Adjoints 2D Hybrid Boundary Value Systems over Spaces of Regulated Functions

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Abstract: A class of 2D hybrid boundary-value time-variable systems is studied, in the general approach of the coefficient matrices, states, inputs and controls over spaces of functions of bounded variation or of regulated functions. A generalized variation-of-parameters formula is obtained for differential-difference equations of the considered type and it is used to derive the formulæ of the state and of the output of these systems. The state space representation of the adjoints of these 2D hybrid systems is introduced and their input-output map is obtained. The duality between the 2D hybrid boundary-value systems and their adjoints is expressed by the means of two bilinear forms.

Key–Words: 2D continuous-discrete systems, input-output map, regulated functions, functions of bounded variation, semiseparable kernels, realizations

1 Introduction

The 2D hybrid continuous-discrete control systems represent an important branch of Systems and Control Theory (see [8], [13], [16], [17] etc). They have both theoretical and practical significance due to their applications in many domains such as linear repetitive processes [2], [18], iterative learning control synthesis [11] or long-wall coal cutting and metal rolling.

This paper extends the study of the continuous-discrete systems to the general framework represented by the space of regulated functions. The topic of regulated functions and of differential equations defined in this approach is studied in a series of monographs or papers (e.g. [3], [7], [12], [20], [21]). The main device is the Perron-Stieltjes integral with respect to regulated functions. The properties of the Perron-Stieltjes integral are reviewed in Section 1 and in Section 2 the solution of the corresponding generalized differential equation is presented on the basis of a generalized fundamental matrix. Section 3 introduces a class of 2D hybrid boundary-value time-variable systems, having the controls over the space of regulated functions, the drift matrix of bounded variation with respect to the continuous variable and the other coefficient matrices being regulated matrix functions.

This class is the 2D hybrid extension of the 1D continuous-time acausal systems introduced by Krener [9], [10] and developed by Gohberg, Kaashoek and Lerer [4], [5], [6]. It represents the time-variable continuous-discrete counterpart of the Attasi's 2D discrete-time time-invariant model. Some extended models were studied in [14] and [17].

A generalized variation-of-parameters formula is obtained for differential-difference equations of the considered type and it is used to derive the formula of the state and the input-output map of these systems. The state space representation of the adjoints of the considered systems is introduced in Section 4 and their input-output map is obtained. The duality between the 2D hybrid boundary-value systems and their adjoints is expressed by the means of two bilinear forms.

The following definitions and notations will be used in the paper. A function $f : [a, b] \to \mathbf{R}$ which posseses finite one sided limits $f(t-)$ and $f(t+)$ for any $t \in [a, b]$ (where by definition $f(a-) = f(a)$ and $f(b+) = f(b)$) is said to be *regulated* on $[a, b]$. The set of all regulated functions denoted by $G(a, b)$, endowed with the supremal norm, is a Banach space; the set $BV(a, b)$ of functions of bounded variation on $[a, b]$ with the norm $||f|| = |f(a)| + \mathrm{var}_a^b f$ is also a Banach space; the Banach space of n-vector valued functions belonging to $G(a, b)$ and $BV(a, b)$ respectively are denoted by $G^n(a, b)$ and $BV^n(a, b)$ (or simply G^n and BV^n); $BV^{n\times m}$ denotes the space of $n \times m$ matrices with entries in $BV(a, b)$. The set of functions $f : [a, b] \times \mathbf{Z} \to \mathbf{R}$ such that $\forall k \in \mathbf{Z}, f(\cdot, k) \in G(a, b)$ $(BV(a, b))$ will be denoted $G_1(a,b)$ ($BV_1(a,b)$) and similar significances will have the above mentioned spaces with subscript 1 ($G_1^n,\ BV_1^n,\ BV_1^{n\times m}$).

A pair $D=(d,s)$ where $d=\{t_0,t_1,\dots,t_m\}$ is a division of $[a,b]$ (i.e. $a=t_0<t_1<\dots<t_m=b$) and $s=\{s_1,\dots,s_m\}$ verifies $t_{j-1}\le s_j\le t_j$, $j=1,\dots,m$ is called a *partition* of $[a,b]$.

A function $\delta:[a,b]\to(0,+\infty)$ is called a *gauge* on $[a,b]$.

Given a gauge δ, the partition (d,s) is said to be *δ-fine* if

$$[t_{j-1},t_j]\subset(s_j-\delta(s_j),s_j+\delta(s_j)),\quad j=1,\dots,m.$$

Given the functions $f,g:[a,b]\to\mathbf{R}$ and a partition $D=(d,s)$ of $[a,b]$ let us associate the integral sum

$$S_D(f\Delta g)=\sum_{j=1}^m f(s_j)(g(t_j)-g(t_{j-1})).$$

Definition 1 The number $I\in\mathbf{R}$ is said to be the *Perron-Stieltjes* (*Kurzweil*) *integral of f with respect to g from a to b* and it is denoted as $\int_a^b f\mathrm{d}g$ or $\int_a^b f(t)\mathrm{d}g(t)$ if for any $\varepsilon>0$ there exists a gauge δ on $[a,b]$ such that $|I-S_D(f\Delta g)|<\varepsilon$ for all δ-fine partitions D of $[a,b]$.

Given $f\in G(a,b)$ and $g\in G([a,b]\times[a,b])$ we define the differences Δ^+,Δ^-,Δ and $\Delta_s^+,\Delta_s^-,\Delta_s$ by $\Delta^+f(t)=f(t+)-f(t)$, $\Delta^-f(t)=f(t)-f(t-)$, $\Delta f(t)=f(t+)-f(t-)$, $\Delta_s^+g(t,s)=g(t,s+)-g(t,s)$, $\Delta_s^-g(t,s)=g(t,s)-g(t,s-)$; $\mathbf{D}^-(f)$, $\mathbf{D}^+(f)$ denote respectively the set of the left and right discontinuities of f in $[a,b]$ and similarly for g we can define $\mathbf{D}_t^-(g)$, $\mathbf{D}_t^+(g)$ with respect to the argument t. We denote by $\sum_t$ the sum $\sum_{t\in\mathbf{D}}$ where $\mathbf{D}=\mathbf{D}^-(f)\cup\mathbf{D}^+(f)\cup\mathbf{D}^-(g)\cup\mathbf{D}^+(g)$.

Let us recall some basic properties of the Perron-Stieltjes integral, by following [18] and [19]. The existence theorem of the Perron-Stieltjes integral $\int_a^b f\mathrm{d}g$ for $f\in BV(a,b)$ and $g\in G(a,b)$, due to Tvrdý [20] is essential for our treatment.

Theorem 2 ([20, Theorems 2.8 and 2.15]) *If $f\in G(a,b)$ and $g\in BV(a,b)$ then the Perron-Stieltjes integrals $\int_a^b f\mathrm{d}g$ and $\int_a^b g\mathrm{d}f$ exist* and

$$\begin{aligned}&\int_a^b f\mathrm{d}g+\int_a^b g\mathrm{d}f=f(b)g(b)-f(a)g(a)+\\&+\sum_t[\Delta^-f(t)\Delta^-g(t)-\Delta^+f(t)\Delta^+g(t)].\end{aligned}\tag{1}$$

Theorem 3 ([20, Proposition 2.16]) *If $\int_a^b f\mathrm{d}g$ exists, then the function $h(t)=\int_a^t f\mathrm{d}g$ is defined on $[a,b]$ and*

i) if $g\in G(a,b)$ then $h\in G(a,b)$ and, for any $t\in[a,b]$

$$\Delta^+h(t)=f(t)\Delta^+g(t),\ \Delta^-h(t)=f(t)\Delta^-g(t)\tag{2}$$

ii) if $g\in BV(a,b)$ and f is bounded on $[a,b]$, then $h\in BV(a,b)$.

