Stability of a Complex Network of Euler-Bernoulli Beams

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Abstract: A complex network of Euler-Bernoulli beams is studied in this paper. As for this network, the boundary vertices are clamped, the displacements of the structure are continuous but the rotations of different beams are not continuous at the interior vertices. The feedback controller are designed at the interior nodes to stabilize the elastic system. The well-posed-ness of the closed loop system is proved by the semigroup theory. By complete spectral analysis of the system operator, the distribution of spectrum, the completeness and the Riesz basis property of the roots vectors of the system operator are given. As a consequence, the asymptotical stability of the system is derived under certain conditions.

Key–Words: Euler-Bernoulli beam, network, spectral analysis, stability

1 Introduction

Many authors have studied the networks of various flexible elements in recent years. For a multi-link flexible structures, one studies not only its global dynamic motion but also has to take into account the interaction and transmission of elastic elements. For examples, Dekoninck and Nicaise in [1] and [2] studied control and eigenvalue problems of network of Euler-Bernoulli beams; Ammari and Jellouli in [3] and [4] studied the stabilization problem of tree-shaped network of strings; Xu et al in [5] studied an abstract second order hyperbolic system and applied the result to controlled network of strings. Xu and Yung in [6] studied a star-shaped network of Euler-Bernoulli beams with boundary dampings. Xu and Mastorakis in [7] and [8] studied the stability and spectral distribution of a hybrid network of strings and Euler beams with boundary dampings. Guo and Wang made contributions to the stabilization of a tree-shaped network with three beams in [9], [10], and [11]. A general flexible network of beams was investigated by Mercier and Regnier in [12] and [13]. The more general differential equation on networks can refer to [14]. In the present paper, we shall study a complex network of Euler-Bernoulli beams which is different from those mentioned above. For the network, we at first design the nodal feedback controllers to stabilize the system, and then analyze the closed loop system. Here we shall carry out a complete spectral analysis for the closed loop system. It is well known that the calculation of the spectrum of a elastic network has been a tough work because of its complexity. In the present paper, we employ the asymptotic analysis technique to get spectral distribution, the completeness and Riesz basis property of the roots vectors of the system operator. Further,we derive the asymptotic stability of the system.

Let us introduce the network under consideration. Let G be a plane graph of the form as shown Fig. 1 in which the edge set $E = \{\gamma_1, \gamma_2, \cdots, \gamma_5\}$. The arc

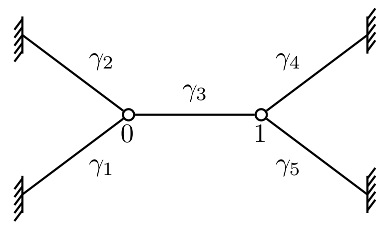

Figure 1: A network of Euler-Bernoulli beams

length of each edge is 1. We parameterize each edge by its arc length s as

$$\pi_j : [0, 1] \to \gamma_j, \quad x|_{\gamma_j} = \pi_j(s)$$

so that the graph becomes a metric graph.

Suppose that the equilibrium position of the elastic structure coincides with G. The elastic structure undergoes a small vibration, denote by $w(x, t)$ the displacement depart from its equilibrium position in position x at time t. On each edge $\gamma_j, j = 1, 2, \cdots, 5,$

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we parameterize it by $w_j(s,t) = w(\pi_j(s),t)$. Then it satisfies the differential equation

$$w_{j,tt}(s,t) + a_j w_{j,ssss}(s,t) = 0, \quad s \in (0,1).$$

Suppose that the boundary vertices of the structure are clamped, and at each interior vertex (i.e., common node) the elastic elements are connected through some conditions on $w_j(s,t)$ and its derivatives $w_{j,s^l}, l = 1,2,3,4$. More precisely, we simply impose the continuity of $w_j(s,t)$ and $w_{j,s}(s,t)$ (representing the rotation of the beam γ_j) is not continuous at the common nodes. There are two exterior forces at the common node $\cap_{j=1}^3 \gamma_j$, which is parameterized as 0; At the common node $\cap_{j=3}^5 \gamma_j$, here we parameterize it as 1, there is only one extra force.

