Energetic instability of a fractional oscillator

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Abstract: We explore the problem of memory-enhanced stability occurring in a fractional oscillator with fluctuating eigenfrequency when the oscillator is driven by external sinusoidal forcing and by an additive internal fractional noise with a power-law type memory. It is established that at intermediate values of the memory exponent the energetic stability of the oscillator is significantly enhanced. We show that, by conveniently choosing the system parameters, the output signal–to–noise ratio is a double peaked function of the memory exponent, thus demonstrating the phenomenon of memory controlled stochastic multiresonance. The connection between the occurrence of energetic instability and the phenomenon of memory-induced multiresonance is established.

Key–Words: Fractional oscillator, stochastic resonance, fractional noise, white noise, energetic instability, viscoelastic friction.

1 Introduction

The idea that noise, via interaction with the deterministic dynamics of the system, can give counterintuitive results, has led to many important discoveries: stochastic resonance [1], noise-induced stability [2, 3], stochastic ratchets (Brownian motors) [4, 5], noise-induced Hopf bifurcations [6], and noise-induced first-order phase transitions in some complex systems [7, 8], to name a few. Active analytical and numerical studies of various models with random perturbations have been stimulated by their possible applications in different fields, ranging from ecosystems [7, 9] to intracellular protein transport in biology [3, 4], or to methods of particle separation in nanotechnology [10, 11]. One of the objects of special attention in this context is the noise-driven harmonic oscillator [12]-[14]. Since non-linearity presents difficulties for theoretical analysis of stochastic resonance phenomena, linear models of oscillators with multiplicative noise are of particular interest. These models, on the one hand, show quasi-non-linear behavior including stochastic resonance [15], while on the other hand, they allow exact analytical treatment. It is shown that the influence of noise on the oscillator eigenfrequency may lead to different resonant phenomena. First, it may cause energetic instability, which manifests itself in an unlimited increase of second-order moments of the output with time, while the mean value of the oscillator displacement remains finite [13, 16]. This phenomenon is a stochastic counterpart of classical parametric resonance [13, 17]. Second, if the oscillator is subjected to an external periodic force and the fluctuations of the oscillator frequency are colored, the behavior of the moments and signal–to–noise ratio (SNR) of the output signal shows a nonmonotonic dependence on noise parameters, i.e., stochastic resonance (SR) [14, 15, 18].

Another popular generalization of the harmonic oscillator, beside inclusion of stochastic terms, consists in the replacement of the usual friction term in the dynamical equation for a harmonic oscillator by a generalized friction term with a power-law type memory [19]-[21]. The main advantage of this equation, also called fractional oscillator, is that it provides a physically transparent and mathematically tractable description of the stochastic dynamics in a system with slow relaxation processes and with anomalously slow diffusion (subdiffusion). Examples of such systems are supercooled liquids, glasses, colloidal suspensions, dense polymer solutions [22, 23], viscoelastic media [24], and amorphous semiconductors [25].

Although the behavior of both above-mentioned generalizations of the harmonic oscillator, i.e., a harmonic oscillator with fluctuating frequency and a fractional oscillator, have been investigated in detail (see, e.g., [14, 26]), it seems that analysis of the potential consequences of interplay between eigenfrequency fluctuations and memory effects is still missing in literature. This is surprising in view of the fact that the importance of fluctuations and viscoelasticity for biological systems, e.g., living cells, has been well recognized [27].

Thus motivated, we consider a fractional oscillator with a power-law memory kernel subjected to an external periodic force. The influence of the fluctuating environment is modeled by a multiplicative white noise (fluctuating eigenfrequency) and by an additive fractional noise with a zero mean.

The main contribution of this paper is as follows. In the long-time limit, $t \to \infty$, we provide exact formulas for the analytic treatment of the dependence of SR characteristics, such as SNR and variance of the output signal, on system parameters. Furthermore, we establish the sufficient conditions for the occurrence of energetic instability and analyze the behavior of critical noise intensity (i.e., the intensity of multiplicative noise above which the system is unstable) on the memory exponent. It is found that the energetic stability of the oscillator is significantly enhanced at intermediate values of the memory exponent. Moreover, we demonstrate that in certain parameter regions stochastic multiresonance is manifested in the dependence of the output SNR of the noisy fractional oscillator upon the memory exponent.

The structure of the paper is as follows. In Section 2 we present, in a general form, the model investigated. In Section 3, using Kubo's second fluctuation-dissipation theorem, some general formulas for the oscillator displacement are derived and a description of the output SR quantifiers is given. In Section 4 we introduce a friction kernel with a power-law type memory and exact formulas are found for the analysis of the long-time behavior of the SR characteristics. Memory-enhanced energetic stability and memory-induced stochastic multiresonance are exposed and discussed in Section 5. Section 6 contains some brief concluding remarks.

2 Model

To model an oscillatory system strongly coupled with a noisy environment, we have chosen an oscillator with a fluctuating eigenfrequency and with a memory friction kernel

$$\ddot{X}(t) + \int_0^t \gamma(t-t')\dot{X}(t')dt' + [\omega^2 + Z(t)]X(t) = \xi(t) + A_0 \sin(\Omega t), \tag{1}$$

where $\dot{X} \equiv dX/dt$, $X(t)$ is the oscillator displacement, and fluctuations of the eigenfrequency ω are expressed as a Gaussian white noise $Z(t)$ with a zero mean and a delta-correlated correlation function:

$$\langle Z(t)Z(t')\rangle = 2D\delta(t-t'), \tag{2}$$

where D is the noise intensity. The driving force consists of an external harmonic force with a frequency Ω and an amplitude A_0, and an additive zero-centered noise $\xi(t)$ which characterizes the influence of a non-Ohmic thermal bath (environment) on the oscillator. In the following, we will treat the random force $\xi(t)$ as a internal noise, i.e., we assume that the fluctuations and dissipation stem from the same source and the system will finally reach an stationary state. In this case the frictional kernel $\gamma(t)$ is related to the correlation function of the noise via Kubo's second fluctuation-dissipation theorem [28]

$$\langle \xi(t)\xi(t')\rangle = k_B T \gamma(|t-t'|), \tag{3}$$

where T is the absolute temperature, and k_B is the Boltzmann constant. Here we emphasize that the noise $Z(t)$ in Eq. (1) is considered as an external noise (e.g., caused by another independent thermal bath). In this case, the fluctuations and dissipation come from different sources, and the friction kernel and the correlation function of the noise are independent.

