A Fuzzy Systems Framework for solving real world problems

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Abstract: Mathematical modelling and Case-Based Reasoning (CBR) are the two principal methods used for the solution of real world problems. In the present paper we attempt a mathematical description of the above processes by using principles of fuzzy logic and of uncertainty theory. Examples are also presented to illustrate our results.

Key words: Fuzzy sets, fuzzy relations, fuzzy models, uncertainty, possibility theory, mathematical modelling, Case-Based Reasoning.

1 Mathematical modelling

Mathematical modelling appears today as a dynamic tool for teaching mathematics, because it connects mathematics with the real world and our everyday life, thus giving students the opportunity to realize its usefulness in practical applications [34]. We recall that the main stages of the mathematical modelling process involve:

Analysis of the given real world problem, i.e. understanding the statement and recognizing limitations, restrictions and requirements of the real system.

Mathematising, i.e. formulation of the real situation in such a way that it will be ready for mathematical treatment, and construction of the model.

Solution of the model, achieved by proper mathematical manipulation.

Validation (control) of the model, usually achieved by reproducing through it the behaviour of the real system under the conditions existing before the solution of the model (empirical results, special cases etc).

Implementation of the final mathematical results to the real system, i.e. "translation" of the mathematical solution obtained in terms of the corresponding real situation in order to reach the solution of the given real problem.

From the above brief description becomes evident that mathematising, solution and validation are the most important stages of the modelling process. In fact, the analysis of the problem, although it deserves some attention as being a prerequisite for mathematising, it could be considered as an introductory stage of the whole process. Further the stage of implementation is not expected to hide any "surprises", at least for the type of modelling problems solved usually by students at school. In other words, a student who obtained a correct mathematical solution is normally expected to be able to "translate" it correctly in terms of the corresponding real situation.

A central object of educational research taking place in the area of Mathematical Modelling and Applications is to recognize the attainment level of students at defined stages of the modelling process. In an earlier paper [29] we presented a stochastic model for the description of the process of mathematical modelling in situations where the teacher provides such modelling problems to students for solution. Namely, we introduced a finite Markov chain having as states the five main stages of the modeling process (analysis, mathematising, solution, validation and implementation). Each state is defined in terms of expected outcomes and transition from one state to the next is wholly dependent upon the successful completion of the previous state. Through this model we succeeded in obtaining a measure of students' mathematical model building abilities. An improved version of the above model has been presented in [35] (see also Figure 2 in [7]).

Models for the mathematical modelling process like the above and the analogous ones described in [17] and the second paragraph of [7], are helpful in understanding what is termed in [7] as *"ideal behaviour"*, in which modellers proceed from real world problems through a mathematical model to

acceptable solutions and report on them. However life in the classroom is not like that. Recent research, ([4], [5] and [10]), reports that students in school take *individual modelling routes* when tackling mathematical modelling problems, associated with their individual learning styles. Students' cognition utilizes in general concepts that are inherently graded and therefore fuzzy. On the other hand, from the teacher's point of view there usually exists vagueness about the degree of success of students in each of the stages of the modelling process. All these gave us the impulsion to introduce principles of fuzzy sets theory in order to describe in a more effective way the process of mathematical modelling in classroom. The concept of uncertainty, which emerges naturally within the broad framework of fuzzy sets theory, is involved in any problem-solving situation, especially when dealing with real-world problems. Uncertainty is a result of some information deficiency. In fact, information pertaining to the model within which a real situation is conceptualized may be incomplete, fragmentary, not full reliable, vague, contradictory, or deficient in some other way. Thus the amount of information obtained by an action can be measured in general by the reduction of uncertainty resulting from the action. In other words the amount of uncertainty regarding some situation represents the total amount of potential information in this situation. Accordingly students' uncertainty during the modelling process is connected to students' capacity in obtaining relevant information. Therefore a measure of uncertainty could be adopted as a measure of students modelling capacities.
For special facts on fuzzy sets and uncertainty theory we refer freely to [11] and [12].

