Noisy fractional oscillator: Temporal behavior of the autocorrelation function

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Abstract: - The influence of external and internal noises on the output signal of a fractional Langevin equation with the friction kernel of a viscoelastic type is considered. The interaction with fluctuations of environmental parameters is modeled by a multiplicative white noise and by an additive noise with a zero mean. In the case of an external harmonic driving force it is shown that additive external and internal noises cause qualitatively different behaviors of the autocorrelation function for the output signal. Particularly, in the case of external noise it is established that at a sufficiently strong memory of the friction kernel the output signal corresponds to trapped dynamics in an effective potential well even if the system is subdiffusive in the case of internal noise.

Key–Words: - fractional Langevin equation, fractional oscillator, energetic stability, internal noise, external noise, viscoelastic friction

1 Introduction

In complex systems conditions far from thermal equilibrium and influence of environmental fluctuations may give rise to unexpected transport phenomena, which are ruled out by the second law of thermodynamics under equilibrium conditions. Among them we can mention the ratchet effect [1, 2], hypersensitive response [3, 4], absolute negative mobility [5–7], noise-enhanced stability [8, 9], noise-supported traveling structures [10], and giant amplification of diffusion [11], to name a few.

Diffusion is one of the fundamental mechanisms for non-equilibrium transport phenomena in physical systems. Normal diffusion is characterized by a mean-square displacement that is asymptotically linear in time and is well described in the theory of Brownian motion as a Gaussian process that is local in both space and time. However, the Brownian motion theory cannot account for anomalous diffusion processes, in which the mean-square displacement is not proportional to time. Examples of such systems are supercooled liquids, glasses, colloidal suspensions, dense polymer solutions [12, 13], viscoelastic media [14, 15], and amorphous semiconductors [16, 17]. Even anomalous diffusive dynamics of atoms in biological macromolecules and intrinsic conformotional dynamics of proteins can be subdiffusive [18–20]. There are several approaches to describe anomalous diffusion processes, where the dynamical origin of the phenomenon is considered as a nonlocality, either in space or time [21]. One of the objects of special attention in this context is the noise-driven fractional oscillator. The dynamical equation for such an oscillator is obtained by replacing the usual friction term in the dynamical equation for a harmonic oscillator by a generalized friction term with a power-law-type memory [18, 21–24].

Although the behavior of the fractional oscillator with an additive noise has been investigated in some detail [22, 24], it seems that analysis of the potential consequences of interplay between eigenfrequency fluctuations and memory effects is rather missing in literature. This is quite surprising in view of the fact that the importance of multiplicative fluctuations and viscoelasticity for biological systems, e.g. living cells, has been well recognized [25, 26]. Thus motivated, the authors of [23, 27] have recently considered a fractional oscillator with fluctuating eigenfrequency subjected to an external periodic force and an additive noise. These models demonstrate that an interplay of noises and memory can generate a variety of cooperative effects, such as memory-enhanced energetic stability [27, 28], stochastic resonance versus noise parameters [23, 29], as well as friction-induced resonance [23]. However, in these works the dependence of the autocorrelation function of the oscillator displacement on the lag-time has not been investigated.

As from an experimental point of view, information about the dynamics of the observed subdiffusive system can be extracted from the normalized autocorrelation function and from the mean-square displacement [18, 20, 24], we investigate the behavior of the autocorrelation function of the output signal of the fractional oscillator with multiplicative white noise subjected to an external periodic force and an additive driving noise (i.e. a model similar to the one presented in [27]). The main purpose of this paper is to demonstrate, partially on the basis of the exact expressions of output characteristics found in [27], that the dependence of the normalized autocorrelation function on the lag-time depends crucially on the physical nature of the additive driving noise; i.e. the results are qualitatively different for internal and external noises. Moreover, we aim at providing a simple criterion that permits to distinguish between these two types of output processes and also to verify the physical nature of the additive noise.

The structure of the paper is as follows. In Section 2 we introduce the fractional Langevin equation (noisy fractional oscillator) as the basic model investigated. Exact formulas for the variance and for the autocorrelation function of the output signal are derived. In Section 3 we analyze the behavior of the output characteristics (such as variance and autocorrelation function), and present the main results of this paper. A simple criterion to determine if the driving noise is internal or external is established. Section 4 contains some brief concluding remarks.