The second-order differential equation (1) can be written as two first-order differential equations

$$\dot{X}(t) = Y(t), \tag{4}$$

$$\dot{Y}(t) + \int_0^t \gamma(t-t')Y(t')dt' + [\omega^2 + Z(t)]X(t) = \xi(t) + A_0 \sin(\Omega t) \tag{5}$$

which, after averaging over the ensemble of realizations of the stochastic processes $Z(t)$ and $\xi(t)$ at a given instant in time t, take the following form:

$$\langle \dot{X}(t)\rangle = \langle Y(t)\rangle, \tag{6}$$

$$\langle \dot{Y}(t)\rangle + \int_0^t \gamma(t-t')\langle Y(t')\rangle dt' + \omega^2\langle X(t)\rangle = A_0 \sin(\Omega t). \tag{7}$$

Here we have used, that from Eq. (4) it follows that the correlator

$$\langle Z(t)X(t)\rangle = 0. \tag{8}$$

Thus, it turns out that fluctuations of frequency do not affect the first moment of the oscillator provided the fluctuations are delta-correlated, and $\langle X(t)\rangle$ remains equal to the noise-free solution. To find the first and second moments of X we use the Laplace transformation technique. By means of the Laplace transformation to Eqs. (4) and (5) one can easily obtain formal expressions for the displacement $X(t)$ and the velocity $\dot{X}(t)$ in the following forms:

$$X(t) = \langle X(t)\rangle + \int_0^t H(t-\tau)[\xi(\tau) - X(\tau)Z(\tau)]d\tau, \tag{9}$$

$$Y(t) = \langle Y(t)\rangle + \int_0^t \dot{H}(t-\tau)[\xi(\tau) - X(\tau)Z(\tau)]d\tau, \tag{10}$$

where the averages $\langle X(t)\rangle$ and $\langle Y(t)\rangle$ are given by

$$\begin{aligned}\langle X(t)\rangle &= \dot{x}_0 H(t) + x_0\left[1 - \omega^2\int_0^t H(\tau)d\tau\right] \\ &\quad + A_0\int_0^t H(t-\tau)\sin(\Omega\tau)d\tau, \qquad (11)\\ \langle Y(t)\rangle &= \dot{x}_0\dot{H}(t) - \omega^2 x_0 H(t) \\ &\quad + A_0\int_0^t \dot{H}(t-\tau)\sin(\Omega\tau)d\tau, \qquad (12)\end{aligned}$$

with deterministic initial conditions such as $X(0) = x_0$ and $\dot{Y}(0) = \dot{x}_0$. The kernel $H(t)$ with the initial condition $H(0) = 0$ is the Laplace inversion of

$$\hat{H}(s) = \frac{1}{s^2 + s\hat{\gamma}(s) + \omega^2}, \tag{13}$$

where

$$\hat{\gamma}(s) = \int_0^\infty e^{-st}\gamma(t)dt. \tag{14}$$

In this work, we will restrict ourselves to the case where

$$\lim_{s\to 0} s\hat{\gamma}(s) = 0. \tag{15}$$

Thus in the long-time limit, $t \to \infty$, the memory about the initial conditions will vanish and the mean oscillator displacement is given by

$$\begin{aligned}&\langle X(t)\rangle_{as} := \langle X(t)\rangle_{|t\to\infty} = \\ &A_0\int_0^t H(t-\tau)\sin(\Omega\tau)d\tau. \qquad (16)\end{aligned}$$

Equation (16) can be written as

$$\langle X(t)\rangle_{as} = A\sin(\Omega t + \varphi). \tag{17}$$

The response A and the phase shift φ are obtained by means of the complex susceptibility

$$\chi = \chi' + i\chi'' = \hat{H}(-i\Omega), \tag{18}$$

where χ' and χ'' are the real and imaginary parts of the susceptibility.

For the response, we have

$$\begin{aligned}&A^2 = A_0^2|\chi|^2 = \\ &A_0^2\left[\left(\omega^2 - \Omega^2 + \Omega\gamma_1(\Omega)\right)^2 + \Omega^2\gamma_2^2(\Omega)\right]^{-1}, \qquad (19)\end{aligned}$$

and the phase shift can be represented as

$$\begin{aligned}\varphi &= \arctan(-\frac{\chi''}{\chi'}) = \\ &\arctan\left[-\frac{\Omega\gamma_2(\Omega)}{\omega^2 - \Omega^2 + \Omega\gamma_1(\Omega)}\right], \qquad (20)\end{aligned}$$

where

$$\begin{aligned}\gamma_1(\Omega) &:= \int_0^\infty \gamma(t)\sin(\Omega t)dt, \\ \gamma_2(\Omega) &:= \int_0^\infty \gamma(t)\cos(\Omega t)dt. \qquad (21)\end{aligned}$$

Note that Eqs. (19) and (20) are independent of the noises $Z(t)$ and $\xi(t)$.

3 Second moments

Here we will evaluate the long-time behavior of the variance $\sigma^2(X)$ and the correlation function $K(\tau,t)$ of the oscillator displacement:

$$\sigma^2(t) \equiv \langle (X(t) - \langle X(t)\rangle)^2\rangle, \tag{22}$$

$$\begin{aligned}K(\tau,t) \equiv \langle (X(t+\tau) - \langle X(t+\tau)\rangle) \\ (X(t) - \langle X(t)\rangle)\rangle. \qquad (23)\end{aligned}$$

Starting from Eq. (9), we obtain for the correlation function

$$K(t,\tau) = \int_0^{t+\tau}\int_0^t H(t+\tau-t_1)H(t-t_2)$$

$$\times\{\langle X(t_1)X(t_2)Z(t_1)Z(t_2)\rangle + \langle \xi(t_1)X(t_2)Z(t_2)\rangle$$

$$+\langle \xi(t_2)X(t_1)Z(t_1)\rangle + \langle \xi(t_1)\xi(t_2)\rangle\}dt_1dt_2. \tag{24}$$

Because of Eq. (8) and statistical independence of the processes $\xi(t)$ and $Z(t)$ it follows that

$$\langle \xi(t_1)X(t_2)Z(t_2)\rangle = \langle \xi(t_2)X(t_1)Z(t_1)\rangle = 0. \tag{25}$$

