\frac{\theta}{k}, \frac{\theta}{k - \frac{1}{2}}\right), \ k \in \mathbb{N}_0.$$

The sine curves may be damped or undamped. A dictinction is made between the major cycle of the first period and the minor cycles The corresponding patterns



Figure C.3: Cyclical pattern

of components are shown in Figure C.4.

region	k=θb	period T	component pattern
А	$k < \frac{1}{e}$	∞	monotonic
В	$k \in (\frac{1}{e}, \frac{\pi}{2})$	T>4θ	damped oscillatory
С	$k=\frac{\pi}{2}$	$T=4\theta$	pure oscillatory
D	$k > \frac{\pi}{2}$	$T \in (2\theta, 4\theta)$	nondamped oscillatory

Figure C.4: Pattern of the components

Abbreviation	Definition	Symbol	Definition
BC DDE DTM LHS LHP MSA NDDE ODE RHS QP RHS RHS RHS	boundary condition delay differential equation differential transform method left hand side left half plane methods of steps algorithm neutral DDE ordinary differential equation right hand side quadratic programming right-hand side right-hand side	$ \begin{array}{l} \mathbb{C} \\ dB \\ deg \\ \mathcal{D}[.] \\ H(s) \\ \text{Im } z \\ j \\ \mathcal{L}^{[.]} \\ \mathcal{L}^{-1}[.] \\ \mathbb{N}_{0} \\ \mathbb{N}_{n} \\ p\text{-zero} \\ \mathbb{R} \\ \text{Re } z \\ [t] \\ y'(t) \\ y^{(k)}(t) \\ \oint_{(c)} \end{array} $	complex plane decibel degree differential transformation transfer function imaginary part of $z$ pure imaginary $\sqrt{-1}$ Laplace transform inverse Laplace transform nonnegative integers n positive integers positive zero real line real part of $z$ largest integer less or equal to $t$ derivative w.r.t. time k-th derivative w.r.t. time contour integral

### References:

- [1] R.G.D. Allen. *Macro-Economic Theory: a Mathematical Treatment*. MacMillan, London– Melbourne–Toronto, 1967.
- [2] A. Arikoglu and I. Ozhol. Solution of differential-difference equations by using differential transform method. *Applied Mathematics and Computation*, 181:1155–1175, 2006.
- [3] B. Balachandran, T. Kalmár-Nagy, and D.E. Gilsinn. *Delay Differential Equations: Recent Advances and New Directions*. Springer Science+Business Media, New York, 2009.
- [4] M. Bambi. Endogenous growth and time-tobuild: the ak case. *Journal of Economic Dynamics & Control*, 32:1015–1040, 2008.
- [5] A. Bellen and M. Zennaro. Numerical Methods for Delay Differential Equations. Clarendon Press, Oxford, 2003.
- [6] R.E. Bellman and K.L. Cooke. Differentialdifference equations. Technical Report R-374-PR, The RAND Corporation, Santa Monica, Cal., 1963.
- [7] G.A. Bocharov and F.A. Rihan. Numerical modelling in biosciences using delay differential equations. *Journal of Computational and Applied Mathematics*, 125:183–199, 2000.
- [8] R. Boucekkine, O. Licandro, and C. Paul. Differential-difference equations in economics: on the numerical solution of vintage capital growth models. *Journal of Economic Dynamics and Control*, 21:347–362, 1997.