Theorem 4 (substitution, [20, Theorem 2.19]) *Let f,g,h be such that h is bounded on $[a,b]$ and the integral $\int_a^b f\mathrm{d}g$ exists. Then the integral $\int_a^b h(t)f(t)\mathrm{d}g(t)$ exists if and only if the integral $\int_a^b h(t)\mathrm{d}\left[\int_a^t f(s)\mathrm{d}g(s)\right]$ exists, and in this case*

$$\int_a^b h(t)f(t)\mathrm{d}g(t)=\int_a^b h(t)\mathrm{d}\left[\int_a^t f(s)\mathrm{d}g(s)\right].\tag{3}$$

Theorem 5 (Dirichlet formula, [19, Theorem I.4.32]) *If $h:[a,b]\times[a,b]\to\mathbf{R}$ is a bounded function and $var_a^b h(s,\cdot)+var_a^b h(\cdot,t)<\infty$, $\forall t,s\in[a,b]$, then for any $f,g\in BV(a,b)$*

$$\begin{aligned}&\int_a^b \mathrm{d}g(t)\left(\int_a^t h(s,t)\mathrm{d}f(s)\right)=\\&=\int_a^b\left(\int_s^b \mathrm{d}g(t)h(s,t)\right)\mathrm{d}f(s)+\\&+\sum_t[\Delta^-g(t)h(t,t)\Delta^-f(t)-\\&-\Delta^+g(t)h(t,t)\Delta^+f(t)].\end{aligned}\tag{4}$$

2 Generalized Linear Differential Equations

The symbol

$$\mathrm{d}x=\mathrm{d}[A]x+\mathrm{d}g\tag{5}$$

where $A\in BV^{n\times n}$ and $g\in G^n(a,b)$ is said to be a *generalized linear differential equation* (GLDE) *in the space of regulated functions*.

Definition 6 A function $x:[a,b]\to\mathbf{R}^n$ is said to be a *solution* of GLDE (5) if for any $t,t_0\in[a,b]$ it verifies the equality

$$x(t)=x(t_0)+\int_{t_0}^t \mathrm{d}[A(s)]x(s)+g(t)-g(t_0).\tag{6}$$

If x satisfies the initial condition

$$x(t_0) = x_0 \tag{7}$$

for given $t_0 \in [a, b]$ and $x_0 \in \mathbf{R}^n$ then x is called the *solution of the initial value problem* (5), (7).

Theorem 7 ([19, Theorem III.2.10]) *Assume that for any* $t \in [a, b]$ *the matrix* $A \in BV^{n\times n}$ *verifies the condition*

$$\det[I + \Delta^+ A(t)] \det[I - \Delta^- A(t)] \neq 0. \tag{8}$$

Then there exists a unique matrix valued function $U : [a, b] \times [a, b] \to \mathbf{R}^{n\times n}$ *such that, for any* $(t, s) \in [a, b] \times [a, b]$

$$U(t, s) = I + \int_s^t d[A(\tau)]U(\tau, s). \tag{9}$$

$U(t, s)$ *is called the fundamental matrix solution of the homogeneous equation*

$$\mathrm{d}x = \mathrm{d}[A]x \tag{10}$$

(or the fundamental matrix of A) and it has the following properties, for any $\tau, t, s \in [a, b]$*:*

$$U(t, s) = U(t, \tau)U(\tau, s); \tag{11}$$

$$U(t, t) = I; \tag{12}$$

$$\begin{aligned} U(t+, s) &= [I + \Delta^+ A(t)]U(t, s),\\ U(t-, s) &= [I - \Delta^- A(t)]U(t, s);\\ U(t, s+) &= U(t, s)[I + \Delta^+ A(s)]^{-1},\\ U(t, s-) &= U(t, s)[I - \Delta^- A(s)]^{-1}; \end{aligned} \tag{13}$$

$$U(t, s)^{-1} = U(s, t); \tag{14}$$

there exists a constant $M > 0$ *such that*

$$|U(t, s)| + \mathrm{var}_a^b U(t, \cdot) + \mathrm{var}_a^b U(\cdot, s) + \mathrm{v}(U) < M \tag{15}$$

where $\mathrm{v}(U)$ *is the twodimensional Vitali variation of* U *on* $[a, b] \times [a, b]$ ([19, *Definition* I.6.1]).

Some methods for the calculus of the fundamental matrix $U(t, s)$ were provided in [12].

From [19, Theorem III.3.1] and [21, Proposition 2.5], one obtains

Theorem 8 (Variation-of-parameters formula) *If* $A \in BV^{n\times n}$ *satisfies the condition* (8)*, then the initial value problem* (5), (7) *has a unique solution given by*

$$\begin{aligned} x(t) &= U(t, t_0)x_0 + g(t) - g(t_0) -\\ &- \int_{t_0}^t \mathrm{d}_s[U(t, s)](g(s) - g(t_0)). \end{aligned} \tag{16}$$

If $g \in G^n$ $(g \in BV^n)$ *then* $x \in G^n$ $(x \in BV^n)$.

3 General response of the 2D hybrid boundary-value systems

The linear spaces $X = G_1^n$, $U = G_1^m$ and $Y = G_1^p$ are called respectively the *state*, *input* and *output spaces*. The *time set* is $T = [a_1, b_1] \times \{a_2, a_2 + 1, \ldots, b_2\}$, where $[a_1, b_1] \subset \mathbf{R}$ and $a_2, b_2 \in \mathbf{Z}$.

Definition 9 A 2D *generalized hybrid boundary value (acausal) system* (2Dghbv) is an ensemble

$$\begin{gathered} \Sigma = (A_1(t, k), A_2(t, k), B(t, k), C(t, k),\\ D(t, k), N_1, N_2, M_1, M_2) \in\\ \in BV_1^{n\times n} \times G_1^{n\times n} \times G_1^{n\times m} \times G_1^{p\times n} \times\\ \times G_1^{p\times m} \times \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times n} \end{gathered}$$

where $A_1(t, k)A_2(t, k) = A_2(t, k)A_1(t, k)$, $\forall (t, k) \in T$, with the following state equation, output equation, boundary condition and output vector equation:

$$\begin{aligned} &\mathrm{d}x(t, k+1) = \mathrm{d}[A_1(t, k+1)]x(t, k+1) +\\ &+ A_2(t, k)\mathrm{d}x(t, k) - \mathrm{d}[A_1(t, k)]A_2(t, k)x(t, k) +\\ &+ B(t, k)\mathrm{d}u(t, k), \end{aligned} \tag{17}$$

$$y(t, k) = C(t, k)x(t, k) + D(t, k)u(t, k), \tag{18}$$

$$N_1 x(a_1, a_2) + N_2 x(b_1, b_2) = v, \tag{19}$$

$$z = M_1 x(a_1, a_2) + M_2 x(b_1, b_2). \tag{20}$$

n is called the *dimension* of the system Σ and it is denoted dimΣ.