2 Model

2.1 Fractional Langevin equation

As a model for an oscillatory system with memory, strongly coupled with a noisy environment, we consider an underdamped fractional Langevin equation with a fluctuating harmonic potential (sometimes referred to as "noisy fractional oscillator" [27, 28]):

$$\ddot{X} + \frac{\gamma}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{X}(t')\,dt'}{(t-t')^{\alpha}} + [\omega^2 + Z(t)]X = \xi(t) + A_0 \sin \Omega t, \tag{1}$$

where $\dot{X} \equiv dX/dt$, $X(t)$ is the oscillator displacement, γ is a friction constant, $\Gamma(y)$ is the Gamma function, A_0 and Ω are the amplitude and the frequency of the harmonic driving force, respectively, and the parameter α, $0 < \alpha < 1$ denotes the memory exponent. Fluctuations of the frequency ω of the binding harmonic field are expressed as Gaussian white noise $Z(t)$ with a zero mean and a delta-correlated correlation function:

$$\langle Z(t) \rangle = 0, \qquad \langle Z(t)Z(t') \rangle = 2D\delta(t-t'), \tag{2}$$

where D is the noise intensity. The zero-centered random force $\xi(t)$ with a stationary correlation function

$$\langle \xi(t)\xi(t') \rangle = \frac{k_B T \gamma}{\Gamma(1-\alpha)|t-t'|^{\alpha}} + 2D_1\delta(t-t'), \qquad \langle \xi(t) \rangle = 0 \tag{3}$$

is assumed as statistically independent from the noise $Z(t)$. If $D_1 = 0$, the driving noise $\xi(t)$ can be regarded as an internal noise, in which case its stationary correlation function satisfies Kubo's second fluctuation-dissipation theorem, where T is the absolute temperature of the heat bath, and k_B is the Boltzmann constant. In the case of $T = 0$ the driving white noise $\xi(t)$ with an intensity D_1 and the dissipation have different origins and $\xi(t)$ will be referred to as "external noise".

2.2 Spectral amplification

The second order differential Eq. (1) can be written as two first-order differential equations:

$$\dot{X}(t) = Y(t), \tag{4}$$

$$\dot{Y}(t) + \frac{\gamma}{\Gamma(1-\alpha)} \int_0^t \frac{Y(t')}{(t-t')^{\alpha}}\,dt' + [\omega^2 + Z(t)]X(t) = \xi(t) + A_0 \sin(\Omega t), \tag{5}$$

which, after averaging over the ensemble of realizations of the random processes $Z(t)$ and $\xi(t)$, take the following form

$$\langle X(t) \rangle^{\cdot} = \langle Y(t) \rangle,$$

$$\langle Y(t) \rangle^{\cdot} + \frac{\gamma}{\Gamma(1-\alpha)} \int_0^t \frac{Y(t')}{(t-t')^{\alpha}}\,dt' + \omega^2 \langle X(t) \rangle = A_0 \sin(\Omega t). \tag{6}$$

Here we have used that, as it follows from Eq. (2), the correlator

$$\langle Z(t)X(t) \rangle = 0. \tag{7}$$

The solution of Eqs. (6) can be represented in the form

$$\langle X(t) \rangle = x_0 \left[1 - \omega^2 \int_0^t H(\tau)d\tau \right] + \dot{x}_0 H(t) + A_0 \int_0^t H(t-\tau)\sin(\Omega\tau)d\tau, \tag{8}$$

$$\langle Y(t)\rangle = \dot{x}_0\dot{H}(t) - \omega^2 x_0 H(t) + \int_0^t \dot{H}(t-\tau)\sin(\Omega\tau)d\tau, \quad (9)$$

where the constants of integration x_0 and $\dot{x}_0$ are determined by the initial conditions and the relaxation function $H(t)$, with $H(0) = 0$, is the inverse form of the Laplace transform $\hat{H}(s)$ given by

$$\hat{H}(s) = \frac{1}{s^2 + \gamma s^\alpha + \omega^2}, \quad (10)$$

where

$$\hat{H}(s) = \int_0^\infty e^{-st} H(t)dt.$$

To evaluate the inverse Laplace transform of Eq. (10) we use the residue theorem method described in [30]. Thus we obtain

$$H(t) = \frac{2}{\sqrt{u^2+v^2}} e^{-\beta t}\sin(\omega^* t + \theta) + \frac{\gamma\sin(\alpha\pi)}{\pi}\int_0^\infty \frac{r^\alpha e^{-rt}dr}{B(r)}, \quad (11)$$

where $s_{1,2} = -\beta \pm i\omega^*$, $(\beta > 0, \omega^* > 0)$, are the pair of conjugate complex zeros of the equation

$$G(s) \equiv s^2 + \gamma s^\alpha + \omega^2 = 0; \quad (12)$$

here, $G(s)$ is defined by the principal branch of s^α. The quantities u, v, θ, and $B(r)$ are determined by

$$u = -2\beta + \frac{\alpha\gamma\cos\{(1-\alpha)[\arctan(-\omega^*/\beta)+\pi]\}}{\left(\beta^2+\omega^{*2}\right)^{(1-\alpha)/2}},$$

$$v = 2\omega^* - \frac{\alpha\gamma\sin\{(1-\alpha)[\arctan(-\omega^*/\beta)+\pi]\}}{\left(\beta^2+\omega^{*2}\right)^{(1-\alpha)/2}},$$

$$\theta = \arctan\left(\frac{u}{v}\right),$$

$$B(r) = \left[r^2 + \gamma r^\alpha\cos(\pi\alpha) + \omega^2\right]^2 + \gamma^2 r^{2\alpha}\sin^2(\pi\alpha). \quad (13)$$

It should be emphasized that the relaxation function $H(t)$ can be represented via Mittag-Leffler-type special functions [31]. But as the numerical calculations are very complicated we suggest, apart from possible representations via Mittag-Leffler functions, a numerical treatment of Eq. (11).