2 The fuzzy model for the modelling process

Let us consider a group of n students, $n \geq 2$, during the modelling process in classroom. Denote by A_i , i=1,2,3 , the stages of mathematising, solution and validation of the model respectively, and by a, b, c, d, and e the linguistic labels of negligible, low, intermediate, high and complete success respectively of a student in each of the A_i's. Set *U={a,d,c,d,e}*. We are going to represent A_i's as fuzzy sets in U. For this, if n_{ia}, n_{ib}, n_{ic}, n_{id} and n_{ie} denote the number of students that had negligible, low, high and complete success at state A_i respectively, i=1,2,3, we define the membership function m_{Ai} in terms of the frequencies, i.e. by $m_{Ai}(x)=\frac{n_{ix}}{n}$ for each x in U. Thus we can write

$$A_i= \{(x, \frac{n_{ix}}{n}) : x \in U\}.$$

In the same way we could also represent the stages of analysis and implementation as fuzzy sets in U. However this, making the presentation of our fuzzy model technically much more complicated, it is not so important, as we have already explained above and therefore we will not attempt it. This manipulation is actually a general technique applied frequently during the modelling process of a real-world problem by eliminating the variables of the real system that are not necessary for the study and solution of it. In this way we transfer from the real system to the, so called, *"assumed real system"*, that helps towards the formulation of the problem in a form ready for mathematical treatment (cf. [35]; section 1).
In order to represent all possible student *profiles (overall states)* during the modelling process we consider a *fuzzy relation*, say R, in U^3 of the form

$$R= \{(s, m_R(s)): s=(x, y ,z) \in U^3\}.$$

To determine properly the membership function m_R we give the following definition:
DEFINITION: A profile s=(x,y,z), with x,y,z in U, is said to be *well ordered* if x corresponds to a degree of success equal or greater than y, and y corresponds to a degree of success equal or greater than z. For example, (c, c, a) is well ordered profile, while (b, a, c) is not.
We define now the membership degree of a profile s to be $m_R(s)=m_{A_1}(x)m_{A_2}(y)m_{A_3}(z)$ (1) if s is well ordered, and zero otherwise. In fact, if for example profile (b, a, c) possessed a nonzero membership degree, how it could be possible for a student, who has failed during mathematisation, to validate satisfactorily the model?
The above definition satisfies the axioms for aggregation operations on fuzzy sets (cf. [11]; p. 58-59, and p. 283).
In the next for reasons of brevity we shall write m_s instead of $m_R(s)$. Then the *possibility* r_s of profile s is defined by $r_s=\frac{m_s}{\max\{m_s\}}$ (2) , where $\max\{m_s\}$ denotes the maximal value of m_s , for all s in U^3. In other words r_s expresses the "relative membership degree" of s with respect to max $\{m_s\}$.
Within the domain of possibility theory (cf. [12]) uncertainty consists of *strife (or discord)*, which expresses conflicts among the various sets of alternatives, and *non-specificity (or imprecision)*, which indicates that some alternatives are left

unspecified, i.e. it expresses conflicts among the sizes (cardinalities) of the various sets of alternatives. Strife is measured by the function ST(r) on the ordered possibility distribution r: $r_1=1\geq r_2 \geq\ldots\ldots\geq r_m \geq r_{m+1}$ of the student group (where m+1 is the total number of all possible students' profiles), defined by

$$ST(r) = \frac{1}{\log 2}\left[\sum_{i=2}^{n}(r_i - r_{i+1})\log\frac{i}{\sum_{j=1}^{i} r_j}\right] \quad (3)$$

while non-specificity is measured by

$$N(r) = \frac{1}{\log 2}\left[\sum_{i=2}^{n}(r_i - r_{i+1})\log i\right] \quad (4).$$

In particular during the modelling process students may use reasoning that involves amplified inferences, whose content is beyond the available evidence, and hence obtain conclusions not entailed in the given premises. These conclusions may produce a generalization whose amount of information will exceed the amount of information in the level of functioning, i.e. an overgeneralization. For mathematical calculations for example, such conclusions could be the illusion that $\sqrt{a+b}=\sqrt{a}+\sqrt{b}$, or that log (a+b)=log a + log b, etc. The appearance of conflict in conclusions requires that the conclusions be appropriately adjusted so that the resulting generalization is free of conflict.

The sum T(r)=ST(r)+N(r) (5) is a measure of the *total possibilistic uncertainty* T(r) for ordered possibility distributions. The total possibilistic uncertainty T(r) of a student group during the modelling process can be adopted as a measure for its modelling capacity (see section 1). This is reinforced by Shackle [23], who argues that human reasoning can be formalized more adequately by possibility theory rather, than by probability theory. The lower is the value of T(r), the better the performance of the student group during the modelling process.