The system

$$\begin{gathered} \Sigma_c = (A_1(t, k), A_2(t, k), B(t, k), C(t, k), D(t, k)) \in\\ \in BV_1^{n\times n} \times G_1^{n\times n} \times G_1^{n\times m} \times G_1^{p\times n} \times \times G_1^{p\times m} \end{gathered}$$

with the state equation (17) and the output equation (18) is said to be a *causal system*.

Let $U(t, t_0; k)$ be the fundamental matrix of $A_1(t, k)$, $k \in \{a_2, a_2 + 1, \ldots, b_2\}$ and $F(t; k, k_0)$ the discrete fundamental matrix of $A_2(t, k)$, $t \in [a, b]$, i.e.

$$F(t; k, k_0) =$$

$$= \begin{cases} A_2(t, k-1)A_2(t, k-2)\cdots A_2(t, k_0) & \text{for } k > k_0\\ I_n & \text{for } k = k_0. \end{cases}$$

Since $A_1(t, k)$ and $A_2(t, k)$ are commutative matrices for any $(t, k) \in T$, by the Peano-Baker type formula for U [12] and by the definition of F it results that $U(t, t_0; k)$ and $F(t; k, k_0)$ are commutative matrices too. We shall use the following notations: $\Delta^+ f(s,l) = f(s+,l) - f(s,l)$, $\Delta_s^+ U(t,s;k) = U(t,s+;k) - U(t,s;k)$ and similarly we define $\Delta^- f(s,l)$ and $\Delta_s^- U(t,s;k)$.

Definition 10 A vector $x_0 \in X$ is called the *initial state* of the causal system Σ_c at the moment $(t_0,k_0) \in T$ if $\forall (t,k) \in T$ with $(t,k) \geq (t_0,k_0)$

$$\begin{aligned} x(t,k_0) &= U(t,t_0;k_0)x_0, \\ x(t_0,k) &= F(t_0;k,k_0)x_0. \end{aligned} \tag{21}$$

Proposition 11 (2D *generalized variation of parameters formula). If*

$$\det[(I-\Delta^- A_i(t,k))(I+\Delta^+ A_i(t,k))] \neq 0,\ i=1,2, \tag{22}$$

$\forall t \in [a,b],\ k \in \mathbf{Z}$, *then the solution of the generalized differential-difference equation*

$$\begin{aligned} \mathrm{d}x(t,k+1) &= \mathrm{d}[A_1(t,k+1)]x(t,k+1)+ \\ &+ A_2(t,k)\mathrm{d}x(t,k)- \\ &- \mathrm{d}[A_1(t,k)]A_2(t,k)x(t,k)+ \\ &+ \mathrm{d}f(t,k) \end{aligned} \tag{23}$$

with the initial conditions (19) *is*

$$\begin{aligned} x(t,k) &= U(t,t_0;k)F(t_0;k,k_0)x_0+ \\ &+ \int_{t_0}^{t}\sum_{l=k_0}^{k-1} U(t,s;k)F(s;k,l+1)\mathrm{d}f(s,l)+ \\ &+ \sum_{a\leq s<t} \Delta_s^+ U(t,s;k)\sum_{l=k_0}^{k-1} F(s;k,l+1)\cdot \\ &\cdot\ \Delta^+ f(s,l)- \\ &- \sum_{a<s\leq t} \Delta_s^- U(t,s;k)\sum_{l=k_0}^{k-1} F(s;k,l+1)\cdot \\ &\cdot\ \Delta^- f(s,l). \end{aligned} \tag{24}$$

Proof. We shall use the notation

$$\mathrm{d}g(t,k) = \mathrm{d}x(t,k) - \mathrm{d}[A_1(t,k)]x(t,k). \tag{25}$$

The equation (24) becomes

$$\mathrm{d}g(t,k+1) = A_2(t,k)\mathrm{d}g(t,k) + \mathrm{d}f(t,k). \tag{26}$$

Then

$$\begin{aligned} \mathrm{d}g(t,k_0+1) &= A_2(t,k_0)\mathrm{d}g(t,k_0) + \mathrm{d}f(t,k_0) = \\ &= F(t;k_0+1,k_0)\mathrm{d}g(t,k_0)+ \\ &+ F(t;k_0+1,k_0+1)\mathrm{d}f(t,k_0). \end{aligned}$$

Let us assume that

$$\begin{aligned} &\mathrm{d}g(t,k) = F(t;k,k_0)\mathrm{d}g(t,k_0)+ \\ &+\sum_{l=k_0}^{k-1} F(t;k,l+1)\mathrm{d}f(t,l). \end{aligned} \tag{27}$$

Then, by (26), (27) and by the definition of $F(t;k,k_0)$, we get

$$\begin{aligned} \mathrm{d}g(t,k+1) &= A_2(t,k)F(t;k,k_0)\mathrm{d}g(t,k_0)+ \\ &+ \sum_{l=k_0}^{k-1} A_2(t,k)F(t;k,l+1)\mathrm{d}f(t,l)+ \\ &+ \mathrm{d}f(t,k) = \\ &= F(t;k+1,k_0)\mathrm{d}g(t,k_0)+ \\ &+ \sum_{l=k_0}^{k} F(t;k+1,l+1)\mathrm{d}f(t,l) \end{aligned}$$

hence (27) is true $\forall k > k_0$. Moreover, from (19), (25) and (10) one obtains

$$\begin{aligned} \mathrm{d}g(t,k_0) &= \mathrm{d}x(t,k_0)- \\ &- \mathrm{d}[A_1(t,k_0)]x(t,k_0) = \\ &= \mathrm{d}[U(t,t_0;k_0)]x_0 - \mathrm{d}[A_1(t,k_0)]x(t,k_0) = \\ &= \mathrm{d}[A_1(t,k_0)]U(t,t_0;k_0)x_0- \\ &- \mathrm{d}[A_1(t,k_0)]U(t,t_0;k_0)x_0 = 0 \end{aligned}$$

hence (27) becomes

$$\mathrm{d}g(t,k) = \sum_{l=k_0}^{k-1} F(t;k,l+1)\mathrm{d}f(t,l). \tag{28}$$

Equation (25) is equivalent to the generalized differential equation

$$\mathrm{d}x(t,k) = \mathrm{d}[A_1(t,k)]x(t,k) + \mathrm{d}g(t,k)$$

with the solution given by Theorem 8

$$\begin{aligned} x(t,k) &= U(t,t_0;k)x(t_0,k)- \\ &- \int_{t_0}^{t}\mathrm{d}_s[U(t,s;k)]\int_{t_0}^{s}\mathrm{d}g(\tau,k)+ \\ &+ \int_{t_0}^{t}\mathrm{d}g(s,k). \end{aligned} \tag{29}$$

By Theorem 3, (29) becomes

$$\begin{aligned} x(t,k) &= U(t,t_0;k)x(t_0,k)+ \\ &+ \int_{t_0}^{t} U(t,s;k)\mathrm{d}\int_{t_0}^{s}\mathrm{d}g(\tau,k)+ \\ &+ \sum_{a\le s<t}\Delta_s^+U(t,s;k)\Delta^+\int_{t_0}^{s}\mathrm{d}g(\tau,k)- \\ &- \sum_{a<s\le t}\Delta_s^-U(t,s;k)\Delta^-\int_{t_0}^{s}\mathrm{d}g(\tau,k). \end{aligned} \tag{30}$$

We replace (28) in (30). One obtains the formula of the state of the system Σ (24) from (30) taking into account the following equality

$$\int_{t_0}^{t}\mathrm{d}g(s,k)=\sum_{l=k_0}^{k-1}\int_{t_0}^{t}F(s;k,l+1)\mathrm{d}f(s,l)$$

and also (19) and Theorem 3, Theorem 4 and Theorem 5.

