Vehicle mathematical model reduction considering the brake system dynamics

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Abstract: - The present work shows a complete model for longitudinal movement of an automobile, starting with the reference coordinate systems; an inertial system, a system considering the vehicle as a rigid body and different coordinate systems for considering the center of mass of the tire and the contact element. Then the dynamics for subsystems involved in the braking process are described. The interest of the model is around the maximum value of the contact force, which physically means that the wheel is closed to a block situation. After this we obtained the characteristic values for every subsystem, and the model is evaluated for a metropolitan transport bus, Finally, considering the constant time for changes on the angular velocity as characteristic time we apply the fractional analysis to obtain a reduced order model of a vehicle, for the simulation the parameters of a bus were considered and the result the dynamics with enough accuracy for the conditions considered.

Key-Words: - Mathematical model, Vehicle, Brake system, Fractional analysis, Model reduction.

1 Introduction

The changes on the automotive industry include not only more powerful engines or improvements on interior or exterior designs, fuel consumption, etc. one very important factor is the security, passive and active systems have been developed.

Passive security is oriented to minimize the consequences on the passenger in case of accident, for example; seat belts, Airbags, ergonomic interiors, etc. the Active security includes devices that the driver can control, for example: ABS, active suspension, active steer, etc.).

The present work deals with a complete vehicle model that is reduced by using the fractional analysis to get a simpler model that describes the brake process, under conditions that can activate the ABS.

The fractional analysis [8] allow us to analyze systems according to specific parameters, with this we can separate variables according to intervals in which the process occurs, in this particular case we separated the variables according to time intervals so we can talk about fast and slow variables.

Many works deal with the problem of modeling, in order to study more complex dynamics, for example vibrations [14].

2 Complete system model

2.1 Coordinate systems and translations

We suppose that the automobile's movement occurs on a horizontal surface, the different coordinate systems are shown in fig. 1 and fig. 2.

We introduce the next coordinate systems: $O\xi\eta\zeta$- Stationary. The plane $O\xi\eta$ coincides with the movement surface. Axis $O\zeta$ is vertical, $O\xi$ - is horizontal

The centre of mass position of the automobile C is given in Cartesian coordinates X,Y,Z on the system $O\xi\eta\zeta$.

The system $Cx_0y_0z_0$ - with origin on the centre of mass, the axis Cz_0 is vertical, the axis Cx_0 lies on the longitudinal plane of symmetry of the automobile. The system $Cx_0y_0z_0$ can be obtained from $O\xi\eta\zeta$ as a result of a translation of the point C and a rotation on ψ over the axis ζ. Where the angle ψ represents the curse angle of the vehicle.

The system Cxyz - related with the vehicle's body coincides with the central inertia axis. A change from the system $Cx_0y_0z_0$ to the system Cxyz is given by the rotation angles, γ over the axis x and ϑ as the attack angle over the axis y.
The systems $A_{ij}x_{ij}y_{ij}z_{ij}$ - related with the wheel. The index "i" has values i = 1 forward, i = 2 rear, while the index "j" has values j = 1 right and j = 2 left, according to the movement's direction. The point A_{ij} lies on the intersection of the turn axis of the ij-th wheel and the longitudinal symmetry plane. The axis $A_{ij}y_{ij}$ coincides with the rotation axis of the wheel, the axis $A_{ij}x_{ij}$ is horizontal.
The system $O_{ij}x_{ij}y_{ij}z_{ij}$ with origin on the point O_{ij}.

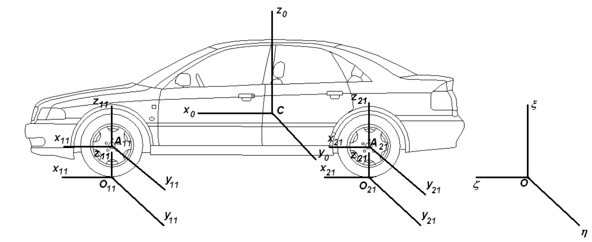

Fig. 1

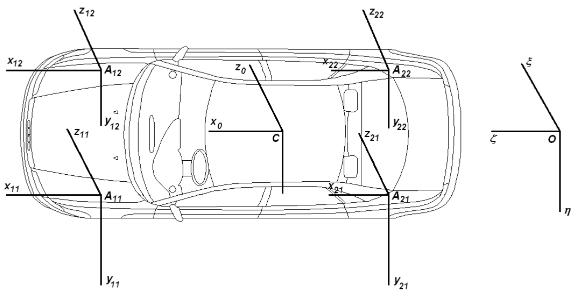

Fig. 2

2.2 Dynamic equations for the movement of the complete vehicle

We propose that the movement occurs only over the plane x0y0. The movement equations for the centre of mass considering a rigid body with six degrees of freedom are written on the tridimensional system $Cx_0y_0z_0$ as follows:

$$M\frac{dV_x}{dT} = \sum_{i,j=1}^{2} P_{ijx} + MV_y\Omega_z + F_{ax} \quad (1)$$

$$M\frac{dV_y}{dT} = \sum_{i,j=1}^{2} P_{ijy} - MV_x\Omega_z + F_{ay} \quad (2)$$

$$M\frac{dV_z}{dT} = \sum_{i,j=1}^{2} N_{ij} - Mg \quad (3)$$

here T - time, M - total vehicle mass (including wheels), - (V_x, V_y, V_z) absolute velocity vector projections of the point C on the system $Cx_0y_0z_0$, Ω_z - absolute angular velocity projection over vertical axis, Mg - weight, $(P_{ijx}; P_{ijy}; N_{ij})$ - contact force projections and normal reaction. $(MV_y\Omega_z, -MV_x\ \Omega_z, 0)$ - inertial force projections, $(F_{ax}; F_{ay})$ – aerodynamic force projections, all the forces acting over the system are shown on fig. 3.

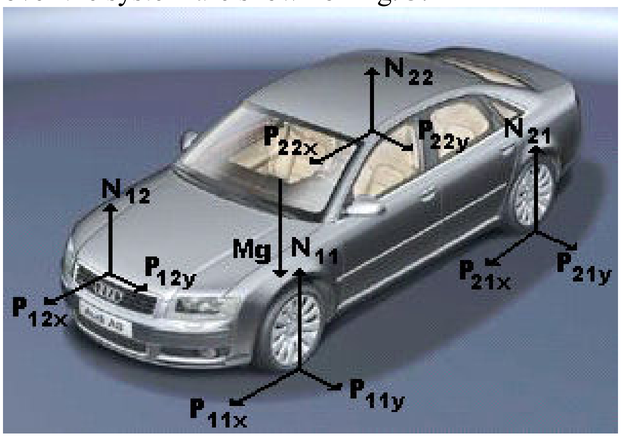

Fig. 3

The kinetic moment change equations related to point C in tridimensional projections $Cx_0y_0z_0$, considering the absences of jumping due to road irregularities are the next:

$$I_x\frac{d\Omega_x}{dT} = \left[(N_{11}+N_{21})-(N_{12}+N_{22})B\right] - Z\sum_{i,j=1}^{2} P_{ijy} - \\ -\sum_{j=1}^{2} I_{1j}\left[\dot{\Omega}_{1j}\sin\Theta + \Omega_{1j}\dot{\Theta}\cos\Theta - \Omega_{1j}\Omega_z\cos\Theta\right] \quad (4)$$

$$I_y\frac{d\Omega_y}{dT} = (N_{21}+N_{22})A_2 - (N_{11}+N_{12})A_1 - Z\sum_{i,j=1}^{2} P_{ijx} - \\ -\sum_{j=1}^{2} I_{1j}\left[\dot{\Omega}_{1j}\cos\Theta + \Omega_{1j}\dot{\Theta}\sin\Theta - \Omega_{1j}\Omega_z\sin\Theta\right] \quad (5)$$

$$I_z\frac{d\Omega_z}{dT} = \left[(P_{12x}+P_{22x})-(P_{11x}+P_{21x})\right]B + \\ +(P_{11y}+P_{12y})A_1 - (P_{21y}+P_{22y})A_2 \quad (6)$$

Where I – represents the inertial moment of the whole vehicle as well as any of the wheels according to the suffixes [3], [11], [5].

2.3 Cinematic equations of the vehicle's movement

We write the cinematic equations of the system on the form:

$$\frac{dX}{dT} = V_x \cos\psi - V_y \sin\psi \tag{7}$$

$$\frac{dY}{dT} = V_x \sin\psi - V_y \cos\psi \tag{8}$$

$$\frac{dZ}{dT} = V_z \tag{9}$$

$$\frac{d\psi}{dT} = \Omega_z \tag{10}$$

and the equations for the angular velocity on a linear approach are:

$$\frac{d\gamma}{dT} = \Omega_x \tag{11}$$

$$\frac{d\vartheta}{dT} = \Omega_y \tag{12}$$

2.4 Relations for the contact force between the wheel and the road

We have the next expressions:

$$P_{11x_0} = P_{11x} \cos\Theta_{11} - P_{11y} \sin\Theta_{11} \tag{13}$$

$$P_{11y_0} = P_{11x} \sin\Theta_{11} - P_{11y} \cos\Theta_{11} \tag{14}$$

$$P_{12x_0} = P_{12x} \cos\Theta_{12} - P_{12y} \sin\Theta_{12} \tag{15}$$

$$P_{12y_0} = P_{12x} \sin\Theta_{12} - P_{12y} \cos\Theta_{12} \tag{16}$$

$$\begin{aligned} P_{21x_0} &= P_{21x}, \quad P_{21y_0} = P_{21y} \\ P_{22x_0} &= P_{22x}, \quad P_{22y_0} = P_{22y} \end{aligned} \tag{17}$$

Fig. 4 shows the forces appearing on the last expressions [13], [6].