Assume finally that one wants to study the combined results of behaviour of k different student groups, $k\geq 2$, during the modelling process of the same real situation. For this we introduce the *fuzzy variables* A_1 (t), A_2 (t) and A_3 (t) with t=1, 2,…, k. The values of the above variables represent the states of the modelling process for each of the k student groups as fuzzy sets in U: e.g. A_1 (2) represents the state of mathematising for the secondgroup (t=2). It becomes evident that, in order to measure the degree of evidence of combined results of the k groups, it is necessary to define the possibility r(s) of each student profile s with respect to the membership degrees of s for all student groups. For this reason we introduce the *pseudo-frequencies* $f(s)=\sum_{t=1}^{k} m_s(t)$ (6) and we define

$$r(s) = \frac{f(s)}{\max\{f(s)\}} \quad (7),$$

where *max {f(s)}* denotes the maximal pseudo-frequency. Obviously the same method could be applied when one wants to study the behaviour of a student group during the modelling process of k different real problems.

3 A classroom experiment

In order to illustrate the results obtained in the previous section we performed the following experiment, which took place recently at the Graduate Technological Educational Institute (T.E.I.) of Patras, Greece. Our subjects were 35 students of the School of Technological Applications, i.e. future engineers, and our basic tool was a list of 10 problems involving mathematical modelling given to students to solve them (time allowed 2 hours). Our characterizations of students' performance at each stage of the modelling process involved:

Negligible success, if they obtained positive results for less than 2 problems.

Low success, if they obtained positive results for 2, 3, or 4 problems.

Intermediate success, if they obtained positive results for 5, 6, or 7 problems.

High success, if they obtained positive results for 8, or 9 problems.

Complete success, if they obtained positive results for all problems.

Examining students' papers we found that 17, 8 and 10 students had intermediate, high and complete success respectively at stage of mathematising. Therefore we obtained that $n_{1a}=n_{1b}=0$, $n_{1c}=17$, $n_{1d}=8$ and $n_{1e}=10$. Thus mathematising was represented as a fuzzy set in U in the form:

$$A_1 = \{(a,0),(b,0),(c,\tfrac{17}{35}),(d,\tfrac{8}{35}),(e,\tfrac{10}{35})\}.$$

In the same way we represented solution and validation of the model as fuzzy sets in U by

$A_2 = \{(a, \frac{6}{35}),(b, \frac{6}{35}),(c, \frac{16}{35}),(d, \frac{7}{35}),(e,0)\}$ and

$A_3 = \{(a, \frac{12}{35}),(b, \frac{10}{35}),(c, \frac{13}{35}),(d,0),(e,0)\}$ respectively.

Using the definition given in the previous section and relation (1), we calculated the membership degrees of the 5^3 in total possible students' profiles (see column of $m_s(1)$ in Table 1). For example, for $s=(c, b, a)$ one finds that

$m_s = m_{A_1}(c).\ m_{A_2}(b).\ m_{A_3}(a) = \frac{17}{35}\frac{6}{35}\frac{12}{35} = \frac{1224}{42875} \approx 0{,}029.$

It turned out that (c, c, c) was the profile of maximal membership degree 0,082 and therefore the possibility of each s in U^3 is given by $r_s=\frac{m_s}{0{,}082}$ (see relation (2)). For example, the possibility of (c, b, a) is $\frac{0{,}029}{0{,}082} \approx 0{,}353$, while the possibility of (c, c, c) is of course 1.

Calculating the possibilities of all profiles (see column of $r_s(1)$ in Table 1) one finds that the ordered possibility distribution for the student group is: $r_1=1$, $r_2=0{,}927$, $r_3=0{,}768$, $r_4=0{,}512$, $r_5=0{,}476$, $r_6=0{,}415$, $r_7=0{,}402$, $r_8=0{,}378$, $r_9=r_{10}=0{,}341$, $r_{11}=0{,}329$, $r_{12}=0{,}317$, $r_{13}=0{,}305$, $r_{14}=0{,}293$, $r_{15}=r_{16}=0{,}256$, $r_{17}=0{,}207$, $r_{18}=0{,}195$, $r_{19}=0{,}171$, $r_{20}=r_{21}=r_{22}=0{,}159$, $r_{23}=0{,}134$, $r_{24}=r_{25}=\ldots\ldots=r_{125}=0$.