From Eqs. (8) and (11) it follows that in the long time limit, $t \to \infty$, the memory about the initial conditions will vanish as

$$x_0\left[1 - \omega^2\int_0^t H(\tau)\,d\tau\right] \approx \frac{\gamma x_0}{\omega^2\Gamma(1-\alpha)t^\alpha} \quad (14)$$

and the average oscillator displacement, $\langle X(t)\rangle_{as} := \langle X(t)\rangle_{|t\to\infty}$ is given by

$$\langle X(t)\rangle_{as} = A_0\int_0^t H(t-\tau)\sin(\Omega\tau)d\tau. \quad (15)$$

Equation (15) can be written by means of the complex susceptibility $\chi(\Omega)$ as [32]

$$\langle X(t)\rangle_{as} = A\sin(\Omega t + \varphi), \quad (16)$$

with the output amplitude

$$A = A_0|\chi(\Omega)|, \quad (17)$$

where the complex susceptibility $\chi(\Omega)$ is defined by

$$\chi(\Omega) = \hat{H}(-i\Omega). \quad (18)$$

For the spectral amplification (SPA) we have

$$\text{SPA} = \frac{A^2}{A_0^2} = \left\{\left[\omega^2 - \Omega^2 + \gamma\Omega^\alpha\cos\left(\frac{\alpha\pi}{2}\right)\right]^2 + \gamma^2\Omega^{2\alpha}\sin^2\left(\frac{\alpha\pi}{2}\right)\right\}^{-1} \quad (19)$$

and the phase shift φ can be represented as

$$\varphi = \arctan\left[\frac{-\Omega^\alpha\gamma\sin\left(\frac{\alpha\pi}{2}\right)}{\omega^2 - \Omega^2 + \gamma\Omega^\alpha\cos\left(\frac{\alpha\pi}{2}\right)}\right]. \quad (20)$$

Here we emphasize that for a deterministic fractional oscillator the formulas (16), (19), and (20) have been previously represented in [22]. To avoid misunderstandings, let us mention that in contrast to the model considered in [23], where the dependence of the SPA on the noise parameters was significant, in the present model the SPA is independent of the noise parameters and remains equal to the noise-free solution. It is important that for any $\alpha < \alpha_c \approx 0.441$ and for any values of other system parameters the dependence of $A(\Omega)$ on Ω is always nonmonotonic with a local minimum and with a resonance peak (see [22, 23]).

2.3 Normalized correlation function

From an experimental point of view the time-homogeneous part of the variance of the oscillator displacement X is an important output characteristic. Alternative information about the experimentally observed stochastic behavior of the output signal can be extracted from the one-time normalized autocorrelation function [20, 24]. To find the above mentioned quantities, we will first consider the long-time behavior of the variance $\sigma^2(t)$ and the autocorrelation function $K(\tau,t)$ of the oscillator displacement:

$$\sigma^2(t) \equiv \left\langle [X(t) - \langle X(t)\rangle]^2 \right\rangle, \quad (21)$$

$$K(\tau,t) \equiv \langle [X(t+\tau) - \langle X(t+\tau)\rangle]\,[X(t) - \langle X(t)\rangle]\rangle. \quad (22)$$

Starting from Eqs. (4) and (5) one can easily obtain a formal expression for the oscillator displacement $X(t)$ in the following form

$$X(t) = \langle X(t)\rangle + \int_0^t H(t-\tau)[\xi(\tau) - X(\tau)Z(\tau)]d\tau. \quad (23)$$

From Eqs. (22) and (23) for the correlation function we have

$$K(\tau,t) = \int_0^{t+\tau}\int_0^t H(t+\tau-t_1)H(t-t_2) \times [\langle X(t_1)X(t_2)Z(t_1)Z(t_2)\rangle + \langle \xi(t_1)X(t_2)Z(t_2)\rangle + \langle \xi(t_2)X(t_1)Z(t_1)\rangle + \langle \xi(t_1)\xi(t_2)\rangle]\, dt_1 dt_2 \quad (24)$$

From Eq. (7) and statistical independence of the processes $\xi(t)$ and $Z(t)$ it follows that

$$\langle \xi(t_1)X(t_2)Z(t_2)\rangle = \langle \xi(t_2)X(t_1)Z(t_1)\rangle = 0. \quad (25)$$