Using the well-known Furutzu-Novikov procedure [29], the new correlator $\langle X(t_1)X(t_2)Z(t_1)Z(t_2)\rangle$ can be given by

$$\langle X(t_1)X(t_2)Z(t_1)Z(t_2)\rangle = 2D\langle X^2(t_2)\rangle\delta(|t_2 - t_1|). \tag{26}$$

In the long-time limit, $t \to \infty$, Eqs. (24)-(26), (17), and (3) yield the following asymptotic formula for the correlation function $K(\tau, t)$:

$$K_{as}(\tau,t) = k_B T \int_0^\infty \int_0^\infty H(t_1)H(t_2)\gamma(|\tau+t_2-t_1|)dt_1 dt_2 + 2D\int_0^\infty H(\tau+t_1)H(t_1) \times \left\{\sigma^2(t-t_1) + A^2 \sin^2\left[\Omega(t-t_1)+\varphi\right]\right\} dt_1, \quad t \to \infty. \tag{27}$$

The two-time correlation function $K_{as}(\tau, t)$ depends on both times t and τ and becomes a periodic function of t with the period of the external driving, $T = 2\pi/\Omega$. Thus, as in [30], we define the one-time correlation function $K_1(\tau)$ as the average of the two-time correlation function over a period of the external driving, i.e.,

$$K_1(\tau) = \frac{1}{T}\int_0^T K_{as}(\tau,t)dt. \tag{28}$$

Using Eq. (27) we obtain

$$K_1(\tau) = D\left(2\bar{\sigma}^2 + A^2\right)\int_0^\infty H(\tau+t)H(t)dt + k_B T \int_0^\infty \int_0^\infty H(t_1)H(t_2)\gamma\left(|\tau+t_2-t_1|\right)dt_1 dt_2, \tag{29}$$

where $\bar{\sigma}^2$ is the time-homogeneous part of the variance of the oscillator displacement X in the asymptotic regime, $t \to \infty$, i.e.,

$$\bar{\sigma}^2 = \frac{1}{T}\int_0^T \sigma^2_{as}(X(t))dt. \tag{30}$$

As $K_1(0) = \bar{\sigma}^2$ and the second term in the right side of Eq. (29) can be simplified at $\tau = 0$ to [31]

$$\int_0^\infty \int_0^\infty H(t_1)H(t_2)\gamma\left(|t_2-t_1|\right)dt_1 dt_2 = \frac{1}{\omega^2} \tag{31}$$

we find from Eq. (29) that

$$\bar{\sigma}^2 = \frac{1}{2[D_{cr}-D]}\left(DA^2 + \frac{2k_B T D_{cr}}{\omega^2}\right), \tag{32}$$

where the critical noise intensity D_{cr} reads

$$D_{cr} = \left[2\int_0^\infty H^2(t)dt\right]^{-1}. \tag{33}$$

From Eq. (32) we can see that the stationary regime is possible only if $D < D_{cr}$. As the noise intensity D tends to the critical value D_{cr} the variance $\bar{\sigma}^2$ increases to infinity. This is the indication that for $D > D_{cr}$ energetic instability appears, which manifests itself in an unlimited increase of second-order moments of the output of the oscillator with time, while the mean value of the oscillator displacement remains finite [13, 16]. This phenomenon is a stochastic counterpart of classical parametric resonance in the case of an ordinary deterministic oscillator (without a memory kernel) with time-dependent frequencies [13, 17].

Turning now to Eq. (29), we consider the signal–to–noise ratio (SNR) of the output signal. The one-time correlator

$$K(\tau) = \frac{1}{T}\int_0^T \langle X(t+\tau)X(t)\rangle_{as} dt \tag{34}$$

can be written exactly as the sum of two contributions:

$$K(\tau) = K_1(\tau) + K_2(\tau), \quad K_2(\tau) = \frac{A^2}{2}\cos(\Omega\tau), \tag{35}$$

i.e., the coherent part $K_2(\tau)$, which is periodic in τ with the period T, and the incoherent part $K_1(\tau)$, which decays to zero for large values of τ. According to [30], the output SNR (R_{out}) at the forcing frequency Ω is defined in terms of the Fourier cosine transform of the coherent and incoherent parts of $K(\tau)$. Namely,

$$R_{out} = \frac{\Gamma_2}{\Gamma_1}, \tag{36}$$

where

$$\Gamma_2 = \frac{2}{T}\int_0^T K_2(\tau)\cos(\Omega\tau)d\tau = \frac{A^2}{2}, \tag{37}$$

and

$$\Gamma_1 = \frac{2}{\pi}\int_0^\infty K_1(\tau)\cos(\Omega\tau)d\tau = \frac{A^2}{\pi A_0^2}\left\{D\left(\bar{\sigma}^2 + \frac{A^2}{2}\right) + 2k_B T\gamma_2(\Omega)\right\}, \tag{38}$$

where the quantity $\gamma_2(\Omega)$ is given by Eq. (21). Thus

$$R_{out} = \frac{A_0^2\pi}{D\left(2\bar{\sigma}^2 + A^2\right) + 4k_B T\gamma_2(\Omega)}. \tag{39}$$

It is worth pointing out that R_{out} tends to zero as the noise intensity D tends to the critical value D_{cr}.

4 Fractional oscillator

To find more physically tractable results, we introduce a noise $\xi(t)$, known as fractional Gaussian noise, in Eq. (1). The fractional Gaussian noise is closely related to the fractional Brownian motion process [20, 32]. That process has two unique properties: self-similarity with the Hurst coefficient $0 < H < 1$, and stationary increments [33]. In the case of the fractional Gaussian noise with $1/2 < H < 1$

$$\langle \xi(t+\tau)\xi(t)\rangle = \frac{k_B T\gamma\tilde{\omega}^{1-\alpha}}{\Gamma(1-\alpha)|\tau|^{\alpha}}, \quad 0<\alpha<1, \tag{40}$$

where the memory exponent $\alpha = 2 - 2H$, $\Gamma(z)$ is the gamma function, and $\tilde{\omega}$ denotes a reference frequency allowing for the friction constant γ to have the dimension of viscosity for any values of the memory exponent. Thus Eq. (1) behaves as a stochastically perturbed fractional oscillator

$$\frac{d^2X}{dt^2} + \gamma\tilde{\omega}^{1-\alpha}\frac{d^{\alpha}X}{dt^{\alpha}} + [\omega^2 + Z(t)]X(t) = \xi(t) + A_0\sin(\Omega t), \tag{41}$$

where

$$\frac{d^{\alpha}X}{dt^{\alpha}} = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\dot{X}(t')}{(t-t')^{\alpha}}dt' \tag{42}$$

is the fractional Caputo derivative [34]. Note that at the same time, Eq. (41) also corresponds to the sub-Ohmic model of coupling with the thermal bath oscillators [35] or with a fracton thermal bath [36].