Therefore, by (3), (4) and (5) and using a calculator we found that the total possibilistic uncertainty of the group was $T(r) \approx 0{,}565+2{,}405=2{,}97$

A few days later we performed the same experiment with a group of 30 students of the School of Management and Economics. The students of the above School study in detail the modelling process within the course of Operations' Research, in contrast to students of School of Technological Applications, who study mathematical modelling only through working examples within the 2 or 3 (it depends upon the corresponding department) mathematics courses that they attend. Working as before we found that

$A_1=\{(a,0),(b, \frac{6}{30}),(c, \frac{15}{30}),(d, \frac{9}{30}),(e,0)\}$,

$A_2=\{(a, \frac{6}{30}),(b, \frac{8}{30}),(c, \frac{16}{30}),(d, 0),(e,0)\}$ and

$A_3=\{(a, \frac{12}{30}),(b, \frac{9}{30}),(c, \frac{9}{30}),(d,0),(e,0)\}$.

Then we calculated the membership degrees of all possible profiles of the student group (see column of $m_s(2)$ in Table 1). It turned out that (c, c, a) was the profile possessing the maximal membership degree 0,107 and therefore the possibility of each s is given by $r_s=\frac{m_s}{0{,}107}$ (see column of $r_s(2)$ in Table 1). Finally we found that $T(r) = 0{,}452+1{,}87 = 2{,}322$.

Thus, since 2,322<2, 97, the second group had in general a slightly better performance than the first one. This happened despite to the fact that profile (c, c, c) with maximal possibility of appearance for the first student group is more satisfactory than the corresponding profile (c, c, a) for the second group. The above result, combined to the fact that the students of School of Management and Economics attend only one course of general Mathematics (they attend also Mathematics of Finance and Statistics) is an indication that a detailed study of the modelling process possibly helps students to have a better performance in solving problems that involve mathematical modelling.

Table 1: Student profiles with non zero pseudo-frequencies

A_1	A_2	A_3	$m_s(1)$	$r_s(1)$	$m_s(2)$	$r_s(2)$	f(s)	r(s)
b	b	b	0	0	0,016	0,150	0,016	0,087
b	b	a	0	0	0,021	0,196	0,021	0,115
b	a	a	0	0	0,016	0,150	0,016	0,087
c	c	c	0,082	1	0,080	0,748	0,162	0,885
c	**c**	**a**	**0,076**	**0,927**	**0,107**	**1**	**0,183**	**1**
c	c	b	0,063	0,768	0,008	0,075	0,071	0,388
c	a	a	0,028	0,341	0,040	0,374	0,068	0,372
c	b	a	0,028	0,341	0,053	0,495	0,081	0,443
c	b	b	0,024	0,293	0,040	0,374	0,064	0,350
d	d	a	0,016	0,495	0	0	0,016	0,087
d	d	b	0,013	0,159	0	0	0,013	0,074
d	d	c	0,021	0,256	0	0	0,021	0,115
d	a	a	0,013	0,159	0,024	0,224	0,037	0,202
d	b	a	0,013	0,159	0,032	0,299	0,045	0,246
d	b	b	0,011	0,134	0,024	0,224	0,035	0,191
d	c	a	0,031	0,378	0,064	0,598	0,095	0,519
d	c	b	0,026	0,317	0,048	0,449	0,074	0,404
d	c	c	0,034	0,415	0,048	0,449	0,082	0,448
e	a	a	0,017	0,207	0	0	0,017	0,093
e	b	b	0,014	0,171	0	0	0,014	0,077
e	c	a	0,039	0,476	0	0	0,039	0,213
e	c	b	0,033	0,402	0	0	0,033	0,180
e	c	c	0,042	0,512	0	0	0,042	0,230
e	d	a	0,025	0,305	0	0	0,025	0,137
e	d	b	0,021	0,256	0	0	0,021	0,115
e	d	c	0,027	0,329	0	0	0,027	0,148

(Note: The outcomes of Table 1 are with accuracy up to the third decimal point)

Of course further research and experiments are needed to validate statistically this conjecture.

Next, in order to study the combined results of behaviours of the two groups, we introduced the fuzzy variables $A_i(t)$, i=1, 2, 3 and t=1, 2. Then the pseudo-frequency of each student profile s is given by $f(s) = m_s(1) + m_s(2)$ (see corresponding column in Table 1). It turned out that (c, c, a) was the profile with highest pseudo-frequency 0,183 and therefore the possibility of each student's profile is given by $r(s) = \frac{f(s)}{0{,}183}$. The possibilities of all profiles having non-zero pseudo-frequencies are presented in the last column of Table 1.