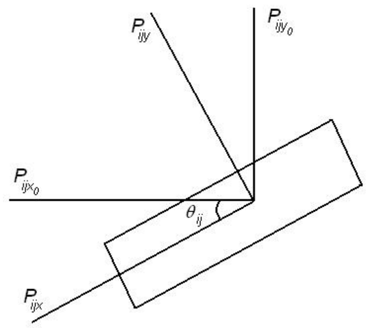

Fig. 4

2.5 Model for the contact element movement

The movement equations are written as projections on $O_{ij}x_{ij}y_{ij}z_{ij}$

$$M_c \frac{d}{dT}\left(V_x + \Omega_{ij} R + \dot{\breve{\xi}}_{ij}\right) = -C_x \dot{\breve{\xi}}_{ij} - K_x \breve{\xi}_{ij} + P_{ijx} \tag{18}$$

$$M_c \frac{d}{dT}\left(V_y + \dot{\breve{\eta}}_{ij}\right) = -C_y \dot{\breve{\eta}}_{ij} - K_y \breve{\eta}_{ij} + P_{ijy} \tag{19}$$

$$0 = -C_z \dot{\breve{\zeta}}_{ij} - K_z \breve{\zeta}_{ij} + N_{ij} \tag{20}$$

Where P_{ijx}, P_{ijy}, N_{ij} - projections of the contact force over the contact element, $-K_x\xi_{ij}$, $-K_y\eta_{ij}$, $-K_z\zeta_{ij}$ - elastic forces vector projections for the contact element, considering the nondeformable part of the tire, K_x, K_y, K_z - elasticity coefficients, $-C_x\xi'_{ij}$, $-C_y\eta'_{ij}$, $-C_z\zeta'_{ij}$,- damping forces vector projections, C_x, C_y, C_z - damping coefficients.

At this point we suppose that there are no vertical movements of the contact element, fig. 5 shows the contact element [9].

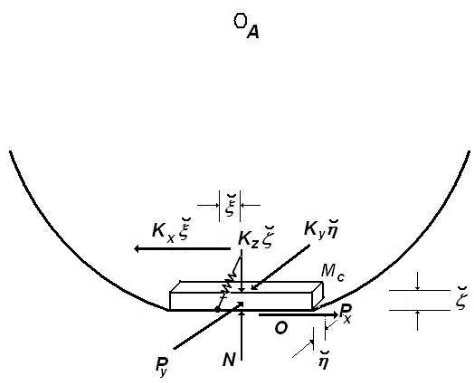

Fig. 5

2.6 Wheel rotation equation

The next equation describes the wheel rotation shown in fig. 6.

$$I_j \frac{d\Omega_{ijy}}{dT} = -P_{ijx}\left(R - \zeta_{ij}\right) + L_{ij} \tag{21}$$

Here I_j - inertial moment for the wheel, L_{ij} - brake moment.

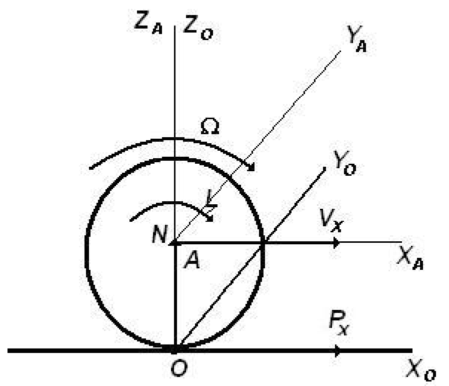

Fig. 6

2.7 Contact force model

From the model obtained on [7]. Slip rate is represented by s

$$s_{x_{ij}} = \frac{V_{Ox_{ij}}}{V_{Ax_{ij}}} \tag{22}$$

$$s_{y_{ij}} = \frac{V_{Oy_{ij}}}{V_{Ax_{ij}}} \tag{23}$$

Where V_{Oxij} - longitudinal velocity of the contact shadow, V_{Axij} - longitudinal velocity of the mass centre of the wheel.
Then we have the next expressions for the contact force:

$$V_{Ox_{ij}} = V_{Ax_{ij}} - \Omega_{ij} R + \xi_{ij} \tag{24}$$

$$V_{Oy_{ij}} = V_{Ay_{ij}} + \eta_{ij} \tag{25}$$

By substituting in the last equations we have:

$$s_{xij} = \frac{V_{Ax_{ij}} - \Omega_{ij} R + \xi_{ij}}{V_{Ax_{ij}}} \tag{26}$$

$$s_{yij} = \frac{V_{Ay_{ij}} + \eta_{ij}}{V_{Ay_{ij}}} \tag{27}$$

For the contact force we have:

$$P_{x_{ij}} = -\nu N \frac{s_{x_{ij}}}{s_{ij}} \varphi\left(s_{ij}\right) \tag{28}$$

$$P_{y_{ij}} = -\nu N \frac{s_{y_{ij}}}{s_{ij}} \varphi\left(s_{ij}\right) \tag{29}$$