Now we consider the dependence of the critical noise intensity D_{cr} on the system parameters. Using the Eqs. (3), (40), and (14) we find from Eq. (13) that

$$\hat{H}(s) = \frac{1}{s^2 + \gamma\tilde{\omega}^{1-\alpha}s^{\alpha} + \omega^2}. \tag{43}$$

To evaluate the inverse Laplace transform of $\hat{H}(s)$ we use the residue theorem method described in [37]. The inverse Laplace transform gives

$$H(t) = \frac{2}{\sqrt{u^2+v^2}}e^{-\beta t}\sin(\omega^* t + \theta) + \frac{\gamma\tilde{\omega}^{1-\alpha}\sin(\alpha\pi)}{\pi}\int_0^{\infty}\frac{r^{\alpha}e^{-rt}dr}{B(r)}, \tag{44}$$

where $s_{1,2} = -\beta \pm i\omega^*$ ($\beta > 0$, $\omega^* > 0$) are the pair of conjugate complex zeros of the equation

$$F(s) \equiv s^2 + \gamma\tilde{\omega}^{1-\alpha}s^{\alpha} + \omega^2 = 0, \tag{45}$$

while, $F(s)$ is defined by the principial branch of s^{α}. The quantities u, v, θ, and $B(r)$ are determined by

$$u = -2\beta + \frac{\alpha\gamma\tilde{\omega}^{1-\alpha}\cos[(1-\alpha)\arctan(-\frac{\omega^*}{\beta})]}{(\beta^2+\omega^{*2})^{\frac{1-\alpha}{2}}},$$

$$v = 2\omega^* - \frac{\alpha\gamma\tilde{\omega}^{1-\alpha}\sin[(1-\alpha)\arctan(-\frac{\omega^*}{\beta})]}{(\beta^2+\omega^{*2})^{\frac{1-\alpha}{2}}},$$

$$\theta = \arctan\left(\frac{u}{v}\right), \tag{46}$$

and

$$B(r) := \left[r^2 + \gamma\tilde{\omega}^{1-\alpha}r^{\alpha}\cos(\pi\alpha) + \omega^2\right]^2 + \gamma^2\tilde{\omega}^{2(1-\alpha)}r^{2\alpha}\sin^2(\alpha\pi). \tag{47}$$

Then, from Eq. (33),

$$D_{cr}^{-1} = \frac{2}{\beta(\beta^2+\omega^{*2})[u^2+v^2]^2} \times\{(\beta^2+\omega^{*2})(u^2+v^2) - \beta[\beta(v^2-u^2) - 2\omega^* uv]\} + \frac{4\gamma^2\tilde{\omega}^{2(1-\alpha)}\sin^2(\alpha\pi)}{\pi^2}\int_0^{\infty}\frac{r^{\alpha}dr}{B(r)}\int_0^{r}\frac{s^{\alpha}ds}{(r+s)B(s)} + \frac{8\gamma\tilde{\omega}^{1-\alpha}\sin(\alpha\pi)}{\pi(u^2+v^2)}\int_0^{\infty}\frac{r^{\alpha}[v\omega^* + u(r+\beta)]dr}{[\omega^{*2}+(r+\beta)^2]B(r)}. \tag{48}$$

It must be emphasized that the relaxation function $H(t)$ as well as D_{cr}^{-1} can be represented via Mittag-Leffler-type special functions [34]. But as the numerical calculations in this case are very complicated, we suggest, apart from possible representations via Mittag-Leffler functions, a numerical treatment of Eq. (48). Finally, to complete the formulas (19), (20), and (39) for the fractional oscillator (41) we evaluate the quantities $\gamma_1(\Omega)$ and $\gamma_2(\Omega)$ from Eq. (21). The results are

$$\gamma_1(\Omega) = \gamma\left(\frac{\tilde{\omega}}{\Omega}\right)^{(1-\alpha)}\cos(\frac{\alpha\pi}{2}),$$
$$\gamma_2(\Omega) = \gamma\left(\frac{\tilde{\omega}}{\Omega}\right)^{(1-\alpha)}\sin(\frac{\alpha\pi}{2}). \tag{49}$$

5 Results

Our next task is to investigate, on the basis of Eq. (48), the dependence of the appearance of energetic instability on the memory exponent α at various values of the friction coefficient γ and a reference frequency $\tilde{\omega}$.

Here we emphasize that for all figures we use a dimensionless formulation of the dynamics with a time scaling of the following form:

$$t^* = \omega t, \quad \gamma^* = \frac{\gamma}{\omega}, \quad \tilde{\omega}^* = \frac{\tilde{\omega}}{\omega}, \quad D^* = \frac{D}{\omega^3}, \\ \Omega^* = \frac{\Omega}{\omega}, \tag{50}$$

i.e. $\omega^* = 1$.

It is well known that in the case of an ordinary oscillator (without memory, $\alpha = 1$) with a fluctuating frequency the critical noise intensity $D_{cr} = \gamma\omega^2$. Thus, when the noise intensity exceeds the value $\gamma\omega^2$, the oscillator becomes energetically unstable [13].