4. Case-Based Reasoning

Case-Based Reasoning (CBR) is a general paradigm for problem-solving and learning from expertise, which is not only a psychological theory of human cognition, but it also provides a foundation for a new technology of intelligent computer systems that can solve problems and adapt to new situations.

Broadly construed CBR is the process of solving new problems based on the solutions of similar past problems. Its coupling to learning occurs as a natural by-product of problem solving. When a problem is successfully solved, the experience is retained in order to solve similar problems in future. When an attempt to solve a problem fails, the reason for the failure is identified and remembered in order to avoid the same mistake in future. Thus CBR is a cyclic and integrated process of solving a problem, learning from this experience, solving a new problem, etc. It must be noticed that the term problem-solving is used here in a wide sense, which means that it is not necessarily the finding of a concrete solution to an application problem, it may be any problem put forth by the user. For example, to justify or criticize a proposed solution, to interpret a problem situation, to generate a set of possible solutions, or generate explanations in observable data, are also problem solving situations.

A lawyer, who advocates a particular outcome in a trial based on legal precedents, or an auto mechanic, who fixes an engine by recalling another car that exhibited similar symptoms, are using CBR; in other words CBR is a prominent kind of analogy making.

All inductive reasoning, where data is too scarce for statistical relevance, is inherently based on anecdotal evidence. Critics of CBR argue that it is an approach that accepts anecdotal evidence as its main operating principle, but without statistically relevant data for backing an implicit generalization, there is no guarantee that the generalization is correct. This criticism has only a theoretical base, because in practice CBR methods give satisfactory results in most cases.

CBR traces its roots in Artificial Intelligence to the work of Roger Schank and his students at Yale University, U.S.A. in early 1980's. Scfhank's model of dynamic memory [22] was the basis of the earliest (in 1983) computer intelligent systems that can be viewed as prototypes for CBR systems, the Kolodner's CYRUS [13] and Lebowitz's IPP [14]. An alternative approach is the category and exemplar model applied first to the PROTOS system of Porter and Bareiss [20], while some other types of memory models, developed later on.

The CBR systems expertise is embodied in general in a collection (library) of past cases rather, than being encoded in classical rules. Each case typically contains a description of the problem plus a solution and/or the outcomes. The knowledge and reasoning process used by an expert to solve the problem is not recorded, but is implicit in the solution.

As an intelligent-systems method CBR has got a lot of attention over the last few years, because it enables the information managers to increase efficiency and reduce cost by substantially automating processes. CBR first appeared in commercial systems in the early 1990's and since then has been sued to create numerous applications in a wide range of domains including diagnosis, help-desk, assessment, decision support, design, etc. Organizations as diverse as IBM, VISA International, Volkswagen, British Airways and NASA have already made use of CBR in applications such as customer support, quality assurance, aircraft maintenance, process planning and many more applications that are easily imaginable.

As a general problem-solving methodology intended to cover a wide range of real-world applications, CBR must face the challenge to deal with uncertain, incomplete and vague information. In fact, uncertainty is already inherent in the basic CBR hypothesis demanding that similar problems have similar solutions. Correspondingly recent years have witnessed an increased interest in formalizing parts of the CBR methodology within different frameworks of reasoning under uncertainty, and in building hybrid approaches by combining CBR with methods of uncertain and approximate reasoning. Fuzzy sets theory can be mentioned as a particularly interesting example. In fact, even though both CBR and fuzzy systems are intended as cognitively more plausible approaches to reasoning and problem-solving, the two corresponding fields have emphasized different aspects that complement each other in a reasonable way. Thus fuzzy set-based concepts and methods can support various aspects of CBR including: Case and knowledge representation, acquisition and modeling, maintenance and management of CBR systems, case indexing and retrieval, similarity assessment and adaptation, instance-based and case-based learning, solution explanation and confidence, and representation of context. On the other way round ideas and techniques for CBR can contribute to fuzzy set-based approximate reasoning.

CBR has been formalized for purposes of computer and human reasoning as a four steps process. These steps involve:

R_1: *Retrieve* the most similar to the new problem past case.

R_2: *Reuse* the information and knowledge of the retrieved case for the solution of the new problem.

R_3: *Revise* the proposed solution.

R_4: *Retain* the part of this experience likely to be useful for future problem-solving.