□

Proposition 12 *If* (22) *holds, then the state of the causal system* Σ_c *at the moment* $(t,k)\in T$*, determined by the initial state* x_0 *at the moment* $(t_0,k_0)\in T$ *and the control* $u:[t_0,t]\times\{k_0,k_0+1,\dots,k-1\}\to \mathbf{R}^m$ *is given by the following formula:*

$$\begin{aligned} x(t,k) &= U(t,t_0;k)F(t_0;k,k_0)x_0+ \\ &+ \int_{t_0}^{t}\sum_{l=k_0}^{k-1}U(t,s;k)\cdot \\ &\cdot\ F(s;k,l+1)B(s,l)\mathrm{d}u(s,l)+ \\ &+ \sum_{a\le s<t}\Delta_s^+U(t,s;k)\sum_{l=k_0}^{k-1}F(s;k,l+1)\cdot \\ &\cdot\ B(s,l)\Delta^+u(s,l)- \\ &- \sum_{a<s\le t}\Delta_s^-U(t,s;k)\sum_{l=k_0}^{k-1}F(s;k,l+1)\cdot \\ &\cdot\ B(s,l)\Delta^-u(s,l). \end{aligned} \tag{31}$$

Proof. The state equation (17) can be obtained from (19) by replacing $f(t,k)$ by

$$f(t,k)=\int_{t_0}^{t}B(s,k)\mathrm{d}u(s,k).$$

Then (31) results from (24) and (2).

□

Now we replace the state $x(t,k)$ given by (31) into the output equation of Σ (18). One obtains the formula of the general response of the system Σ

Theorem 13 *Under the hypothesis* (22) *the general response of the* 2Dgh *causal system* Σ_c (17), (18) *is*

$$\begin{aligned} & y(t,k)=C(t,k)U(t,t_0;k)F(t_0;k,k_0)x_0+ \\ & +\int_{t_0}^{t}\sum_{l=k_0}^{k-1}C(t,k)U(t,s;k)F(s;k,l+1)\cdot \\ & \cdot B(s,l)\mathrm{d}u(s,l)+D(t,k)u(t,k)+ \\ & +\sum_{a\le s<t}C(t,k)\Delta_s^+U(t,s;k)\cdot \\ & \cdot\sum_{l=k_0}^{k-1}F(s;k,l+1)B(s,l)\Delta^+u(s,l)- \\ & -\sum_{a<s\le t}C(t,k)\Delta_s^-U(t,s;k)\cdot \\ & \cdot\sum_{l=k_0}^{k-1}F(s;k,l+1)B(s,l)\Delta^-u(s,l). \end{aligned} \tag{32}$$

Corollary 14 *If* $u\in G_1^m$ $(u\in BV_1^m)$ *then* $x\in G_1^n$ *and* $y\in G_1^p$ $(x\in BV_1^n$ *and* $y\in BV_1^p)$.

Proof. We apply Theorems 8 and 13 and Proposition 12.

□

Definition 15 The boundary condition (7) is said to be *well-posed* if the homogeneous problem corresponding to (17) and (19) (i.e. with $u\equiv 0$ and $v=0$) has the unique solution $x=0$.

Proposition 16 *The boundary condition* (19) *is well-posed if and only if the matrix*

$$R=N_1+N_2U(b_1,a_1;b_2)F(a_1;b_2,a_2)$$

is nonsingular.

Proof: By (31) with $u\equiv 0$ we get

$$x(b_1,b_2)=U(b_1,a_1;b_2)F(a_1;b_2,a_2)x(a_1,a_2);$$

we replace $x(b_1,b_2)$ and $v=0$ in (19). It results that (19) is well-posed if and only if the equation $[N_1+N_2U(b_1,a_1;b_2)F(a_1;b_2,a_2)]x(a_1,a_2)=0$ has the unique solution $x(a_1,a_2)=0$, condition which is equivalent to R nonsingular.

□

In the sequel we shall consider boundary value systems Σ with well-posed boundary condition (19) and which verify (22). Moreover, the discrete-time character of Σ with respect to the variable k imposes the following assumption: the matrices A_2 depend only on k and $A_2(k)$ are nonsingular for any $k \in \{a_2, a_2+1, \ldots, b_2\}$.

Then the discrete fundamental matrix of A_2 does not depends on the real variable t and it becomes $F(k,l)$. In this case we can define this fundamental matrix even for the case $k < l$, by the following formula:

$$F(k,l) = [A_2(l-1)A_2(l-2)\cdots A_2(k+1)A_2(k)]^{-1}.$$

In this case the semigroup property

$$F(k,l)F(l,i) = F(k,i)$$

is true for any $k, l, i \in \{a_2, a_2+1, \ldots, b_2\}$.

Definition 17 The matrix

$$P = P_\Sigma = R^{-1}N_2U(b_1, a_1; b_2)F(b_2, a_2)$$

is called the *canonical boundary value operator* of the 2Dghbv system Σ with well-posed boundary condition .

Theorem 18 *If the system is with well-posed boundary condition then the state of the 2Dghbv system Σ determined by the control $u : T \to \mathbf{R}^m$ and by the input vector $v \in \mathbf{R^n}$ is*

$$\begin{aligned}
&x(t,k) = U(t,a_1;k)F(k,a_2)R^{-1}v-\\
&-\int_{a_1}^{b_1}\sum_{l=a_2}^{b_2-1} U(t,a_1;k)F(k,a_2)\cdot\\
&\cdot PU(a_1,s;b_2)F(a_2,l+1)B(s,l)\mathrm{d}u(s,l)+\\
&+\int_{a_1}^{t}\sum_{l=a_2}^{k-1} U(t,s;k)F(k,l+1)B(s,l)\mathrm{d}u(s,l)+\\
&-U(t,a_1;k)F(k,a_2)P\cdot\\
&\cdot\Bigg(\sum_{a_1\le s<b_1}\Delta_s^+U(a_1,s;b_2)\sum_{l=a_2}^{b_2-1}F(a_2,l+1)\cdot\\
&\cdot B(s,l)\Delta^+u(s,l) - \sum_{a_1<s\le t}\Delta_s^-U(a_1,s;b_2)\cdot\\
&\cdot\sum_{l=a_2}^{b_2-1}F(a_2,l+1)B(s,l)\Delta^-u(s,l)\Bigg)+\\
&+\sum_{a_1\le s<t}\Delta_s^+U(t,s;k)\sum_{l=a_2}^{k-1}F(k,l+1)\cdot\\
&\cdot B(s,l)\Delta^+u(s,l) - \sum_{a_1<s\le t}\Delta_s^-U(t,s;k)\cdot\\
&\cdot\sum_{l=a_2}^{k-1}F(k,l+1)B(s,l)\Delta^-u(s,l).
\end{aligned} \tag{33}$$