In Fig. 1, several graphs depict the behavior of D_{cr}/γ versus the memory exponent α for different representative values of the parameters γ and $\tilde{\omega}$. These graphs show a typical resonance-like behavior of $D_{cr}(\alpha)$. As a rule, the maximal value of D_{cr}/γ increases as the values of the friction coefficient γ or the reference frequency $\tilde{\omega}$ increases, while the positions of the maxima are monotonically shifted to a lower α with a rise in γ. Moreover, for some values of the parameters γ and $\tilde{\omega}$ a multiresonance with two maxima appears [see the solid lines in Fig. 1]. Thus, at intermediate values of the memory exponent α the energetic stability of the fractional oscillator is significantly enhanced in comparison with an ordinary oscillator (without memory, $\alpha = 1$). The effect is very pronounced at high values of the damping γ. A physical explanation for the behavior of $D_{cr}(\alpha)$ in the case of strong memory, $\alpha \to 0$, is based on the cage effect [26]. For small α the friction force induced by the medium is not just slowing down the particle but also causing the particle to develop a rattling motion. To see this consider Eq. (41) together with Eq. (42) in the limit $\alpha \to 0$,

$$\ddot{X} + [\gamma\tilde{\omega} + \omega^2 + Z(t)]X = \xi(t) + \gamma\tilde{\omega}X(0) + A_0\sin(\Omega t). \tag{51}$$

Equation (51) describes a stochastically perturbed harmonic motion with the zero value effective friction coefficient, $\gamma_{ef} = 0$, and with the effective eigenfrequency $\omega_{ef} = \sqrt{\gamma\tilde{\omega} + \omega^2}$. Evidently, the corresponding $D_{cr} = \gamma_{ef}\omega_{ef}^2$ tends to zero. In this sense the medium is binding the particle, preventing dissipation but forcing oscillations.

To clarify the behavior of $D_{cr}(\alpha)$ at the low memory limit, $\alpha \to 1$, let us take a closer look to the parameter regime $\omega = \tilde{\omega} = 1$ and $\gamma < 2$, (cf. Fig. 1(b)). In this case it turns out that due to the values of the parameters considered the integral term of the relaxation function $H(t)$ [see Eq. (44)] is, at sufficiently low memory, $\alpha \approx 1$, so small (though it will be dominant at later moments) that it can be neglected together

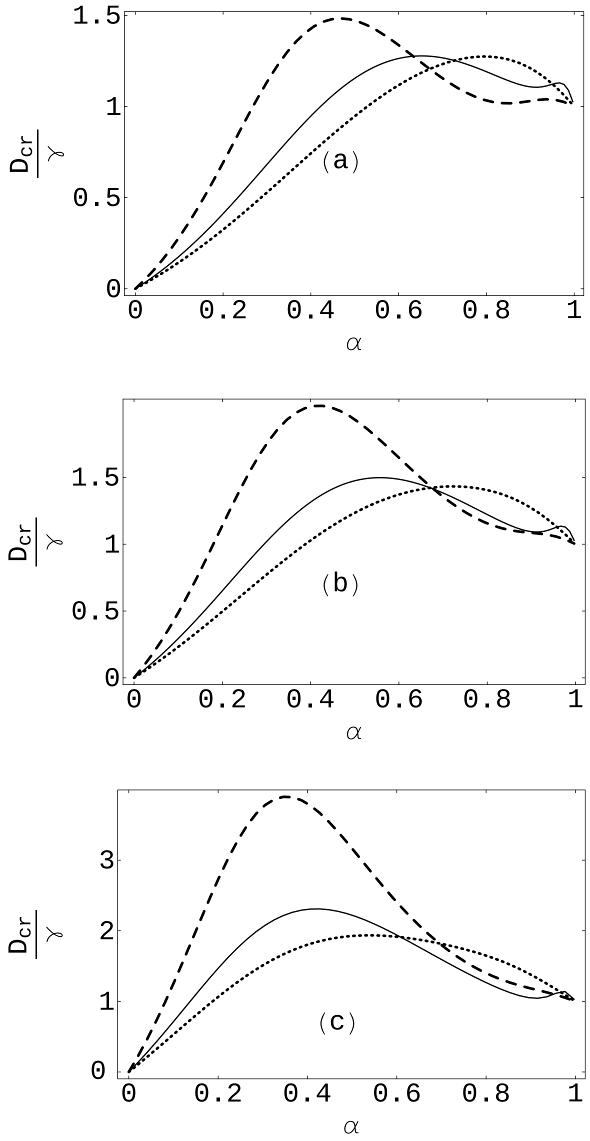

Figure 1: Critical noise intensity D_{cr} versus the memory exponent α at several values of the parameters γ and $\tilde{\omega}$. All quantities are dimensionless with a time scaling $\omega = 1$. Solid line, $\gamma = 1.62$; dashed line, $\gamma = 5$; dotted line, $\gamma = 0.8$. Panel (a): $\tilde{\omega} = 2/3$; panel (b): $\tilde{\omega} = 1$; panel (c): $\tilde{\omega} = 2$.

with the phaseshift θ. So Eq. (44) turns to

$$H(t) \approx \frac{2}{\sqrt{u^2 + v^2}} e^{-\beta t} \sin(\omega^* t). \tag{52}$$

To get an idea about the influence of the memory exponent α on the critical noise intensity D_{cr} at $\alpha = 1$ we compare the result (52) with those from the classical damped oscillator with fluctuating frequency

$$\ddot{X} + \gamma_{ef}\dot{X} + \left[\omega_{ef}^2 + Z(t)\right] X = 0, \tag{53}$$

where

$$\gamma_{ef} = 2\beta, \quad \omega_{ef}^2 = \omega^{*2} + \beta^2. \tag{54}$$

Note that the relaxation function $H(t)$ for the oscillator (53) is the same as that presented by Eq. (52). Hence, the critical noise intensity $D_{cr}(\alpha)$ behaves as

$$D_{cr}(\alpha) = \gamma_{ef}\omega_{ef}^2. \tag{55}$$

For the parameter regime $\omega = \tilde{\omega} = 1$ and $\gamma < 2$ it follows from Eqs. (45) and (54) that in the limit $\epsilon \to 0$, where $\epsilon = 1 - \alpha$, the effective parameters $\gamma_{ef}(\epsilon)$ and $\omega_{ef}(\epsilon)$ are increasing functions of ϵ

$$\begin{aligned} \left(\frac{d\gamma_{ef}}{d\epsilon}\right)_{|\epsilon=0} &= \frac{2\gamma^2}{\sqrt{(4-\gamma^2)}}\left[\pi - \arccos\frac{\gamma}{2}\right] > 0, \\ \left(\frac{d\omega_{ef}}{d\epsilon}\right)_{|\epsilon=0} &= \frac{1}{\gamma}\left(\frac{d\gamma_{ef}}{d\epsilon}\right)_{|\epsilon=0} > 0. \end{aligned} \tag{56}$$

Thus, in the case of very low memory a decrease of the memory exponent causes increase of the effective eigenfrequency and also enhancement of dissipation.