Where $s_{ij} = \sqrt{s_{x_{ij}}^2 + s_{y_{ij}}^2}$

Function φ(s) is defined experimentally and it will be described further.
When the vehicle moves only over x axis we have $s_{yij} = 0$ and $P_{yij} = 0$, then the equation takes the form:

$$P_{x_{ij}} = -\nu N \varphi\left(s_{ij}\right) sign\left(s_{x_{ij}}\right) \tag{30}$$

2.8 Model for the vertical vibrations of the wheel

We study the mechanical system composed by the elements of the suspension, the chassis and the support elements [1], [2]. The movement equation for the centre of mass of such system is:

$$m_A \frac{d^2 z_{Aij}}{dT^2} = N_{ij} + C_{jz}\Delta z_{ij} + D_j \frac{d\Delta z_{ij}}{dT} \tag{31}$$

Here m_A - equivalent mass of the system, $z_{Aij}=R-\zeta_{ij}+const$ - centre of mass coordinates, Δz_{ij} - springs deformation, D_j, C_{jz} - damping coefficients, that is in general different for front and rear wheels.

2.9 Model for the brake system

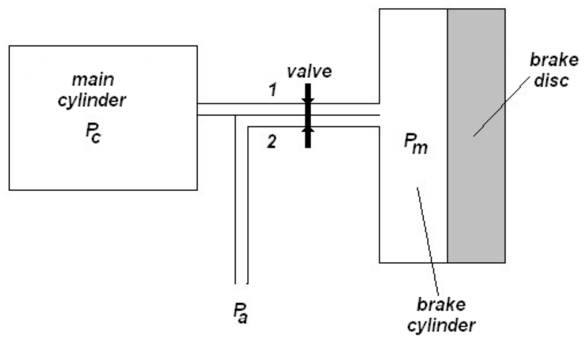

Fig. 7

The brake system shown in fig. 7 includes two different tanks, the main cylinder and the brake

cylinder, from each one comes out a pipe, from the main cylinder to the brake cylinder (1) and from the brake cylinder to the atmosphere (2). We analyze the system with a single valve that acts on both pipes, when 1 is closed 2 is opened and when 1 is opened 2 is closed [12].
We suppose that the brake force is proportional to the pressure on the brake cylinder, and the direction is opposite to the angular velocity of the wheel.

$$L_{ij} = -K_L P_{m_{ij}} sign\Omega_{ij}$$

By using a first order approximation for the pressure change on the brake cylinder we have:

$$\begin{cases} T_{e_{in}} \dfrac{dP_{m_{ij}}}{dT} + P_{m_{ij}} = P_c\left(T + \Delta t_b\right) & \text{fill} \\ T_{e_{in}} \dfrac{dP_{m_{ij}}}{dT} + P_{m_{ij}} = P_a & \text{exhaust} \end{cases} \quad (32)$$

Where P_{mij} - pressure on the pipes arriving to the brake cylinder, T_e - characteristic time of the pipe, and Δt_b - time delay due to the valve's movement.

2.10 Pade approximation of the characteristic φ(s)

We study the behavior of the system on the region around the maximum value of the brake moment, that means the maximum value for the function φ(s), on the range $0.12 \leq s \leq 0.52$, We propose the next approximation function [10], [13]:

$$\varphi(s) = \frac{a_1 s^2 + a_2 s + a_3}{s^2 + a_4 s + a_5} \quad (33)$$

in order to define the coefficients the equation is written in the next form:

$$a_1 s_i^{\,2} + a_2 s_i + a_3 - \varphi(s_i)(a_4 s_i + a_5) = \varphi(s_i) s_i^{\,2} \quad (34)$$

Where "i" indicates the value for experimental tests.
Then we can build the next system:

$$PA = B$$

Where

$$P = \begin{pmatrix} s_1^2 & s_1 & 1 & -\varphi(s_1)s_1 & -\varphi(s_1) \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ s_i^2 & s_i & 1 & -\varphi(s_i)s_i & -\varphi(s_i) \end{pmatrix}$$

Matrix P is not square and it size depends on the number of experimental tests.