Proof: We replace $x(b_1, b_2)$ given by (31) in the boundary condition (19). We get

$$\begin{aligned}
&[N_1 + N_2U(b_1,a_1;b_2)F(b_2,a_2)]x_0+\\
&+N_2\int_{a_1}^{b_1}\sum_{l=a_2}^{b_2-1}U(b_1,s;b_2)F(b_2,l+1)B(s,l)\mathrm{d}u(s,l)+\\
&+\sum_{a_1\le s<b_1}\Delta_s^+U(b_1,s;b_2)\sum_{l=a_2}^{b_2-1}F(b_2,l+1)\cdot\\
&\cdot B(s,l)\Delta^+u(s,l) - \sum_{a_1<s\le b_1}\Delta_s^-U(b_1,s;b_2)\cdot\\
&\cdot\sum_{l=a_2}^{b_2-1}F(b_2,l+1)B(s,l)\Delta^-u(s,l) = v
\end{aligned}$$

hence, by the semigroup properties of the fundamental matrices $U(t,s;k)$ and $F(k,l)$, we obtain the initial state of the system Σ

$$x_0 = R^{-1}v - P\int_{a_1}^{b_1}\sum_{l=a_2}^{b_2-1} U(a_1,s;b_2)F(a_2,l+1)\cdot$$
$$\cdot B(s,l)\mathrm{d}u(s,l) - P\sum_{a_1\le s<b_1}\Delta_s^+U(a_1,s;b_2)\cdot$$
$$\cdot\sum_{l=a_2}^{b_2-1}F(a_2,l+1)B(s,l)\Delta^+u(s,l)+ \quad (34)$$
$$+P\sum_{a_1<s\le b_1}\Delta_s^-U(a_1,s;b_2)\cdot$$
$$\cdot\sum_{l=a_2}^{b_2-1}F(a_2,l+1)B(s,l)\Delta^-u(s,l).$$

We replace the initial state $x_0 = x(a_1,a_2)$ given by (34) in (31); then (33) results by using the semigroup property of the fundamental matrices $U(b_1,s;b_2)$ and $F(b_2,l+1)$, i.e.

$$U(b_1,s;b_2) = U(b_1,a_1;b_2)U(a_1,s;b_2)$$

and

$$F(b_2,l+1) = F(b_2,a_2)F(a_2,l+1)$$

.

□

Theorem 19 *The input-output map of the* 2Dghbv *system* Σ *is*

$$H : G_1^m\times\mathbf{R}^n \to G_1^p\times\mathbf{R}^n,$$
$$H(u,v) = (y,z)$$

where

$$y(t,k) = C(t,k)U(t,a_1;k)F(k,a_2)R^{-1}v-$$
$$-\int_{a_1}^{b_1}\sum_{l=a_2}^{b_2-1}C(t,k)U(t,a_1;k)F(k,a_2)\cdot$$
$$\cdot PU(a_1,s;b_2)F(a_2,l+1)B(s,l)\mathrm{d}u(s,l)+$$
$$+\int_{a_1}^{t}\sum_{l=a_2}^{k-1}C(t,k)U(t,s;k)F(k,l+1)B(s,l)\mathrm{d}u(s,l)+$$
$$+D(t,k)u(t,k) - C(t,k)U(t,a_1;k)F(k,a_2)P\cdot$$
$$\cdot\left(\sum_{a_1\le s<b_1}\Delta_s^+U(a_1,s;b_2)\sum_{l=a_2}^{b_2-1}F(a_2,l+1)\cdot\right.$$
$$\cdot B(s,l)\Delta^+u(s,l) - \sum_{a_1<s\le t}\Delta_s^-U(a_1,s;b_2)\cdot \quad (35)$$
$$\left.\cdot\sum_{l=a_2}^{b_2-1}F(a_2,l+1)B(s,l)\Delta^-u(s,l)\right)+$$
$$+C(t,k)\sum_{a_1\le s<t}\Delta_s^+U(t,s;k)\sum_{l=a_2}^{k-1}F(k,l+1)\cdot$$
$$\cdot B(s,l)\Delta^+u(s,l) - C(t,k)\sum_{a_1<s\le t}\Delta_s^-U(t,s;k)\cdot$$
$$\cdot\sum_{l=a_2}^{k-1}F(k,l+1)B(s,l)\Delta^-u(s,l).$$

and, by denoting $Q = M_1 + M_2U(b_1,a_1;b_2)\cdot$ $\cdot F(b_2,a_2), S = Q(I-P) - M_1,$

$$z = QR^{-1}v + S\int_{a_1}^{b_1}\sum_{l=a_2}^{b_2-1}U(a_1,s;b_2)F(a_2,l+1)\cdot$$
$$\cdot B(s,l)\mathrm{d}u(s,l) + S\left(\sum_{a_1\le s<b_1}\Delta_s^+U(a_1,s;b_2)\cdot\right.$$
$$\cdot\sum_{l=a_2}^{b_2-1}F(a_2,l+1)B(s,l)\Delta^+u(s,l)- \quad (36)$$
$$-\sum_{a_1<s\le b_1}\Delta_s^-U(a_1,s;b_2)\cdot$$
$$\left.\cdot\sum_{l=a_2}^{b_2-1}F(a_2,l+1)B(s,l)\Delta^-u(s,l)\right).$$

Proof: We obtain (35) by replacing the state $x(t,k)$ given by (33) in the output equation (18). Then, by replacing $x(a_1,a_2) = x_0$ (34) and $x(b_1,b_2)$ given by (33) in (20) and by a long calculus which uses the semigroup property and which is omitted, we get (36). □

Corollary 20 *If $u \in G_1^m$ then $x \in G_1^n$ and $y \in G_1^p$. If $A_2 \in BV_1^{n\times n}, B \in BV_1^{n\times m}, C \in BV_1^{p\times n},\ D \in BV_1^{n\times m}$ and $u \in BV_1^m$ then $x \in BV_1^n$ and $y \in BV_1^p$.*

Proof: We apply Theorems 8, 18 and 19.

□

Definition 21 The *space of admissible controls* is the set

$$\mathcal{U} = \{u \in G_1^m(a,b) | \mathbf{D}_t^+(A_i(\cdot,k) \cap \mathbf{D}_t^+(u(\cdot,k)) = \emptyset,$$
$$\mathbf{D}_t^-(A_i(\cdot,k)) \cap \mathbf{D}_t^-(u(\cdot,k)) = \emptyset,\ i=1,2,\ \forall k \in \mathbf{Z}\}.$$

Corollary 22 *If $u \in \mathcal{U}$, then the state and the output of the system Σ are given by the following formulæ:*

$$\begin{aligned} x(t,k) &= U(t,a_1;k)F(k,a_2)R^{-1}v- \\ &-\int_{a_1}^{b_1}\sum_{l=a_2}^{b_2-1} U(t,a_1;k)F(k,a_2)\cdot \\ &\cdot PU(a_1,s;b_2)F(a_2,l+1)B(s,l)\mathrm{d}u(s,l)+ \\ &+\int_{a_1}^{t}\sum_{l=a_2}^{k-1} U(t,s;k)F(k,l+1)B(s,l)\mathrm{d}u(s,l), \end{aligned} \tag{37}$$

$$\begin{aligned} y(t,k) &= C(t,k)U(t,a_1;k)F(k,a_2)R^{-1}v- \\ &-\int_{a_1}^{b_1}\sum_{l=a_2}^{b_2-1} C(t,k)U(t,a_1;k)F(k,a_2)\cdot \\ &\cdot PU(a_1,s;b_2)F(a_2,l+1)B(s,l)\mathrm{d}u(s,l)+ \\ &+\int_{a_1}^{t}\sum_{l=a_2}^{k-1} C(t,k)U(t,s;k)F(k,l+1)\cdot \\ &\cdot B(s,l)\mathrm{d}u(s,l)+D(t,k)u(t,k). \end{aligned} \tag{38}$$

Remark 23 The 2D "classical" continuous-discrete systems [15] with the state equation

$$\begin{aligned} &\frac{\partial x}{\partial t}(t,k+1) = \tilde{A}_1(t,k+1)x(t,k+1)+ \\ &+\tilde{A}_2(t,k)\frac{\partial x}{\partial t}(t,k) - \tilde{A}_1(t,k)\tilde{A}_2(t,k)x(t,k)+ \\ &+\tilde{B}(t,k)\tilde{u}(t,k) \end{aligned}$$

represent particular cases of 2Dghbv (18) with absolutely continuous matrices $A_i(t,k) = \int_a^t \tilde{A}_i(s,k)\mathrm{d}s$, $i=1,2$ and controls $u(t,k) = \int_a^t \tilde{u}(s,k)\mathrm{d}s$.