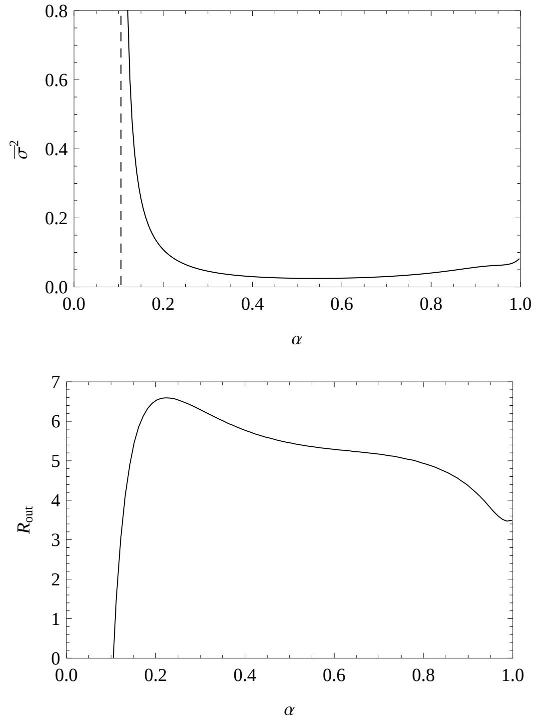

Figure 2: Dependence of the variance $(\bar{\sigma}^2)$ and the output SNR (R_{out}) computed from Eqs. (32) and (39) on the memory exponent α. The parameter values: $\gamma = 1.62$, $T = 0$, $A_0 = \Omega = \omega = \tilde{\omega} = 1$, and $D = 0.5$. All quantities are dimensionless with a scaling by Eq. (50). The dashed line depicts the position of the critical memory exponent $\alpha_1 \approx 0.10$ below which the oscillator is unstable.

Consequently, at $\alpha \approx 1$ the critical noise intensity $D_{cr}(\alpha)$ is a decreasing function of α (cf. Fig. 1). Bearing in mind the behavior of $D_{cr}(\alpha)$ at both of the limiting cases (i.e., at low and strong memory) the occurrence of a maximum of D_{cr} at intermediate values of the memory exponent is not surprising.

Next we consider the dependence of SR characteristics (SNR and variance) on the memory exponent α. In Figs. 2, 3, and 4 we depict, on two panels, the behavior of $\bar{\sigma}^2$ and R_{out} for various values of the noise intensity D. In the case considered in Fig. 2 the noise intensity is lower as the critical value at $\alpha = 1$, i.e., $D < \omega^2\gamma$. As a rule (except for some special parameter combinations) in this case the variance $\bar{\sigma}^2$ decreases rapidly from infinity at α_1, $D_{cr}(\alpha_1) = D$, to a minimum and next increases slowly to a finite value at $\alpha = 1$ as the memory exponent increases. Note that for $\alpha < \alpha_1$ the system is energetically unstable. The dependence of SNR on α corresponds to the behavior of the variance [cf. Eq. (39)] and thus demonstrates a resonance–like maximum in the interval $(\alpha_1, 1)$.

In the case of $\omega^2\gamma < D < D_{crmax}$, where D_{crmax} is the maximal value of $D_{cr}(\alpha)$ by variations

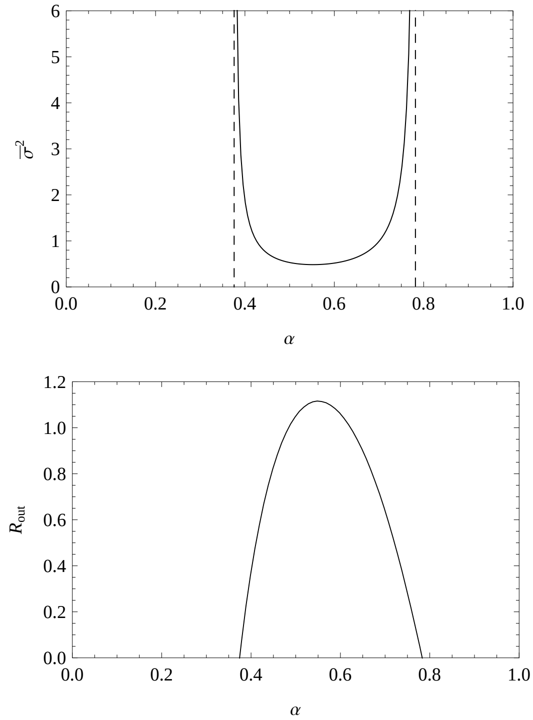

Figure 3: SR characteristics [SNR (R_{out}) and variance $(\bar{\sigma}^2)$] as functions of the memory exponent α at the noise intensity $D = 2.025$. Other parameter values are the same as in Fig. 2. The dashed lines depict the position of the critical memory exponents $\alpha_1 \approx 0.37$ and $\alpha_2 \approx 0.78$: for $\alpha > \alpha_2$ and for $\alpha < \alpha_1$ the oscillator is unstable.

of α, the SR characteristics exhibit a very strong non-monotonic dependence on the memory exponent, i.e., a typical resonance phenomenon continues to increase α. As shown in Fig. 3, clearly the resonance peak of R_{out} corresponds to the value of α at which the variance $\bar{\sigma}^2(\alpha)$ is maximally suppressed.

As mentioned above, for some parameter regimes the dependence of the critical noise intensity on the memory exponent exhibits a double peak structure (see Fig. 1). If $D_{crmin} < D < D^*_{crmax}$, where D_{crmin} and D^*_{crmax} are the values of $D_{cr}(\alpha)$ at the local minimum and the local maximum, respectively, this case is characterized by the following scenario (see Fig. 4): For small values of the memory exponent, $\alpha < \alpha_1$, where $D > D_{cr}(\alpha_1)$, the oscillator is energetically unstable. At $\alpha = \alpha_1$, i.e., $D = D_{cr}(\alpha_1)$, the system becomes stable. In the interval $\alpha_1 < \alpha < \alpha_2$ there appears a stable regime, where SNR exhibits a resonance-like behavior versus α. At $\alpha = \alpha_2$, where $D = D_{cr}(\alpha_2)$, the energetic stability disappears (variance tends to infinity) and the system approaches an unstable regime, thus making a reentrant transition. With increasing the memory exponent, one observes another region of the values of α, $\alpha_3 < \alpha < \alpha_4$, where the oscillator is energetically stable and the above scenario is repeated. Thus, the appearance of multiresonace of SNR versus α is due to a strong suppression of the output variance in the stability regions of the system.