More specifically, the retrieve task starts with the description of the new problem, and ends when a best matching previous case has been found. The subtasks of the retrieving procedure involve: Identifying a set of relevant problem descriptors, matching the case and returning a set of sufficiently similar cases given a similarity threshold of some kind, and selecting the best case from the set of cases returned. Some systems retrieve cases based largely on superficial syntactic similarities among problem descriptors, while advanced systems use semantic similarities.

The reuse of the solution of the retrieved case in the context of the new problem focuses on two aspects: The differences between the past and the current case, and what part of the retrieved case can be transferred to the new case. Usually in non trivial situations part of the solution of the retrieved case cannot be directly transferred to the new case, but requires an adaptation process that takes into account the above differences.

Through the revision the solution generated by reuse is tested for success – e.g. by being applied to the real world environment, or to a simulation of it, or evaluated by a specialist – and repaired, if failed. When a failure is encountered, the system can then get a reminding of a previous similar failure and use the failure case in order to improve its understanding of the present failure, and correct it. The revised task can then be retained directly (if the revision process assures its correctness), or it can be evaluated and repaired again.

The final step R_4 involves selecting which information from the new case to retain, in what form to retain it, how to index the case for better retrieval in future for similar problems, and how to integrate the new case in the memory structure.Notice that Slade ([24]; Figure 1), Lei et al ([15]; Figure 1), Aamodt and Plaza ([1]; Figures 1 and 2) and others have presented detailed flowcharts illustrating the basic steps of the CBR process. In an earlier paper [36] we have also presented a detailed analysis of the CBR methodology.

The general knowledge usually plays a part in the CBR cycle by supporting the CBR process. This support however may range from very weak (or none) to very strong, depending on the type of the CBR method. By general knowledge we here mean general, domain-dependent knowledge, as opposed to specific knowledge embodied by cases. For example, in the case of a lawyer, mentioned in our introduction, who advocates a particular outcome in a trial based on legal precedents, the general knowledge is expressed through the knowledge of the existing relevant laws and the correlations among them and the case of the trial. A set of rules may have the same role in other CBR cases.

5. A fuzzy pepresentation of a CBR system

Let us consider a CBR system whose library contains n past cases, $n \geq 2$. We denote by R_i , i=1,2,3 , the steps of retrieval, reuse and revision respectively, and by a, b, c, d, and e the linguistic labels of negligible, low, intermediate, high and complete degree of success respectively for each of the R_i's. Set U={a,d,c,d,e}; then we are going to represent the R_i's as fuzzy sets in U. For this, if n_{ia}, n_{ib}, n_{ic}, n_{id} and n_{ie} denote the number of cases where it has been achieved negligible, low, intermediate, high and complete degree of success for the state R_i respectively, i=1,2,3, we define the membership function m_{Ri} as follows:

$$m_{R_i}(x) = \rightarrow \begin{cases} 1, & \text{if } \frac{4n}{5} < n_{ix} \leq n \\ 0{,}75, & \text{if } \frac{3n}{5} < n_{ix} \leq \frac{4n}{5} \\ 0{,}5, & \text{if } \frac{2n}{5} < n_{ix} \leq \frac{3n}{5} \\ 0{,}25, & \text{if } \frac{n}{5} < n_{ix} \leq \frac{2n}{5} \\ 0, & \text{if } 0 \leq n_{ix} \leq \frac{n}{5} \end{cases},$$

Therefore, we can write R_i as a fuzzy set in U in the

form : $R_i = \{(x, m_{Ai}(x)) : x \in U\}$, i=1,2,3.

Notice that there is no need to include step R_4 of CBR process in our fuzzy representation, because all the past cases, either successful, or not, are retained in the system's library and therefore there is no fuzziness in this case. In other words, keeping the same notation, we have that $n_{4a}=n_{4b}=n_{4c}=n_{4d}=0$ and $n_{4e}=1$.

In order to represent all possible profiles of a case during the CBR process, we consider a fuzzy relation, say R, in U^3 of the form

R= {(s, $m_R(s)$): s=(x, y ,z) $\in U^3$}.

and we work in the same way as we have done in section 2 for the modelling process to calculate the membership degrees and the possibilities r_s of all profiles s and the total possibilistic uncertainty T(r) of the ordered possibility distribution r. The lower is the value of T(r), the higher is the effectiveness of the corresponding CBR system in solving new related problems.