4 Adjoints 2D generalized hybrid boundary value systems

We consider the 2Dghbv system $\Sigma = (A_1(t), A_2(k), B(t,k),\ C(t,k),\ D(t,k),\ N_1, N_2)$ with well-posed boundary conditions, given by (17)-(19), where the matrices $A_2(k)$ are nonsingular $\forall k \in \{a_2, a_2+1, \ldots, b_2\}$. In order to cover the ground of systems over $\mathbf{C}$ we shall denote by A^* the adjoint of a matrix A. Obviously, if A is a real matrix, $A^* = A^T$. By A^{-*} we denote $(A^*)^{-1}$.

Let us assume that

$$\begin{aligned} &\det[(I-\Delta^- A_1(t))(I-\Delta^+ A_1(t))\cdot \\ &\cdot(I+\Delta^- A_1(t))(I+\Delta^+ A_1(t))] \neq 0 \end{aligned} \tag{39}$$

Definition 24 The 2Dghbv system $\tilde{\Sigma}$ having the state space representation

$$\begin{aligned} &\mathrm{d}\tilde{x}(t,k+1) = -\mathrm{d}[A_1(t)^*]\tilde{x}(t,k+1)+ \\ &+A_2(k)^{-*}\mathrm{d}\tilde{x}(t,k)+ \\ &+\mathrm{d}[A_1(t)^*]A_2(k)^{-*}\tilde{x}(t,k) - C(t,k)^*\mathrm{d}\tilde{u}(t,k), \end{aligned} \tag{40}$$

$$\tilde{y}(t,k) = B(t,k)^*\tilde{x}(t,k) + D(t,k)^*\tilde{u}(t,k), \tag{41}$$

$$\tilde{x}(a_1,a_2) = N_1^*\lambda, \quad \tilde{x}(b_1,b_2) = -N_2^*\lambda \tag{42}$$

where $\tilde{u} \in G_1^p,\ \tilde{x} \in G_1^n,\ \tilde{y} \in G_1^m$ is called the *adjoint* of Σ.

Therefore, the system $\tilde{\Sigma}$ is characterized by the matrices $\tilde{A}_1(t) = -A_1(t)^*$, $\tilde{A}_2(k) = A_2(k)^{-*}$ $(= (A_2(k)^*)^{-1})$, $\tilde{B}(t,k) = -C(t,k)^*$, $\tilde{C}(t,k) = B(t,k)^*$, N_1^* and $-N_2^*$.

From [19, §III 4] one obtains, by denoting by $V(t,s)$ the corresponding fundamental matrix, the following result:

Proposition 25 *The general linear differential equation*

$$\mathrm{d}x = \mathrm{d}[-A^*]x + \mathrm{d}g,\ x(t_0) = x_0 \tag{43}$$

where $A \in BV^{n\times n}$ and $g \in G^n(a,b)$, has the solution

$$\begin{aligned} &x(t)^* = x(t_0)^*V(t_0,t) + \int_{t_0}^{t}(g(s)^*- \\ &-g(t_0)^*)\mathrm{d}_s[V(s,t)] + g(t)^* - g(t_0)^* \end{aligned} \tag{44}$$

and

$$\begin{aligned}
&V(t,s)=U(t,s)+V(t,s)(\Delta^+A(s))^2-\\
&-(\Delta^-A(s))^2U(t,s)+\sum_{s<\tau<t}V(t,\tau)[(\Delta^+A(\tau))^2-\\
&-(\Delta^-A(\tau))^2]U(\tau,s),\quad \text{if}\quad t>s,\\
&V(t,s)=U(t,s)+V(t,s)(\Delta^-A(s))^2-\\
&-(\Delta^+A(s))^2U(t,s)+\sum_{t<\tau<s}V(t,\tau)[(\Delta^-A(\tau))^2-\\
&-(\Delta^+A(\tau))^2]U(\tau,s),\quad \text{if}\quad t<s,\\
&V(t,t)=U(t,t)=I
\end{aligned}\tag{45}$$

We shall consider for the system $\tilde{\Sigma}$ boundary conditions of the form (21):

$$\tilde{x}(t,a_2)=V(a_1,t)^*x_0,\ \tilde{x}(a_1,k)=F(a_2,k)^*x_0. \tag{46}$$

We assume that $\tilde{R}:=N_1+N_2V(b_1,a_1)\ F(b_2,a_2)$ is a nonsingular matrix and we denote $\tilde{P}=R^{-1}N_2V(b_1,a_1)F(b_2,a_2)$.

Theorem 26 *The input-output map of the adjoint system $\tilde{\Sigma}$ is the operator $\tilde{H}:G_1^p\to G_1^m\times\mathbf{R}^n$ given by $\tilde{H}(\tilde{u})=(\tilde{y},\tilde{\lambda})$, where*

$$\begin{aligned}
&\tilde{y}(t,k)^*=\int_{a_1}^{b_1}\sum_{l=a_2}^{b_2-1}\mathrm{d}\tilde{u}(s,l)^*C(s,l)V(s,a_1)\cdot\\
&\cdot F(l+1,a_2)(I-\tilde{P})F(a_2,k)V(a_1,t)B(t,k)-\\
&-\int_{a_1}^{t}\sum_{l=a_2}^{k-1}\mathrm{d}\tilde{u}(s,l)^*C(s,l)V(s,t)F(l+1,k)B(t,k)+\\
&+\tilde{u}(t,k)^*D(t,k)+\\
&+[\sum_{a_1\le s<b_1}\sum_{l=a_2}^{b_2-1}\Delta^+\tilde{u}(s,l)^*C(s,l)\cdot\\
&\cdot F(l+1,a_2)\Delta_s^+V(s,a_1)-\\
&-\sum_{a_1<s\le b_1}\sum_{l=a_2}^{b_2-1}\Delta^-\tilde{u}(s,l)^*C(s,l)F(l+1,a_2)\cdot\\
&\cdot\Delta_s^-V(s,a_1)](I-\tilde{P})F(a_2,k)V(a_1,t)B(t,k)-\\
&-\sum_{a_1\le s<t}\sum_{l=a_2}^{k-1}\Delta^+\tilde{u}(s,l)^*C(s,l)F(l+1,k)\cdot\\
&\cdot\Delta_s^+V(s,t)B(t,k)+\\
&+\sum_{a_1<s\le t}\sum_{l=a_2}^{k-1}\Delta^-\tilde{u}(s,l)^*C(s,l)F(l+1,k)\cdot\\
&\cdot\Delta_s^-V(s,t)B(t,k),
\end{aligned}\tag{47}$$