$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_5 \end{pmatrix}$$

$$B = \begin{pmatrix} \varphi(s_1)s_1^2 \\ \vdots \\ \varphi(s_i)s_i^2 \end{pmatrix}$$

Calculating A by least square method we have:

$$A = \left(P^T P\right)^{-1} P^T B$$

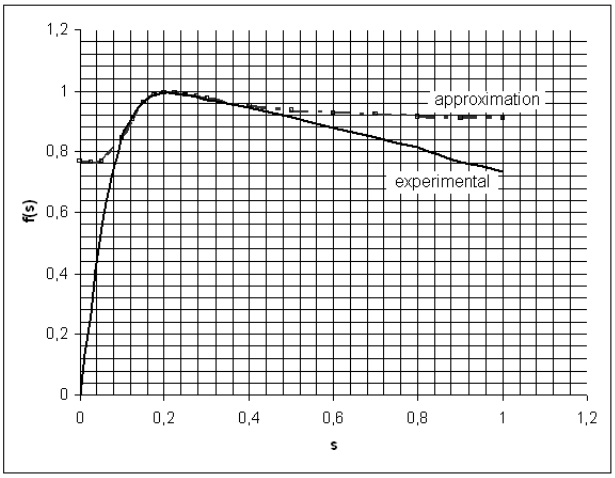

Fig. 8

3 Fractional Analysis

3.1 Time constants description

3.1.1 Characteristic time for longitudinal movement of the vehicle

We study the longitudinal movement of the vehicle, considering a symmetrical system and the absence of any lateral force
From (1) we have

$$M \frac{dV_x}{dT} = \sum_{i,j=1}^{2} P_{ijx}$$

Analyzing the movement for the case when
$P_{ijx} = P_*$
for every i, j. The equation takes the form form

$$\frac{dV_x}{dT} = \frac{4P_*}{M}$$

and

$$V_x = \frac{4P_*}{M}\left(T - T_0\right) - V_{x_0}$$

Then the characteristic time shall be the time for wich V_x increces from 0 to a given value V_{x*}, considering $V_{x0} = 0$, $T_0 = 0$ and $V_x = V_{x*}$ we have:

$$T_0 = \frac{MV_{x*}}{4P_*} \tag{35}$$

T_0 – Characteristic time for the longitudinal movement of the vehicle.

3.1.2 Characteristic time for vertical vibrations of the vehicle's body

We study the elastic vertical vibrations of the vehicle's body on the springs; we take the model shown on fig. 9. The damping force for both front and rear is not considered.

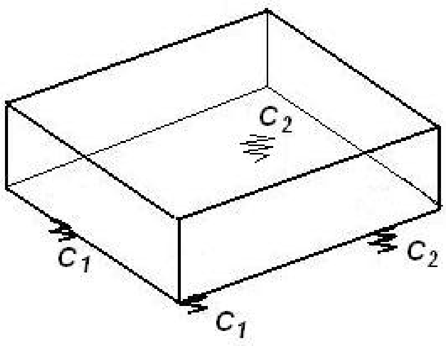

Fig. 9

$$M\frac{d^2z}{dT^2} = 2(C_1 + C_2)z - Mg$$

Rewriting we have:

$$\frac{M}{2(C_1 + C_2)}\frac{d^2z}{dT^2} = z - \frac{Mg}{2(C_1 + C_2)}$$

Taking as a constant time the vibration's period of the chassis over the spring we have

$$T_1 = \sqrt{\frac{M}{2(C_1 + C_2)}} \tag{36}$$

T_1 – Time constant for vertical vibrations of the body due to the suspension springs.

3.1.3 Characteristic time for vibrations of the nondeformable mass of the tire and suspension

We study the vertical vibrations due to the longitudinal movement of the vehicle over the road surface

Writing the equations for vertical vibrations of the wheel as in (28) for a stationary body with a flat contact surface.

$$m_A\frac{d^2z_{A_{ij}}}{dT^2} = K_z\zeta_{ij} + C_{jz}\Delta z_{ij}$$

Since z_{Aij}=R- ζ_{ij} and Δz_{ij}= z_{Aij}- z_{Aij0} we can write the next

$$m_A\frac{d^2\zeta_{ij}}{dT^2} = K_z\zeta_{ij} - C_{jz}\dot{\zeta}_{ij}$$

Then the period of vertical vibrations is

$$T_2 = \sqrt{\frac{m_A}{K_z + C_{jz}}} \tag{37}$$

T_2 – Characteristic time for the vibrations of the nondeformable mass.

3.1.4 Characteristic time for velocity changes on the angular velocity of the wheel due to a longitudinal contact force

We study the movement of the wheel in horizontal direction. Considering that neither blocking situations nor slipping are present, we also consider that the deformable part of the tire is very small, compared with the whole tire, the vehicle moves with a constant longitudinal velocity V_{x*}.