- [9] K. Jayanthakumaran (Editor). *Advanced Technologies*. In-teh, Vukovar, Croatia, 2009.
- [10] L.E. El'sgol'ts and S.B. Norkin. Introduction to the Theory and Application of Differential Equations with Deviating Arguments, volume 105 of Mathematics in Science and Engineering (Bellman, Editor). Academic Press, New York– London, 1973.
- [11] K. Engelborghs and D. Roose. On stability of lms methods and characteristic roots of delay differential equations. *SIAM Journal of Numerical Analysis*, 40(2):629–650, 2002.
- [12] R. Frisch and H. Holme. The characteristic solutions of mixed difference and differential equations occuring in economic dynamics. *Econometrica*, 3(2):225–239, 1935.
- [13] L. Glass, A. Reuter, and D. Larocque. Time delays, oscillations, and chaos in physiological control systems. *Mathematical Biosciences*, 90(1-2):111–125, 1988.
- [14] E. Hairer, S.P. Nørsett, and G. Wanner. Solving Ordinary Differential Equations I: Nonstiff Problems, Second Revised Edition. Springer Series in Computational Mathematics. Springer-Verlag, Berlin–Heidelberg–New York, 1993.
- [15] A. Halanay. Differential Equations: Stability, Oscillations, Time Lags. Springer Series in Computational Mathematics. Academic Press, New York–London, 1993.
- [16] J.K. Hale and S.M. Verduyn Lunel. Introduction to Functional Differential Equations. Springer Series in Computational Mathematics. Springer-Verlag, Berlin–Heidelberg–New York, 1993.
- [17] N.D. Hayes. Roots of the transcendental equation associated with a certain differencedifferential equation. *Journal of the London Mathematical Society*, 25:226–232, 1950.
- [18] G.E. Hutchinson. Circular causal systems in ecology. *Annals of the New York Academy of Sciences*, 50:221–246, 1948.
- [19] K. Ikeda and K. Matsumoto. High-dimensional chaotic behavior in systems with time-delayed feedback. *Physica D.*, 29:223–235, 1987.
- [20] M. Kalecki. A macroeconomic theory of business cycle. *Econometrica*, 3(3):327–344, 1935.

- [21] A.A. Keller. *Optimal economic stabilization policy under uncertainty*. In Jayanthakumaran [9], 2009.
- [22] V. Kolmanovskii and V.R. Nosov. Stability of Functional Differential Equations, volume 463 of Mathematics and Its Applications. Academic Press, London–Orlando–San Diego–New York, 1999.
- [23] V.B. Kolmanovskii and A. Myshkis. Applied Theory of Functional Differential Equations, volume 85 of Mathematics and Its Applications. Kluwer Academic Publishers, Dordrecht– Boston–London, 1992.
- [24] V.B. Kolmanovskii and A. Myshkis. Introduction to the Theory and Applications of Functional Differential Equations, volume 463 of Mathematics and Its Applications. Kluwer Academic Publishers, Dordrecht–Boston–London, 1999.
- [25] Y. Kuang. Delay Differential Equations with Applications in Population Dynamics, volume 191 of Mathematics in Science and Engineering, W.F. Ames Editor. Academic Press, Boston–San Doego–New York, 1993.
- [26] E. Liz, C. Martínez, and S. Trofimchuk. Attractivity properties of infinite delay Mackey-glass type equations. *Differential and Integral Equations*, 15(7):875–896, 2002.
- [27] A.J. Lotka. *Elements of Mathematical Biology*. Dover Publications, New York, 1956.
- [28] M.C. Mackey and L. Glass. Oscillation and chaos in physiological control systems. *Science*, 197(4300):287–289, 1977.
- [29] E. Pinney. Ordinary Difference-Differential Equations. University of California Press, Berkeley–Los Angeles, 1959.
- [30] R. Shone. *Economic Dynamics: Phase Diagrams and their Economic Applications, second edition.* Cambridge University Press, Cambridge, UK–New York, 2002.
- [31] J.C. Sprott. A simple chaotic delay differential equation. *Physics Letters A*, 366:397–402, 2007.
- [32] J. Tinbergen. Ein Schiffbauzyklus ? Weltwirtschaftliches Archiv, 34(2):152–164, 1931.

- [33] J. Volterra. *Theory of functionals and of integral and integro-differential equations*. Dover Publications, New York, 1959.
- [34] E.M. Wright. A non-linear differencedifferential equation. *Journal für die Reine und Angewandte Mathematik*, pages 66–87, 1955.