It is also possible to study the combined results of the behaviour of k different CBR systems $k \geq 2$, designed for the solution of the same type of problems. For this, we must introduce the fuzzy variables $R_i(t)$, with i=1,2,3 and t=1,2,…,k, and determine the possibilities of the profiles s(t) through the pseudofrequences, as we have done in sections 2 and 3 for the modelling process. These possibilities measure the degree of evidence of the combined results of the k CBR systems.

As an example let us consider a CBR system with an existing library of 105 past cases, where in no case there was a failure at the step of retrieval of a past case for the solution of the corresponding problem. In fact, let us assume that in 51 cases we had an intermediate success in retrieving a suitable past case, in 24 cases high, and in 30 cases we had a complete success respectively. Thus the state of retrieval is represented as a fuzzy set in U as

R_1={(a,0),(b,0),(c, 0,5),(d, 0,25),(e, 0,25)}.

In the same way we assume that we found

*R_2={(a,0),(b,0),(c,0,5),(d,0,25),(e,0)}*and

R_3={(a,0,25),(b,0,25),(c, 0,25),(d,0),(e,0)}.

Then we calculate the membership degrees of the 5^3 in total possible profiles (see column of m_s in Table 2). For example, for *s=(c, c, a)* one finds that $m_s = m_{A_1}(c).\ m_{A_2}(c).\ m_{A_3}(a=0,5.0,5.0,25=0,06225$. It turns out that (c, c, a) is one of the profiles of maximal membership degree and therefore the possibility of each s in U^3 is given by $r_s=\frac{m_s}{0,06225}$.

Calculating the possibilities of all profiles (see column of r_s in Table 2) one finds that the ordered possibility distribution is:

$r_1=r_2=1, r_3=r_4=r_5=r_6=r_7=r_8=0,5, r_9=r_{10}=r_{11}=r_{12}=r_{13}=r_{14}=0,258$, $r_{15}=r_{16}=\ldots\ldots.=r_{125}=0$. Thus using calculator we find that $ST(r)=\frac{1}{\log 2}[\sum_{i=2}^{14}(r_i-r_{i+1})\log\frac{i}{\sum_{j=1}^{i} r_j}] \approx$

$$\frac{1}{0,301}\ \log\frac{2}{2}+0,242\log\frac{8}{5}+0,258\log\frac{14}{6,548}] \approx$$

3,32(0,242.0,204+0,258.0,33) $\approx$ 0,445 and

$$N(r)= \frac{1}{\log 2}[\sum_{i=2}^{n}(r_i-r_{i+1})\log i\,]=$$

$$\frac{1}{\log 2}[0,5\log 2+0,242\log 8+0,258\log 14] \approx$$

0,5+3.0,242+0,857.1,146 $\approx$ 2,2. Therefore we finally find that T(r) $\approx$ 2,653.

Table 2: Profiles with non zero possibilities

R_1	R_2	R_3	m_s	r
c	c	c	0,062	1
c	c	a	0,062	1
d	d	a	0,016	0,258
d	d	b	0,016	0,258
d	d	c	0,016	0,258
d	c	a	0,031	0,5
d	c	b	0,031	0,5
d	c	c	0,031	0,5
e	c	a	0,031	0,5
e	c	b	0,031	0,5
e	c	c	0,031	0,5
e	d	a	0,016	0,258
e	d	b	0,016	0,258
e	d	c	0,016	0,258

(Note: The outcomes of Table 2 are with accuracy up to the third decimal point)

6. Discussion and Conclusions

The application research currently taking place in the field of fuzzy sets covers almost all sectors of human activities, such as natural, life and social sciences, engineering, medicine, management and decision making, operational research, computer science and systems' analysis, education, etc; e.g. see [9] ([11]; Chapter 6), [3], [8], [18], [21],etc.

Our fuzzy models provide useful quantitative information for the process of mathematical modelling in classroom and for a CBR system:

possibilities, value of T(r) etc. They also provide a qualitative view of behaviours' of student groups and CBR systems: profiles that they give, in terms of the linguistic labels, a comprehensive idea about the degree of students' success at the successive stages of the modelling process and of the effectiveness of a CBR system in solving new related problems.

All these enable the instructor to get a concentrating view of his (her) students' cognitive status, that helps him (her) to adapt properly teaching methods, plans and targets according to each particular class. They also help the manager of a CBR system to make the proper modifications in order to increase its efficiency.