From equations (4), (28) and (30) we have

$$I_j\frac{d\Omega_{ijy}}{dT} = -P_{ijx}R + L_{ij}$$

Where $P_{ijx} = -\nu N_{ij}K_0\varepsilon_{ijx}$

Rewriting we have

$$\frac{I_jV_{x*}}{RT}\frac{d\Omega_{ijy}}{dT} = -R\nu N_*K_0\varepsilon_{ijx}\sum_{i,j=1}^{2}N_{ij} + L_{ij}$$

or

$$\frac{I_jV_{x*}}{R^2\nu N_*K_0T}\frac{d\Omega_{ijy}}{dT} = -\varepsilon_{ijx}\sum_{i,j=1}^{2}N_{ij} + L_{ij}$$

Then we have

$$T_3 = \frac{I_jV_{x*}}{R^2\nu N_*K_0} \tag{38}$$

T_3 – Time constant for changes on the angular velocity of the wheel due to a longitudinal contact force.

3.1.5 Characteristic time the tire's deformation in the longitudinal direction

We study the changes on the deformation of the tire, considering that no vertical vibrations are present, as well as lateral forces and the movement occurs in longitudinal direction only. We also consider V_{ijx} y Ω_{ijy} as constants.

From equations (15), (28) y (30) we have

$$M_c \frac{d^2 \breve{\xi}_{ij}}{dT^2} = -K_x \breve{\xi}_{ij} - C_x \frac{d\breve{\xi}_{ij}}{dT} + P_{ijx}$$

Where the contact force for small slipping is considered lineal.

$$P_{ijx} = -\nu N_{ij} K_0 \left(\frac{V_{ijx} + \Omega_{ijy} R + \frac{d\breve{\xi}_{ij}}{dT}}{V_{ijx}} \right)$$

then

$$M_c \frac{d^2 \breve{\xi}_{ij}}{dT^2} + K_x \breve{\xi}_{ij} + C_x \frac{d\breve{\xi}_{ij}}{dT} =$$

$$-\nu N_{ij} K_0 \left(\frac{V_{ijx} + \Omega_{ijy} R + \frac{d\breve{\xi}_{ij}}{dT}}{V_{ijx}} \right)$$

Rewriting

$$M_c \frac{d^2 \breve{\xi}_{ij}}{dT^2} + \left(C_x + \frac{\nu N_{ij} K_0}{V_{ijx}} \right) \frac{d\breve{\xi}_{ij}}{dT} + K_x \breve{\xi}_{ij} =$$

$$-\nu N_{ij} K_0 \left(\frac{V_{ijx} + \Omega_{ijy} R}{V_{ijx}} \right)$$

The change period for ξ is equal to

$$T_{4*} = \frac{1}{\sqrt{\frac{K_x}{M_c} - \left(\frac{C_x V_{ijx} + \nu N_{ij} K_0}{2 M_c V_{ijx}} \right)^2}}$$

If $\frac{K_x}{M_c} >> \left(\frac{C_x V_{ijx} + \nu N_{ij} K_0}{2 M_c V_{ijx}} \right)^2$ we can write

$$T_4 = \sqrt{\frac{K_x}{M_c}} \qquad (39)$$

T_4 – Time constant for tire's partial deformation.

3.1.6 Characteristic time of the tire's deformation in lateral diurection

We study the tire's deformation in the lateral direction, considering the absence of vertical vibrations and external lateral forces. The vehicle's movement ocuurs only in the longitudinal direction. From equations (16), (28) and (30) we have

$$M_c \frac{d^2 \breve{\eta}_{ij}}{dT^2} = -K_y \breve{\eta}_{ij} - C_y \frac{d\breve{\eta}_{ij}}{dT} + P_{ijy}$$

For the lineal dependency of $\varphi(s)$ we have

$$P_{ijy} = -\nu N_{ij} \frac{s_{ijy}}{s_{ij}} \varphi(s_{ij}) = -\nu N_{ij} K_0 \frac{d\breve{\eta}_{ij}}{dT}$$

then

$$M_c \frac{d^2 \breve{\eta}_{ij}}{dT^2} + C_y \frac{d\breve{\eta}_{ij}}{dT} + K_y \breve{\eta}_{ij} = -\nu N_{ij} K_0 \frac{d\breve{\eta}_{ij}}{dT}$$

the change period η ↻←

$$T_{5*} = \frac{1}{\sqrt{\frac{K_y}{M_c} - \left(\frac{C_y + \nu N_{ij} K_0}{2 M_c} \right)^2}}$$

Since $\frac{K_y}{M_c} >> \left(\frac{C_y + \nu N_{ij} K_0}{2 M_c} \right)^2$ we can write

$$T_5 = \sqrt{\frac{K_y}{M_c}} \qquad (48)$$

T_5 – characteristic time of the lateral deformation of the tire.