Using the well-known Furutzu-Novikov procedure [33], the correlator $\langle X(t_1)X(t_2)Z(t_1)Z(t_2)\rangle$ can be given by [28]

$$\langle X(t_1)X(t_2)Z(t_1)Z(t_2)\rangle = 2D\left\langle X^2(t_2)\right\rangle \delta(|t_2 - t_1|). \quad (26)$$

In the long-time limit, $t \to \infty$, Eqs. (24) – (26), (21) and (16) yield the following asymptotic formula for the correlation function $K(\tau,t)$:

$$K_{as}(\tau,t) = \int_0^\infty\int_0^\infty H(t_1)H(t_2)\langle \xi(\tau+t_2)\xi(t_1)\rangle\, dt_1 dt_2 + 2D\int_0^\infty H(\tau+t_1)H(t_1)\left\{\sigma^2(t-t_1) + A^2\sin^2[\Omega(t-t_1)+\varphi]\right\} dt_1, \quad t\to\infty. \quad (27)$$

The two-time asymptotic correlation function $K_{as}(\tau,t)$ depends on both times t and τ and becomes a periodic function of t with the period of the external driving, $T = 2\pi/\Omega$. Thus as in [34, 35], we define the one-time correlation function $K_1(\tau)$ as the average of the two-time correlation function over a period of external driving, i.e.,

$$K_1(\tau) = \frac{1}{T}\int_0^T K_{as}(\tau,t)dt. \quad (28)$$

Using Eqs. (3), (27), and (28) we obtain

$$K_1(\tau) = \left[D\left(2\sigma_h^2 + A^2\right) + 2D_1\right]\Psi(\tau) + \frac{k_B T}{\omega^2}F(\tau), \quad (29)$$

where σ_h^2 is the time-homogeneous part of the variance of the oscillator displacement X in the asymptotic regime, $t\to\infty$, i.e.,

$$\sigma_h^2 = \frac{1}{T}\int_0^T \sigma_{as}^2(t)dt. \quad (30)$$

It should be emphasized that the functions $\Psi(\tau)$ and $F(\tau)$, defined as

$$\Psi(\tau) := \int_0^\infty H(t+\tau)H(t)dt, \quad (31)$$

and

$$F(\tau) := \frac{\gamma\omega^2}{\Gamma(1-\alpha)}\int_0^\infty\int_0^\infty H(t_1)H(t_2)\frac{dt_1 dt_2}{|\tau + t_2 - t_1|^\alpha}, \quad (32)$$

respectively, are independent of the driving force parameters A_0 and Ω as well as of the noise intensities D, D_1, and T. Using the formula (11) and the results of [27], one gets

$$\Psi(\tau) = \frac{e^{-\beta\tau}}{u^2+v^2}\left\{\frac{1}{\beta}\cos(\omega^*\tau) - \frac{1}{\beta^2+\omega^{*2}}\left[\beta\cos(\omega^*\tau+2\theta) - \omega^*\sin(\omega^*\tau+2\theta)\right]\right\} + \frac{\gamma\sin(\alpha\pi)}{\pi}\int_0^\infty \frac{r^\alpha dr}{B(r)}\left\{\frac{e^{-r\tau}}{r^2+\gamma r^\alpha+\omega^2} + \frac{2e^{-\beta\tau}[\omega^*\cos(\omega^*\tau+\theta) + (r+\beta)\sin(\omega^*\tau+\theta)]}{\sqrt{u^2+v^2}\left[(r+\beta)^2+\omega^{*2}\right]}\right\}, \quad (33)$$

$$F(\tau) = \frac{2\omega^2 e^{-\beta\tau}}{\sqrt{u^2+v^2}\left(\beta^2+\omega^{*2}\right)} \times [\omega^* \cos(\omega^*\tau+\theta)+\beta\sin(\omega^*\tau+\theta)] + \frac{\omega^2\gamma}{\pi}\sin(\alpha\pi)\int_0^\infty \frac{e^{-r\tau}dr}{r^{1-\alpha}B(r)}. \quad (34)$$

Turning now to Eq. (29) we consider the time-homogeneous part of the variance σ_h^2. As $K_1(0) = \sigma_h^2$ and $F(0) = 1$ we find from Eq. (29) that

$$\sigma_h^2 = \frac{1}{D_{cr}-D}\left(\frac{A^2}{2}D + D_1 + \frac{k_B T D_{cr}}{\omega^2}\right), \quad (35)$$

where the critical noise intensity D_{cr} reads:

$$D_{cr} = \frac{1}{2\Psi(0)}. \quad (36)$$

From Eq. (35) we can see that the stationary regime is possible only if $D < D_{cr}$. As the intensity of the multiplicative noise D tends to the critical value D_{cr} the variance σ_h^2 increases to infinity. This is an indication that for $D > D_{cr}$ energetic instability appears, which manifests itself in an unlimited increase of second-order moments of the output of the oscillator with time, while the mean value of the oscillator displacement remains finite [35, 36]. Thus, in the stationary case the normalized one-time correlation function

$$K_n(\tau) = \frac{K_1(\tau)}{\sigma_h^2} \quad (37)$$

is given by

$$K_n(\tau) = 2D_{cr}\Psi(\tau)\left(1-\frac{k_B T}{\omega^2\sigma_h^2}\right) + \frac{k_B T}{\omega^2\sigma_h^2}F(\tau). \quad (38)$$

The analytical expressions (33) – (35) and (38) belong to the main results of this work.