$$\begin{aligned}
&\lambda^*=\int_{a_1}^{b_1}\sum_{l=a_2}^{b_2-1}\mathrm{d}\tilde{u}(s,l)^*C(s,l)\cdot\\
&\cdot F(l+1,a_2)V(s,a_1)\tilde{R}^{-1}+\\
&+\sum_{a_1\le s<b_1}\sum_{l=a_2}^{b_2-1}\Delta^+\tilde{u}(s,l)^*C(s,l)\cdot\\
&\cdot F(l+1,a_2)\Delta_s^+V(s,a_1)\tilde{R}^{-1}-\\
&-\sum_{a_1<s\le b_1}\sum_{l=a_2}^{b_2-1}\Delta^-\tilde{u}(s,l)^*C(s,l)\cdot\\
&\cdot F(l+1,a_2)\Delta_s^-V(s,a_1)\tilde{R}^{-1}.
\end{aligned}\tag{48}$$

Proof: By Proposition 25 the fundamental matrix of $\tilde{A}_1=-A_1^*$ is $U_{\tilde{A}_1}(t,s)=V(s,t)^*$. Then the (discrete) fundamental matrix of $\tilde{A}_2=(A_2^*)^{-1}$ is, for $k>l$, $\forall k,l\in\mathbf{Z}$

$$\begin{aligned}
&F_{\tilde{A}_2}(k,l)=[\tilde{A}_2(k-1)\tilde{A}_2(k-2)\cdots\tilde{A}_2(l)]=\\
&=[(A_2(k-1)^*)^{-1}(A_2(k-2)^*)^{-1}\cdots(A_2(l)^*)^{-1}]=\\
&=([A_2(k-1)A_2(k-2)\cdots A_2(l)]^{-1})^*=F_{A_2}(l,k)^*
\end{aligned}$$

and similarly we can prove that $F_{\tilde{A}_2}(k,l)=F_{A_2}(l,k)^*$ for $k<l$.

By applying (31) to the adjoint system $\tilde{\Sigma}$ one obtains:

$$\begin{aligned}
&\tilde{x}(t,k)^*=\tilde{x}(a_1,a_2)^*V(a_1,t)F(a_2,k)-\\
&-\int_{a_1}^{t}\sum_{l=a_2}^{k-1}\mathrm{d}\tilde{u}(s,l)^*C(s,l)V(s,t)F(l+1,k)-\\
&-\sum_{a_1\le s<b_1}\sum_{l=a_2}^{b_2-1}\Delta^+\tilde{u}(s,l)^*C(s,l)\cdot\\
&\cdot F(l+1,k)\Delta_s^+V(s,l)-\\
&-\sum_{a_1<s\le b_1}\sum_{l=a_2}^{b_2-1}\Delta^-\tilde{u}(s,l)^*C(s,l)\cdot\\
&\cdot F(l+1,k)\Delta_s^-V(s,l).
\end{aligned}\tag{49}$$

By (4.4) and (4.10) we have the following equalities:

$$\begin{aligned}&-\lambda^* N_2 = \tilde{x}(b_1, b_2)^* = \\ &= \tilde{x}(a_1, a_2)^* V(a_1, b_1) F(a_2, b_2) - \\ &- \int_{a_1}^{b_1} \sum_{l=a_2}^{b_2-1} \mathrm{d}\tilde{u}(s, l)^* \cdot \\ &\cdot C(s, l) V(s, b_1) F(l+1, b_2) - \\ &- \sum_{a_1 \le s < b_1} \sum_{l=a_2}^{b_2-1} \Delta^+ \tilde{u}(s, l)^* C(s, l) \cdot \\ &\cdot F(l+1, b_2) \Delta_s^+ V(s, b_1) + \\ &+ \sum_{a_1 < s \le b_1} \sum_{l=a_2}^{b_2-1} \Delta^- \tilde{u}(s, l)^* C(s, l) \cdot \\ &\cdot F(l+1, b_2) \Delta_s^- V(s, b_1).\end{aligned} \tag{50}$$

We replace the initial state $\tilde{x}(a_1, a_2)^*$ by $\lambda^* N_1$ and we postmultiply the obtained equality by $V(b_1, a_1) F(b_2, a_2)$; by applying the semigroup property of the matrices $V(s, t)$ and $F(k, l$, we get

$$\begin{aligned}&\lambda^*[N_1 + N_2 V(b_1, a_1) F(b_2, a_2)] = \\ &= \int_{a_1}^{b_1} \sum_{l=a_2}^{b_2-1} \mathrm{d}\tilde{u}(s, l)^* C(s, l) V(s, a_1) F(l+1, a_2) - \\ &- \sum_{a_1 \le s < b_1} \sum_{l=a_2}^{b_2-1} \Delta^+ \tilde{u}(s, l)^* C(s, l) \cdot \\ &\cdot F(l+1, a_2) \Delta_s^+ V(s, a_1) + \\ &+ \sum_{a_1 < s \le b_1} \sum_{l=a_2}^{b_2-1} \Delta^- \tilde{u}(s, l)^* C(s, l) \cdot \\ &\cdot F(l+1, a_2) \Delta_s^- V(s, a_1).\end{aligned}$$

From this equality we obtain the expression (48) of λ_*. Now we postmultiply again (50) by $V(b_1, a_1) F(b_2, a_2)$ and we replace λ^* by (48). We obtain

$$\begin{aligned}&\tilde{x}(a_1, a_2)^* = \int_{a_1}^{b_1} \sum_{l=a_2}^{b_2-1} \mathrm{d}\tilde{u}(s, l)^* \cdot \\ &\cdot C(s, l) V(s, a_1) F(l+1, a_2) - \\ &- \int_{a_1}^{b_1} \sum_{l=a_2}^{b_2-1} \mathrm{d}\tilde{u}(s, l)^* \cdot \\ &\cdot C(s, l) V(s, a_1) F(l+1, a_2) \tilde{P} - \\ &- \sum_{a_1 \le s < b_1} \sum_{l=a_2}^{b_2-1} \Delta^+ \tilde{u}(s, l)^* \cdot \\ &\cdot C(s, l) F(l+1, a_2) \Delta_s^+ V(s, a_1) + \\ &+ \sum_{a_1 < s \le b_1} \sum_{l=a_2}^{b_2-1} \Delta^- \tilde{u}(s, l)^* \cdot \\ &\cdot C(s, l) F(l+1, a_2) \Delta_s^- V(s, a_1) + \\ &+ \sum_{a_1 \le s < b_1} \sum_{l=a_2}^{b_2-1} \Delta^+ \tilde{u}(s, l)^* \cdot \\ &\cdot C(s, l) F(l+1, a_2) \Delta_s^+ V(s, a_1) \tilde{P} - \\ &- \sum_{a_1 < s \le b_1} \sum_{l=a_2}^{b_2-1} \Delta^- \tilde{u}(s, l)^* \cdot \\ &\cdot C(s, l) F(l+1, a_2) \Delta_s^- V(s, a_1) \tilde{P},\end{aligned}$$

hence the initial state $\tilde{x}(a_1, a_2)^*$ becomes

$$\begin{aligned}&\tilde{x}(a_1, a_2)^* = \int_{a_1}^{b_1} \sum_{l=a_2}^{b_2-1} \mathrm{d}\tilde{u}(s, l)^* C(s, l) \cdot \\ &\cdot V(s, a_1) F(l+1, a_2)(I - \tilde{P}) + \\ &+ \sum_{a_1 \le s < b_1} \sum_{l=a_2}^{b_2-1} \Delta^+ \tilde{u}(s, l)^* C(s, l) \cdot \\ &\cdot F(l+1, a_2) \Delta_s^+ V(s, a_1)(I - \tilde{P}) - \\ &- \sum_{a_1 \le s < b_1} \sum_{l=a_2}^{b_2-1} \Delta^+ \tilde{u}(s, l)^* C(s, l) \cdot \\ &\cdot F(l+1, a_2) \Delta_s^+ V(s, a_1)(I - \tilde{P}).\end{aligned} \tag{51}$$