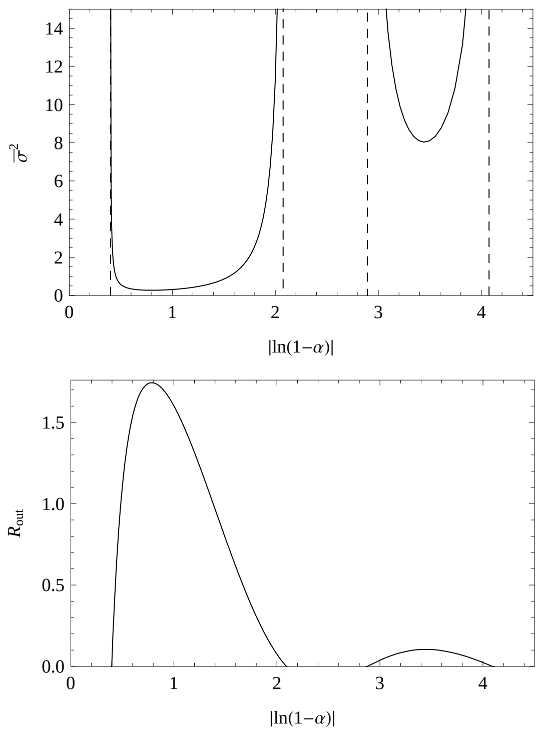

Figure 4: Multiresonance of SR characteristics [SNR (R_{out}) and variance ($\bar{\sigma}^2$)] versus the memory exponent α at the noise intensity $D = 1.8$. Other parameter values are the same as in Fig. 2. The dashed lines depict the position of the critical memory exponents $\alpha_1 \approx 0.33$, $\alpha_2 \approx 0.87$, $\alpha_3 \approx 0.95$, and $\alpha_4 \approx 0.98$, at which $\bar{\sigma}^2$ tends to infinity.

6 Conclusions

In this work, we have analysed the phenomena of SR and stochastic parametric resonance within the context of a noisy, fractional oscillator with a fluctuating eigenfrequency driven by external sinusoidal forcing and by an additive internal fractional noise. The frequency fluctuations are modeled as a Gaussian white noise. According to Kubo's second fluctuation-dissipation theorem, the viscoelastic type friction kernel with memory is assumed as a power-law function of time. The Laplace transformation technique with the Furutzu–Novikov procedure allows us to find exact formulas for the output variance and for the signal–to–noise ratio (SNR) at the long-time limit.

A major virtue of the investigated model is that interplay of parametric fluctuations, external periodic forcing, and the memory of the friction kernel can generate a variety of nonequilibrium cooperation effects.

One of our main results is that the energetic stability of the noisy fractional oscillator can be significantly enhanced at intermediate values of the memory exponent. This result is highly unexpected in view of the cage effect observable at strong memory [26].

As another main result we have found a multiresonance versus the memory exponent of the output SNR, and we show that this phenomenon is significantly associated with the critical characteristics of stochastic parametric resonance. Furthermore, the memory of the friction kernel can induce repeated reentrant transitions between different dynamical regimes of the oscillator. Namely, in some cases an increase of the memory exponent induces transitions from a regime where the system is energetically unstable to a stable regime, but instability appears again through a reentrant transition at higher values of the memory exponent.

We believe that the results of this paper not only supply new issues for theoretical investigations of fractional dynamics in stochastic systems, but also suggest new possibilities for interpreting experimental subdiffusion results in biological applications, where issues of memory and multiplicative noise can be crucial [20, 27, 38, 39, 40].

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References:

[1] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Stochastic resonance, *Rev. Mod. Phys.*, Vol. 70, 1998, pp. 223–287.

[2] A. Mielke, Noise Induced Stability in Fluctuating, Bistable Potentials, *Phys. Rev. Lett.*, Vol. 84, 2000, pp. 818–821.

[3] R. Mankin, E. Soika, and A. Sauga, Multiple noise-enhanced stability versus temperature in asymmetric bistable potentials, WSEAS Transactions on Systems, Vol. 7, 2008, pp. 239–250.

[4] M. O. Magnasco, Forced thermal ratchets, *Phys. Rev. Lett.*, Vol. 71, 1993, pp. 1477-1481.

[5] P. Reimann, Brownian motors: noisy transport far from equilibrium, *Phys. Rep.*, Vol. 361, 2002, pp. 57–265.

[6] R. Mankin, T. Laas, A. Sauga, A. Ainsaar, and E. Reiter, Colored-noise-induced Hopf bifurcations in predator-prey communities, *Phys. Rev. E*, Vol. 74, 2006, pp. 021101(1)–(10).

[7] A. Sauga and R. Mankin, Addendum to ”Colored-noise-induced discontinuous transitions in symbiotic ecosystems”, *Phys. Rev. E*, Vol. 71, 2005, pp. 062103(1)–(4).

[8] R. Mankin, A. Sauga, T. Laas, and E. Soika, Environmental-Fluctuations-Induced Slow-Fast Cycles in Ratio-Dependent Predator-Prey Systems. *WSEAS Transactions on Systems*, Vol. 6, pp. 934 - 941.

[9] R. Mankin, T. Laas, E. Soika, and A. Ainsaar, Noise-controlled slow-fast oscillations in predator-prey models with the Beddington functional response, *Eur. Phys. Jour. B*, Vol. 59, 2007, pp. 259–269.

[10] H. Linke (ed), Special issue on ”Ratchets and Brownian motors: basics, experiments and applications”, *Appl. Phys. A*, Vol. 75, 2002, pp. 167–352.

[11] R. Tammelo, R. Mankin, and D. Martila, Three and four current reversals versus temperature in correlation ratchets with a simple sawtooth potential, *Phys. Rev. E*, Vol. 66, 2002, pp. 051101(1)–(5).