3 Results

3.1 Variance of the output signal

In Figs. 1 and 2 we depict the behavior of the critical noise intensity D_{cr} and the variance σ_h^2 by variations of the memory exponent α. Fig. 1 shows a typical resonance-like behavior of $D_{cr}(\alpha)$. As a rule, the maximal value of D_{cr}/γ increases as the value of the friction coefficient γ increases, while the positions of the maxima are monotonically shifted to a lower α as γ rises. In the case considered in Fig.2

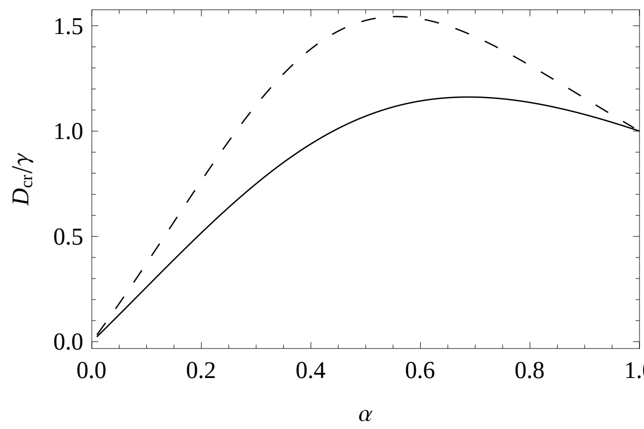

Fig. 1. Critical noise intensity D_{cr} as function of the memory exponent α, obtained from Eqs. (19), (33), and (36) for $A_0 = \omega = 1$. Solid line, $\gamma = 4$; dashed line $\gamma = 1.62$.

the intensity of the multiplicative noise is in the interval $\omega^2\gamma < D < D_{cr\,\max}$, where $D_{cr\,\max}$ is the maximal value of $D_{cr}(\alpha)$ by variations of α. In this case the variance σ_h^2 decreases rapidly from infinity at α_1, $D_{cr}(\alpha_1) = D$, to a minimum and next increases to infinity at α_2, $D_{cr}(\alpha_2) = D$. Thus the fractional oscillator is energetically stable only in the interval $\alpha_1 < \alpha < \alpha_2$.

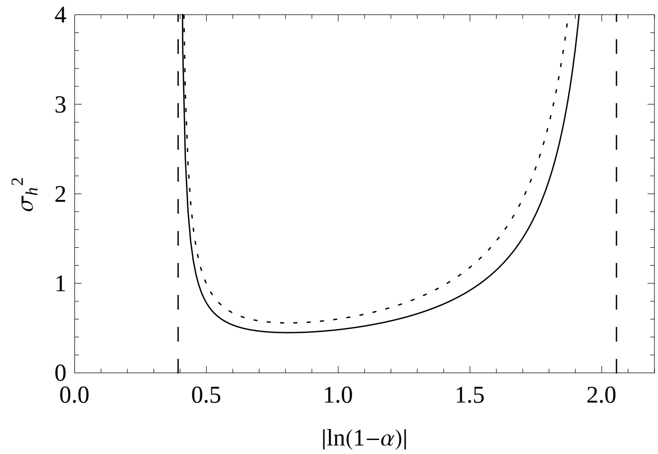

Fig. 2. Dependence of the variance σ_h^2 on the memory exponent α, computed from Eqs. (19), (33), (35), and (36). The parameter values: $A_0 = \omega = 1$, $\gamma = 4$, $D = 4.8$, $D_1 = 0$, and $k_BT/\omega^2 = 0.1$. Solid line, $\Omega = 10$; dotted line, $\Omega = 1$. The dashed lines depict the positions of the critical memory exponents $\alpha_1 \approx 0.325$ and $\alpha_2 \approx 0.872$ between which the oscillator is energetically stable.