We replace $\tilde{x}(a_1, a_2)^*$ (51) in (49) and we obtain the formula of the state state of the adjoint system $\tilde{\Sigma}$:

$$\tilde{x}(t,k)^* = \int_{a_1}^{b_1}\sum_{l=a_2}^{b_2-1} \mathrm{d}\tilde{u}(s,l)^* C(s,l)V(s,a_1)\cdot$$
$$\cdot F(l+1,a_2)(I-\tilde{P})F(a_2,k)V(a_1,t)-$$
$$-\int_{a_1}^{t}\sum_{l=a_2}^{k-1} \mathrm{d}\tilde{u}(s,l)^* C(s,l)V(s,t)F(l+1,k)+$$
$$+[\sum_{a_1\le s<b_1}\sum_{l=a_2}^{b_2-1}\Delta^+\tilde{u}(s,l)^*C(s,l)\cdot$$
$$\cdot F(l+1,a_2)\Delta_s^+V(s,a_1)-$$
$$-\sum_{a_1< s\le b_1}\sum_{l=a_2}^{b_2-1}\Delta^-\tilde{u}(s,l)^*C(s,l)\cdot \quad (52)$$
$$\cdot F(l+1,a_2)\Delta_s^-V(s,a_1)](I-\tilde{P})\cdot$$
$$\cdot F(a_2,k)V(a_1,t)-$$
$$-\sum_{a_1\le s<t}\sum_{l=a_2}^{k-1}\Delta^+\tilde{u}(s,l)^*C(s,l)\cdot$$
$$\cdot F(l+1,k)\Delta_s^+V(s,t)+$$
$$+\sum_{a_1< s\le t}\sum_{l=a_2}^{k-1}\Delta^-\tilde{u}(s,l)^*C(s,l)\cdot$$
$$\cdot F(l+1,k)\Delta_s^-V(s,t),$$

and (47) results by replacing $\tilde{x}(t,k)^*$ given by (52) in the output equation (41).

□

Now let us consider the Banach spaces $NBV_{1,r}^p$ and $NBV_{1,l}^p$ of the functions $f\in BV_1^p$ which verify $f(a_1,k)=0,\ \forall k\in\{a_2,\dots,b_2-1\}$ and are continuous on the right and left respectively. One obtains the dual pairs $(BV_1^p, NBV_{1,r}^p)$ and $(NBV_{1,l}^m\times \mathbf{R}^n, BV_1^m\times\mathbf{R}^n)$ with respect to the bilinear forms

$$< y,\tilde{u} >_1 = \int_{a_1}^{b_1}\sum_{l=a_2}^{b_2-1}\mathrm{d}\tilde{u}(s,l)^*y(s,l)$$

and

$$< (u,v),(\tilde{y},\lambda) >_2 = \lambda^*v+\int_{a_1}^{b_1}\sum_{l=a_2}^{b_2-1}\tilde{y}(s,l)^*\mathrm{d}u(s,l)$$

respectively.

We shall emphasize the duality relationship between the 2Dghbv system Σ and its adjoint $\tilde{\Sigma}$ with respect to these bilinear forms.

In order to simplify, we shall consider the following sets of *admissible controls*:

$$\mathcal{U}=\{u\in NBV_{1,l}^m(a,b)|\mathbf{D}_t^+(A_i(\cdot)\cap$$
$$\cap\mathbf{D}_t^+(u(\cdot,k))=\emptyset\ i=1,2,\ \forall k\in\mathbf{Z}\}.$$

$$\tilde{\mathcal{U}}=\{\tilde{u}\in NBV_{1,r}^p(a,b)|\mathbf{D}_t^-(A_i(\cdot))\cap$$
$$\cap\mathbf{D}_t^-(\tilde{u}(\cdot,k))=\emptyset,\ i=1,2,\ \forall k\in\mathbf{Z}\}$$

and the following assumption:

$$(\Delta^+A_1(t))^2=(\Delta^-A_1(t))^2=0,\ \forall t\in[a_1,b_1]. \quad (53)$$

Theorem 27 *If* (39) *and* (53) *hold, then* $\forall u\in\mathcal{U},\ \forall\tilde{u}\in\tilde{\mathcal{U}}$, $\forall v\in\mathbf{R}^n$

$$< H_1(u,v),\tilde{u} >_1 = < (u,v),\tilde{H}(\tilde{u}) >_2, \quad (54)$$

where H_1 *and* $\tilde{H}$ *are the input-output operators* $H_1(u,v)=y,\ \tilde{H}(\tilde{u})=(\tilde{y},\lambda)$.

Proof: By (45) and (53) we get $V(t,s)=U(t,s)$. For admissible controls u and $\tilde{u}$, the formulæ (47) and (48) become

$$\tilde{y}(t,k)^*=\int_{a_1}^{b_1}\sum_{l=a_2}^{b_2-1}\mathrm{d}\tilde{u}(s,l)^*C(s,l)U(s,a_1)\cdot$$
$$\cdot F(l+1,a_2)(I-\tilde{P})F(a_2,k)U(a_1,t)B(t,k)- \quad (55)$$
$$-\int_{a_1}^{t}\sum_{l=a_2}^{k-1}\mathrm{d}\tilde{u}(s,l)^*C(s,l)U(s,t)\cdot$$
$$\cdot F(l+1,k)B(t,k)+\tilde{u}(t,k)^*D(t,k),$$

$$\lambda^*=\int_{a_1}^{b_1}\sum_{l=a_2}^{b_2-1}\mathrm{d}\tilde{u}(s,l)^*C(s,l)\cdot \quad (56)$$
$$\cdot F(l+1,a_2)U(s,a_1)\tilde{R}^{-1}.$$

By a long calculus which is omitted, we obtain by (38), (55) and (56) the following equality, which is equivalent with (54):

$$\int_{a_1}^{b_1}\sum_{l=a_2}^{b_2-1}\mathrm{d}\tilde{u}(t,k)^*y(t,k)=$$

$$=\lambda^*v+\int_{a_1}^{b_1}\sum_{l=a_2}^{b_2-1}\tilde{y}(t,k)^*\mathrm{d}u(t,k).$$

5 Conclusion

The state space representation was studied for a class of time-varying 2D hybrid boundary-value time-variable systems in the general framework of the coefficient matrices, states, inputs and outputs over the space of regulated functions. The adjoints of these systems were introduced and the duality between the 2D hybrid systems and their adjoints was expressed by the means of two bilinear forms.

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