3.1.7 Characteristic time of the lateral vibration

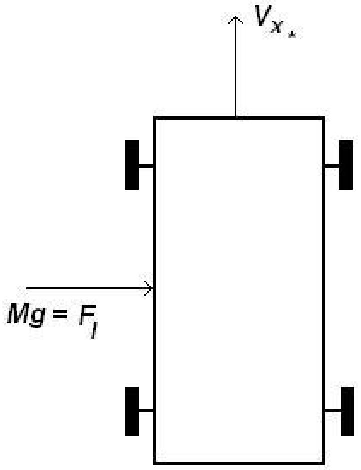

Fig. 10

We study the lateral movement of the vehicle for the case when the lateral forces reach a value around the maximum of φ(s), as is shown in fig. 10, then we can model this characteristic by using the next lineal function.

$$\varphi(s) = K_0 s \tag{49}$$

We consider that the vehicle is moving in horizontal direction due to a constant force

F_1~Mg.

This is true when V_y, Ωz (A_1+A_2) << V_x

then we have

$$M\frac{dV_y}{dT} = \sum_{i,j=1}^{2} P_{ijy} + F_1$$

Considering P_{ijy} = -νNK_0s_y we can write

$$M\frac{dV_y}{dT} = -\nu K_0 \frac{V_y}{V_{x*}} \sum_{i,j=1}^{2} N_{ij} + Mg$$

Rewriting

$$\frac{V_{x*}}{\nu g K_0}\frac{dV_y}{dT} + V_y = \frac{V_{x*}}{\nu K_0}$$

The time constant for this equation is

$$T_6 = \frac{V_{x*}}{\nu g K_0} \tag{50}$$

3.1.8 Characteristic time of the brake system

We suppose the pressure on the brake cylinder P_* = const, for the fill phase P_* = P_c, and for the exhaust phase P_*= P_a.
Then we choose T_{ein} as the characteristic time for fill phase and T_{eout} as the characteristic time for exhaust phase.
We choose the smaller of them as the characteristic time for brake system.

$$T_7 = T_{e_{in}} \tag{51}$$

Since we want to describe a specific behavior of the system, it is necessary to select the time interval according to that specific behavior; in this case, we are interested in the movements related with the brake process we choose the time constant for changes on the angular velocity due to the longitudinal contact force.

3.2 Fractional analysis of the movement equations

We introduce dimensionless variables (symbol "*" means the characteristic dimensional value of the chosen variable).

$$V_x = V_{x*} v_x \tag{52}$$

Here V_{x*} - characteristic velocity on longitudinal direction. For V_{x*} we choose the initial velocity in longitudinal direction. If the time interval analyzed is small enough the this velocity doesn't have a significant change and v_x~1

$$V_y = V_{y*} v_y \tag{53}$$

Where V_{y*} - characteristic velocity on lateral direction. For V_{y*} we choose the initial velocity in lateral direction. We consider that the movement on lateral direction is not bigger than $0.1V_{x*}$

$$V_z = V_{z*} v_z \tag{54}$$

V_{z*} - characteristic velocity on vertical direction. For V_{z*} we choose the characteristic amplitude for the change on velocity due to the vertical vibrations of the suspension at the characteristic frequency.

$$V_{z*} = \frac{Z_* 2\pi}{T_1}$$

$$N_{ij} = N_* n_{ij} \tag{55}$$

N_* - characteristic value for normal reaction, for N_* we choose the value $N_* = \frac{Mg}{4}$

$$P_{ijx} = P_* p_{ijx} \quad (56)$$

$$P_{ijy} = P_* p_{ijy} \quad (57)$$

Here P_* - characteristic value of the contact force. We consider movement only in longitudinal direction and absence of vertical vibrations. Then $P_*=\nu_* N_*$, where $\nu_* =1$

$$\Omega_x = \Omega_{x*}\omega_x \quad (58)$$

Here Ω_{x*} - characteristic value for the angular velocity over axis x, in absence of vertical vibrations. The spin occurs around the center of mass. We choose $\Omega_{x*} = \frac{V_{z*}}{B}$

$$\Omega_y = \Omega_{y*}\omega_y \quad (59)$$

Here Ω_{y*} - characteristic value for the angular velocity over axis y, in absence of vertical vibrations. The spin occurs around the center of mass. We choose $\Omega_{y*} = \frac{V_{z*}}{A_2}$

$$\Omega_z = \Omega_{z*}\omega_z \quad (60)$$

Ω_{z*} - characteristic angular velocity for vehicle's steering. We have $\Omega_{z*} = \frac{V_{y*}}{A_1}$

$$\Omega_{ij} = \Omega_* \omega_{ij} \quad (61)$$

Where Ω_* - characteristic angular velocity for wheel's rotation. For Ω_* we choose the angular velocity in absence of blocking and slip. The vehicle moves longitudinally with velocity V_{x*}, then $\Omega_* = \frac{V_{x*}}{R}$

$$\Theta_{ij} = \Theta_* \theta_{ij} \quad (62)$$

Aquí Θ_* - characteristic angle for rotation of forward wheels around the vertical axis. We choose the maximum possible value for this angle.