From Fig. 2 one can see that the values of the variance σ_h^2 depend on the frequency Ω of the harmonic driving force. As in the case without multiplicative

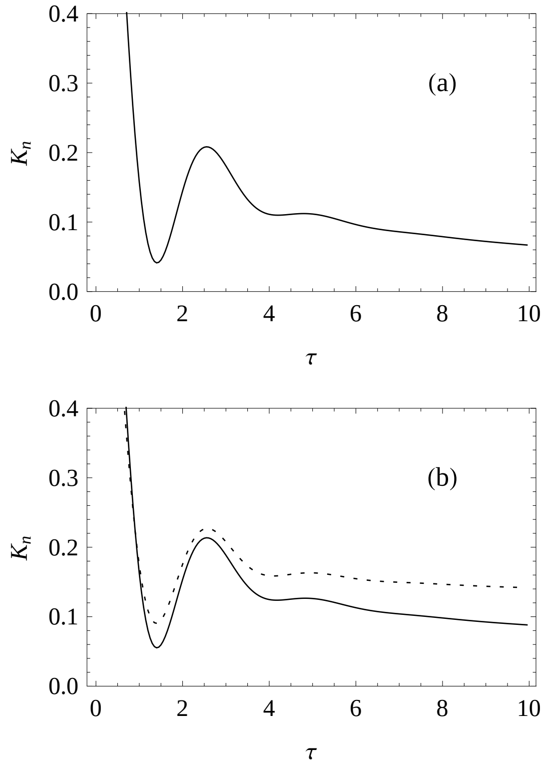

Fig. 3. Normalized autocorrelation function $K_n(\tau)$ vs. the time lag τ computed from Eq. (38) for $\omega = 1$, $\gamma = 4$, and $\alpha = 0.45$. Panel (a): external noise (i.e., $T = 0$). Panel (b): internal noise (i.e., $D_1 = 0$). System parameter values: $A_0 = 1$, $D = 4.8$, and $k_B T/\omega^2 = 0.01$. Solid line, $\Omega = 1$; dotted line, $\Omega = 10$.

noise such a dependence is absent (cf. also Eq. (35)), this phenomenon offers a simple experimental possibility to verify the existence of a multiplicative noise in oscillatory systems described by model (1).

3.2 Temporal behavior of the autocorrelation function

Now we will consider the behavior of the normalized autocorrelation function $K_n(\tau)$ (see Eqs. (33), (34) and (38)). In contrast to the results for variance σ_h^2, here the role of the additive driving noise $\xi(t)$ is crucial. If the driving noise is external (i.e. $T = 0$), the typical form of the graph $K_n(\tau)$ is represented in Fig. 3(a). Note that the exact solution exhibits exponentially damped oscillations around a curve which for large τ decays totally monotonically like a power-law. Consequently, the normalized autocorrelation function has only a finite number of zeros and decays, in the end, as $\tau^{-(1+\alpha)}$ tends to the τ-axis. Note that in this case the function $K_n(\tau)$ is independent of the driving force parameters A_0 and Ω.

In the case of an internal noise $\xi(t)$ (i.e., $D_1 = 0$) the picture of the dependence of $K_n(\tau)$ on τ is different (see Fig. 3(b)). First, the autocorrelation function $K_n(\tau)$ relaxes asymptotically like $\tau^{-\alpha}$. This is in sharp contrast with the result for the external noise that exhibits a much faster decay. Second, the most important difference is the dependence of $K_n(\tau)$ on the amplitude A of the output signal (cf. Eqs. (19), (35), and (38)). Thus, in the case of internal noise the exact form of $K_n(\tau)$ is sensitive to the values of the frequency Ω of the external harmonic driving force.

3.3 Memory-induced trapping

Next we consider the behavior of output characteristics (K_n and σ_h^2) without the harmonic field, i.e., that of Eq. (1) with a zero eigenfrequency, $\omega = 0$. It is seen from Eq. (35) that the time-homogeneous part of the variance of $X(t)$ depends on the quantity

$$\sigma_0^2 = \frac{k_B T}{\omega^2} + \frac{D_1}{D_{cr}}, \tag{39}$$

which is determined as the stationary asymptotic value, $t \to \infty$, of the variance of $X(t)$ in the case where the multiplicative fluctuations and the external sinusoidal force are both absent. The last mentioned particular case without the harmonic field (i.e., $\omega = A_0 = D = 0$ in Eq. (1)) will be referred to in this section as the basic system. It is well known that in the case of internal noise the output process of the basic system is always subdiffusive, i.e., $\sigma_0^2 \sim t^\alpha$, and a stationary regime is impossible [37]. Thus, as the driving noise includes an internal component, ($T \neq 0$), the behavior of the model (1) with $\omega = 0$ is subdiffusive and that renders formulas (35) and (38) physically meaningless.