[12] S. Chandrasekhar, Stochastic Problems in Physics and Astronomy, *Rev. Mod. Phys.*, Vol. 15, 1943, pp. 1–89.

[13] K. Lindenberg, V. Seshadri, and B. J. West, Brownian motion of harmonic systems with fluctuating parameters. II. Relation between moment instabilities and parametric resonance, *Phys. Rev. A*, Vol. 22, 1980, pp. 2171–2179.

[14] R. Mankin, K. Laas, T. Laas, and E. Reiter, Stochastic multiresonance and correlation-time-controlled stability for a harmonic oscillator with fluctuating frequency, *Phys. Rev. E*, Vol. 78, 2008, pp. 031120(1)–(11).

[15] M. Gitterman, Classical harmonic oscillator with multiplicative noise, *Physica A*, Vol. 352, No. 2–4, 2005, pp. 309–334.

[16] R. C. Bourret, U. Frisch, and A. Pouquet, Brownian motion of harmonic oscillator with stochastic frequency, *Physica*, Vol. 65, 1973, pp. 303–320.

[17] I. Bena, C. Van den Broeck, R. Kawai, M. Copelli, and K. Lindenberg, Collective behavior of parametric oscillators, *Phys. Rev. E*, Vol. 65, 2002, pp. 036611(1)–(18).

[18] K. Laas, R. Mankin, and A. Rekker, Hypersensitive response of a harmonic oscillator with fluctuating frequency to noise amplitude. In: Recent Advances in Mathematical Biology and Ecology. *Proceedings of the 5th WSEAS International Conference on Mathematical Biology and Ecology: MABE09.* Lifeng Xi. Ningbo, China: WSEAS, 2009, (Book Series: Recent Advances in Biology and Biomedicine), pp. 15 - 20.

[19] A. D. Viñales and M. A. Despósito, Anomalous diffusion: Exact solution of the generalized Langevin equation for harmonically bounded particle, *Phys. Rev. E*, Vol. 73, 2006, pp. 016111(1)–(4).

[20] S. C. Kou and X. Sunney Xie, Generalized Langevin Equation with Fractional Gaussian Noise: Subdiffusion within a Single Protein Molecule, *Phys. Rev. Lett.*, Vol. 93, 2004, pp. 180603(1)-(4).

[21] E. Lutz, Fractional Langevin equation, *Phys. Rev. E*, Vol. 64, 2001, pp. 051106(1)-(4).

[22] W. Götze and L. Sjögren, Relaxation processes in supercooled liquids, *Rep. Prog. Phys.*, Vol. 55, 1992, pp. 241-376.

[23] T. Carlsson, L. Sjögren, E. Mamontov, and K. Psiuk-Maksymowicz, Irreducible memory function and slow dynamics in disordered systems, *Phys. Rev. E*, Vol. 75, 2007, pp. 031109(1)–(8).

[24] T. G. Mason and D. A. Weitz, Optical Measurements of Frequency-Dependent Linear Viscoelastic Moduli of Complex Fluids, *Phys. Rev. Lett.*, Vol. 74, 1995, pp. 1250–1253.

[25] Qing Gu, E. A. Schiff, S. Grebner, F. Wang, and R. Schwarz, Non-Gaussian Transport Measurements and the Einstein Relation in Amorphous Silicon, *Phys. Rev. Lett.*, Vol. 76, 1996, pp. 3196–3199.

[26] S. Burov and E. Barkai, Fractional Langevin equation: Overdamped, underdamped, and critical behaviors, *Phys. Rev. E*, Vol. 78, 2008, pp. 031112(1)–(18).

[27] I. M. Tolić-Nørrelykke, E.-L. Munteanu, G. Thon, L. Oddershede, and K. Berg-Sørensen, Anomalous Diffusion in Living Yeast Cells, *Phys. Rev. Lett.*, Vol. 93, 2004, pp. 078102(1)–(4).

[28] R. Kubo, The fluctuation-dissipation theorem, *Rep. Prog. Phys.*, Vol. 29, 1966, pp. 255-284.

[29] E. A. Novikov, Functionals and the random-force method in turbulence theory, *Soviet Physics JETP*, Vol. 20, 1965, pp. 1290–1299.

[30] J. Casado-Pascual, J. Gómez-Ordóñez, M. Morillo, and P. Hänggi, *Phys. Rev. E*, Vol. 68, 2003, pp. 061104(1)–(7).

[31] K. G. Wang and M. Tokuyama, Nonequilibrium statistical description of anomalous diffusion, *Physica A*, Vol. 265, 1999, pp. 341-351.

[32] B. Mandelbrot and J.W. Van Ness, Fractional Brownian motions, fractional noises and applications, *SIAM Rev.*, Vol. 10, 1968, pp. 422-437.

[33] P. Embrechts and M. Maejima, *Selfsimilar Processes*, Princeton University Press, Princeston, 2002.

[34] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[35] U. Weiss, *Quantum Dissipative Systems*, 2nd ed., World Scientific, Singapore, 1999.

[36] R. Granek and J. Klafter, Fractons in Proteins: Can They Lead to Anomalously Decaying Time Autocorrelations, *Phys. Rev. Lett.*, Vol. 95, 2005, pp. 098106(1)–(4).

[37] S. Kempfle, I. Schafer, and H. Beyer, Fractional calculus via functional calculus: theory and applications., *Nonlinear Dyn.*, Vol. 29, 2002, pp. 99-12.

[38] W. Min, G. Luo, B. J. Cherayil, S. C. Kou, and X. S. Xie, Observation of a Power-Law Memory Kernel for Fluctuations within a Single Protein Molecule, *Phys. Rev. Lett.*, Vol. 94, 2005, pp. 198302(1)–(4).

[39] R D. Astumian and M. Bier, Mechanochemical coupling of the motion of molecular motors to ATP hydrolysis, *Biophys. J.*, Vol. 70, 1996, pp. 637–653.

[40] A. Rekker and R. Mankin, Stochastic parametric resonance of a fractional oscillator. In: Recent Advances in Apllied Mathematics. *Proceedengs of the 14th WSEAS International Conference on Applied Mathematics: MATH09.* Puerto de La Cruz, Spain: WSEAS, 2009, (Book Series: Mathematics and Computers in Science and Engineering), pp. 93–98.