There is a lot of work in the area of student modelling in general and student diagnosis in particular and our fuzzy models for the processes of mathematical modelling and CBR, combined with an analogous model presented in earlier papers for the process of learning a subject matter ([32], [37]) give a new approach for a deeper study of this area. Analogous efforts to use fuzzy logic in education have been attempted by other researchers as well; e.g. [2], [6], [16], [19], [25], [26] etc.

We must finally underline the importance of use of stochastic methods (Markov chain models) as an alternative approach for the same purposes; e.g. [27]-[31], [33], [35], [38] etc. Nevertheless Markov models, although easier sometimes to be applied in practice by a non expert (e.g. the teacher), they are self-restricted to provide *quantitative information* only for the situations that they represent, e.g. measures for the problem-solving, or model-building abilities of student groups, short and long-run forecasts (probabilities) for the evolution of various phenomena, etc. Therefore, one could claim that a fuzzy model, like those presented in this paper, is more useful for a deeper study of the corresponding real situation, because, apart from the quantitative information, it gives also the possibility of a qualitative analysis of the problems involved. In particular our fuzzy model for the modelling process has also the extra advantage of giving the opportunity for a combined study of the modelling performance of several student groups, or of the same group during the modelling process of different real problems. The same could happen also for a combined study of different CBR systems designed for the solution of the same type of problems

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Appendix

List of the problems given for solution to students in our classroom experiment

Problem 1: We want to construct a channel to run water by folding the two edges of an orthogonal metallic leaf having sides of length 20cm and 32 cm, in such a way that they will be perpendicular to the other parts of the leaf. Assuming that the flow of the water is constant, how we can run the maximum possible quantity of the water?

(*Remark:* The correct solution is obtained by folding the edges of the longer side of the leaf)

Problem 2: A car dealer has a mean annual demand of 250 cars, while he receives 30 new cars per month. The annual cost of storing a car is 100 euros and each time he makes a new order he pays an extra amount of 2200 euros for general expenses (transportation, insurance etc). The first cars of a new order arrive at the time when the last car of the previous order has been sold. How many cars must he order in order to achieve the minimum total cost?

Problem 3: An importation company codes the messages for the arrivals of its orders in terms of characters consisting of a combination of the binary elements 0 and 1. If it is known that the arrival of a certain order will take place from 1st until the 16th of March, find the minimal number of the binary elements of each character required for coding this message.

Problem 4: Let us correspond to each letter the number showing its order into the alphabet (A=1, B=2, C=3 etc). Let us correspond also to each word consisting of 4 letters a 2X2 matrix in the obvious way; e.g. the matrix $\begin{bmatrix} 19 & 15 \\ 13 & 5 \end{bmatrix}$ corresponds to the word SOME. Using the matrix E=$\begin{bmatrix} 8 & 5 \\ 11 & 7 \end{bmatrix}$ as an encoding matrix how you could send the message LATE in the form of a camouflaged matrix to a receiver knowing the above process and how he (she) could decode your message?

Problem 5: The demand function $P(Q_d)=25-Q_d^2$ represents the different prices that consumers willing to pay for different quantities Q_d of a good. On the other hand the supply function $P(Q_s)=2Q_s+1$ represents the prices at which different quantities Q_s of the same good will be supplied. If the market's equilibrium occurs at (Q_0, P_0) producers who would supply at lower price than P_0 benefit. Find the total gain to producers'.

Problem 6: A ballot box contains 8 balls numbered from 1 to 8. One makes 3 successive drawings of a lottery, putting back the corresponding ball to the box before the next lottery. Find the probability of getting all the balls that he draws out of the box different.

Problem 7: A box contains 3 white, 4 blue and 6 black balls. If we put out 2 balls, what is the probability of choosing 2 balls of the same colour?

Problem 8: The population of a country is increased proportionally. If the population is doubled in 50 years, in how many years it will be tripled?

Problem 9: A wine producer has a stock of wine greater than 500 and less than 750 kilos. He has calculated that, if he had the double quantity of wine and transferred it to bottles of 12, 25, or 40 kilos, it would be left over 6 kilos each time. Find the quantity of stock.

Problem 10: Among all cylindrical towers having a total surface of 180π m^2, which one has the maximal volume?

(*Remark*: Some students didn't include to the total surface the one base (ground-floor) and they found another solution, while some others didn't include both bases (roof and ground-floor) and they found no solution, since we cannot construct cylinder with maximal volume from its surrounding surface.)