If the driving noise $\xi(t)$ is external, ($T = 0$, $D_1 \neq 0$), two different behaviors of the basic system arise depending on the values of the memory exponent α. Namely, in this case the variance σ_0^2 is a finite constant at $t \to \infty$, if $\alpha < 1/2$, and diverges with time as $\sigma_0^2 \sim t^{2\alpha-1}$ if $\alpha > 1/2$. Thus, in the case of an external additive noise $\xi(t)$ with $\alpha < 1/2$ the formulas (35) and (38) for the output characteristics (K_n and σ_h^2) of Eq. (1) without a harmonic trap ($\omega = 0$) are applicable. It is remarkable that in this case the corresponding quantities u, v, β, θ, and ω^* in Eqs. (33) and (34) can be reduced to the following more convenient form:

$$\begin{aligned} \beta &= -\gamma^{1/(2-\alpha)} \cos\left(\frac{\pi}{2-\alpha}\right), \\ \omega^* &= \gamma^{1/(2-\alpha)} \sin\left(\frac{\pi}{2-\alpha}\right), \\ \theta &= \frac{\alpha\pi}{2(\alpha-2)}, \\ u &= (\alpha-2)\beta, \quad v = (2-\alpha)\omega^*. \end{aligned} \tag{40}$$

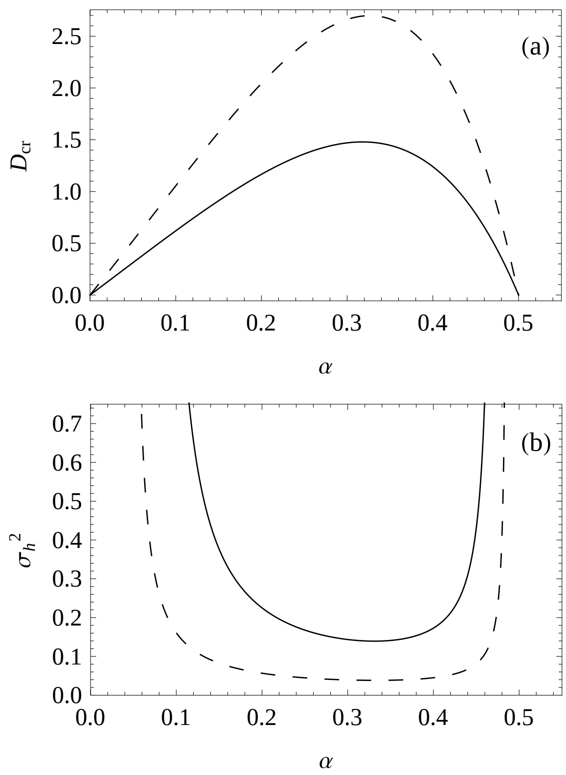

Fig. 4. The critical noise intensity D_{cr} and the variance σ_h^2 as functions of the memory exponent α by absence of the harmonic potential, $\omega = 0$, (Eqs. (33), (35), and (36)). The parameter values: $A_0 = \Omega = 1$, $T = 0$, $D_1 = 0.05$, and $D = 0.5$. Dashed line, $\gamma = 3.5$; solid line, $\gamma = 2.5$. Note that the critical memory exponent $\alpha_{cr} = 1/2$, at which the critical noise intensity D_{cr} tends to zero.

It is interesting that although the critical intensity of the multiplicative noise D_{cr} (see Eqs. (33) and (36)) is independent of the driving noise $\xi(t)$, the necessary and sufficient condition for the existence of energetic stability (i.e., $D_{cr} \neq 0$) is exactly the same as the condition for the existence of a stationary variance σ_0^2 in the case of external noise, i.e., $0 < \alpha < 1/2$. The above described behavior of an unbounded system (Eq. (1)) in the case of a delta-correlated external driving noise with the intensity D_1 is illustrated in Figs. 4 and 5.

Furthermore, due to the cage effect the dependence of SPA on the frequency Ω exposes a bona fide resonance for any γ and $\alpha < \alpha_c \approx 0.441$ (see Fig. 6), even when the binding harmonic filed is absent (i.e. $\omega = 0$).

In the case of $\omega = 0$ the positions of extrema $\Omega_\pm$

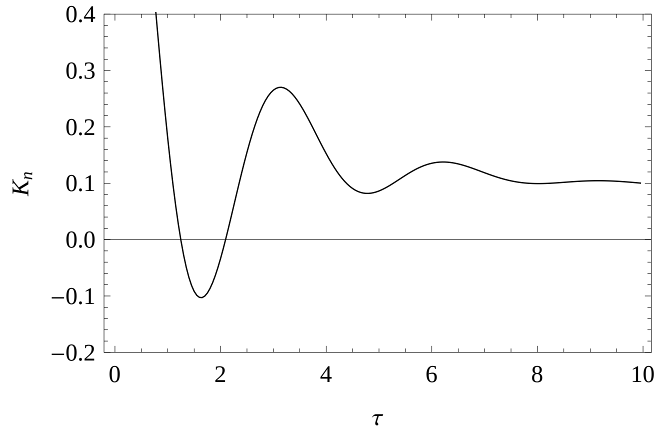

Fig. 5. Normalized one-time correlation function $K_n(\tau)$ versus the time lag τ computed from Eqs. (33), (36), (38), and (40) for $T = \omega = 0$, $\alpha = 0.3$, and $\gamma = 3.5$.