$$\Psi = \Psi_* \psi \quad (63)$$

Ψ_* - characteristic angle for vehicle's steering. We choose Ψ_*, as the change on the angle Ψ due to the chassis rotation with an angular velocity Ω_{z*} in a time T_6, then $\Psi_*= \Omega_{z*} T_6$

$$\Gamma = \Gamma_* \gamma \quad (64)$$

Γ_* - characteristic yaw angle of the vehicle. We choose Γ_* as the change on the angle Γ due to the chassis movement during the vertical vibrations time, we have $\Gamma_*= \Omega_{x*}T_1$

$$\Xi = \Xi_* \vartheta \quad (65)$$

Ξ_* - characteristic curse angle for the vehicle. We choose the change on the angle after the time for the vertical vibrations, then we have $\Xi_*=\Omega_{y*}T_1$

$$\xi_{ij} = \xi_* \xi_{ij} \quad (66)$$

ξ_* - characteristic value for tire deformation in the longitudinal direction. We choose the deformation due to the longitudinal velocity during the characteristic time T_3, then we have $\xi_* = V_{x*}T_3$

$$\eta_{ij} = \eta_{ij*}\eta_{ij} \quad (67)$$

η_* - characteristic value for tire deformation in the lateral direction. We choose as η_* the deformation due to a lateral velocity V_{y*} during the characteristic time T_5, we have $\eta_* = V_{y*}T_5$

$$\zeta_{ij} = \zeta_* \zeta_{ij} \quad (68)$$

ζ_* - characteristic value for tire deformation in the vertical direction. We choose as ζ_* the deformation due to the normal reaction, then we have $\zeta_* = \frac{Mg}{4K_z}$

$$T = T_* t \quad (69)$$

T_* - characteristic time, for our case it shall be T_3.

3.3 Simulation

We used the parameters for a metropolitan transport bus with the next numerical values:
$M=9584kg \sim 10^4$,
$K_0=10$, $\nu \sim 1$ (it changes according to road situation, dry road 0.8, and wet road 0.5),
$C_1=248487Ns/m \sim 2.5\times10^5$,
$C_2=407115Ns/m \sim 4\times10^5$,
$m_A=390kg$,
$K_z=841960N/m \sim 8.5\times10^5$,
$I_j=18.9kgm^2 \sim 20$,
$R=0.53m \sim 0.5m$,
with a longitudinal velocity $V_x=20m/s$
T_* from 1/300 til 1/60 depending on ν
With these parameter th characteristic times have the next values:

$$T_0 \sim 1$$

$$T_1 \sim \frac{1}{10}$$

$$T_2 \sim \frac{1}{50}$$

$$T_3 \quad from \quad \frac{1}{300} \quad to \quad \frac{1}{60} \quad depending \quad on \quad v$$

$$T_4 \sim \frac{1}{10000}$$

$$T_5 \sim \frac{1}{10000}$$

$$T_6 \sim \frac{1}{10}$$

$$T_7 \sim \frac{1}{30}$$

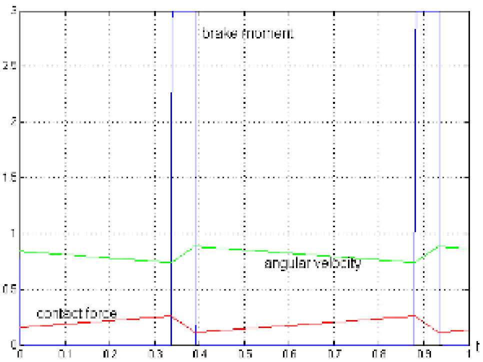

Fig. 11

On fig. 11 we show the response of the reduced system, for the main output parameters: angular velocity, contact force and brake torque.

4 Conclusion

A complete vehicle's model was presented, considering different dynamics that affect the movement, first, was considered a rigid body with six freedom degrees, and then some subsystems were considered as single cases.

For the fractional analysis, every characteristic time was obtained, so that the model can be reduced considering different dynamics.

The resultant complete model was reduced by fractional analysis method, and, since the interest here is the study of the brake system dynamics we considered a characteristic time related with the changes on the angular velocity due to the contact force between the wheel and the road, finally a four equation model was obtained, with angular velocity, longitudinal deformation of the contact element, pressure on the brake cylinder and slip rate as variables of the system. This model is normalized and dimensionless and can be use to design a control for the brake system.

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