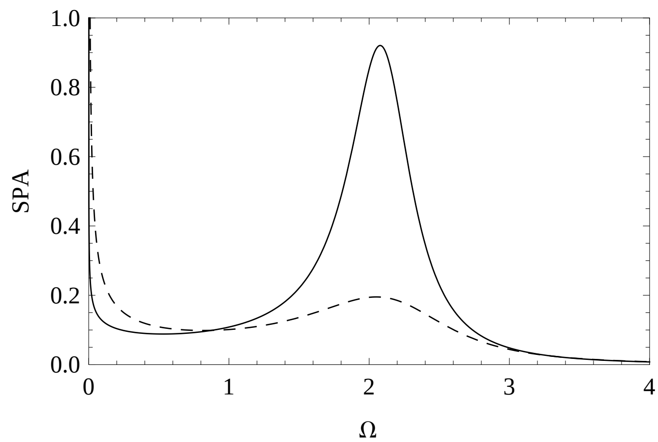

Fig. 6. The SPA versus the driving frequency Ω computed from Eq. (19) for $T = \omega = 0$ and $\gamma = 4$. Solid line, $\alpha = 0.15$; dashed line, $\alpha = 0.3$.

are determined by

$$\Omega_\pm = \left\{ \frac{\gamma}{4} \left[(\alpha+2)\cos\left(\frac{\alpha\pi}{2}\right) \pm \sqrt{(\alpha+2)^2\cos^2\left(\frac{\alpha\pi}{2}\right) - 8\alpha} \right] \right\}^{\frac{1}{2-\alpha}}. \quad (41)$$

As memory exponent α decreases, the positions of the minimum and of the maximum tends to values 0 and $\sqrt{\gamma}$, respectively (see Fig. 7). This demonstrates the binding role of the fractional derivative in Eq. (1) at strong memory.

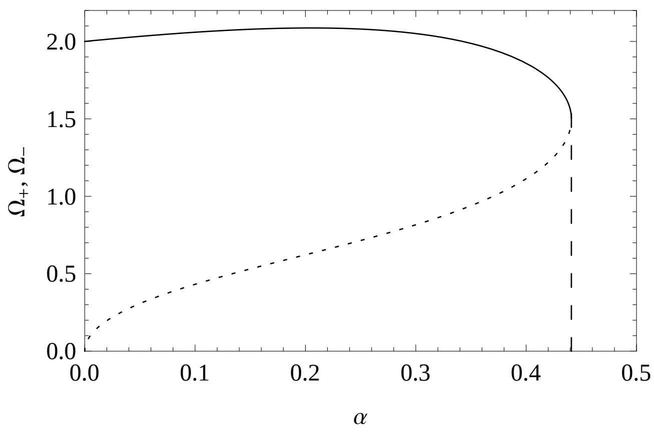

Fig. 7. Positions of the SPA maximum (Ω_+, solid line) and minimum (Ω_-, dotted line) as functions of the memory exponent α, computed from Eq. (41) for $\gamma = 4$. The dashed line depicts the position of the critical memory exponent $\alpha_{cr} \approx 0.441$.

4 Conclusions

We have studied, in the long-time regime, the variance and the autocorrelation function of particle displacement for a harmonically trapped Brownian particle in a viscoelastic media. Starting from a suitable generalized Langevin equation with memory, i.e., a fractional oscillator with a fluctuating eigenfrequency driven by external sinusoidal forcing and by an additive noise, we have been able to derive an exact analytic expression for the one-time autocorrelation function in the case of a power-law-type friction kernel.

As one of our main results we have established that in the presence of a multiplicative noise the output variance depends on the parameters of external sinusoidal forcing. Since without a multiplicative noise such a dependence is absent, this effect gives a simple criterion to determine if a multiplicative noise is present in the dynamics of the system. Moreover, it is remarkable that in the case of an additive external noise and a sufficiently strong memory, a related phenomenon involving memory-induced trapping occurs for an unbound system (i.e., in Eq. (1), the harmonic binding potential is absent). Note that for internal noise the behavior of such a system is always subdiffusive [37].

As another main result we have shown that in the case of an additive external noise the dependence of the normalized autocorrelation function $K_n(\tau)$ on the time lag τ is independent of external periodic forcing. This contrasts the behavior for the case of internal noise, where the dependence of $K_n(\tau)$ on periodic forcing is significant.

Thus we have found convenient criteria that enable us to distinguish the presence of a multiplicative and an internal noise in systems described by Eq. (1). The advantage of these criteria is that the control parameter is the frequency (or the amplitude) of the external periodic force, which can be easily varied in possible experiments as well as potential technological applications, e.g., in electric oscillator devices with circuit elements of a fractional type (i.e., tree or chain fractances) [31].

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