By the action of the extra forces, the motion of the elastic structure are governed by the partial deferential equations

$$\begin{cases} w_{j,tt}(s,t) + a_j w_{j,ssss}(s,t) = 0,\ j = 1,2,\cdots,5 \\ w_1(0,t) = w_2(0,t) = w_3(0,t) \\ \sum\limits_{i=1}^{3} a_i w_{i,sss}(0,t) = 0 \\ w_3(1,t) = w_4(1,t) = w_5(1,t) \\ \sum\limits_{i=3}^{5} a_i w_{i,sss}(1,t) = 0 \\ w_{1,ss}(0,t) = u_0(t),\ w_1(1,t) = w_{1,s}(1,t) = 0 \\ w_{2,ss}(0,t) = w_2(1,t) = w_{2,s}(1,t) = 0 \\ w_{3,ss}(0,t) = u_1(t),\ w_{3,ss}(1,t) = u_2(t) \\ w_{4,ss}(1,t) = w_4(0,t) = w_{4,s}(0,t) = 0 \\ w_{5,ss}(1,t) = w_5(0,t) = w_{5,s}(0,t) = 0 \end{cases} \tag{1}$$

Obviously, this is a complex network of Euler-Bernoulli beams. Our purpose in the present paper is to design the feedback controllers and to stabilize the system.

The rest is organized as follows. In next section we design the feedback controllers and then discuss the well-posed-ness of the closed loop system. In section 3, we carry out a complete asymptotic analysis for the spectrum of operator determined by the closed loop system. we get the spectral distribution, and then prove the completeness and the Riesz basis property of root vectors (the eigenvectors and generalized eigenvectors) of the system operator. Finally, in section 4, we discuss the stability of the system. Under the certain conditions we derive the asymptotic stability of the system.

2 Design of Controllers and Well-posed-ness

In this section, we shall design the feedback controllers for the system (1) and then study the well-posed-ness of the closed loop system.

Let us consider the energy function of (1), which is given by

$$E(t) = \frac{1}{2}\sum_{j=1}^{5}\int_0^1 [a_j w_{j,ss}^2(s,t) + w_{j,t}^2(s,t)]ds$$

Differentiate it formally with respect to time t, we have

$$\begin{aligned} \frac{dE(t)}{dt} &= -a_1 w_{1,ss}(0,t) w_{1,st}(0,t) \\ &\quad -a_3 w_{3,ss}(0,t) w_{3,st}(0,t) + a_3 w_{3,ss}(1,t) w_{3,st}(1,t) \\ &= -a_1 u_0(t) w_{1,st}(0,t) - a_3 u_1(t) w_{3,st}(0,t) \\ &\quad + a_3 u_2(t) w_{3,st}(1,t). \end{aligned}$$

So designing controllers as

$$\begin{aligned} &u_0(t) = w_{1,st}(0,t),\ u_1(t) = w_{3,st}(0,t), \\ &u_2(t) = -w_{3,st}(1,t) \end{aligned}$$

yields $\dfrac{dE(t)}{dt} \le 0$.

With these feedback controllers, the model (1) becomes a closed-loop system

$$\begin{cases} w_{j,tt}(s,t) + a_j w_{j,ssss}(s,t) = 0, j = 1,2,\cdots,5 \\ w_1(0,t) = w_2(0,t) = w_3(0,t) \\ \sum\limits_{i=1}^{3} a_i w_{i,sss}(0,t) = 0 \\ w_3(1,t) = w_4(1,t) = w_5(1,t) \\ \sum\limits_{i=3}^{5} a_i w_{i,sss}(1,t) = 0 \\ w_{1,ss}(0,t) = w_{1,st}(0,t), \\ w_1(1,t) = w_{1,s}(1,t) = 0 \\ w_{2,ss}(0,t) = w_2(1,t) = w_{2,s}(1,t) = 0 \\ w_{3,ss}(0,t) = w_{3,st}(0,t), \\ w_{3,ss}(1,t) = -w_{3,st}(1,t) \\ w_{4,ss}(1,t) = w_4(0,t) = w_{4,s}(0,t) = 0 \\ w_{5,ss}(1,t) = w_5(0,t) = w_{5,s}(0,t) = 0 \end{cases} \tag{2}$$

Now we formulate the system (2) into an appropriate Hilbert state space.

Let $L^2(G)$ and $C(G)$ be defined as usual, we define linear space $H^k(G_a), k \in \mathbb{N}$, by

$$H^k(G_a) = \left\{ f \in L^2(G) \;\middle|\; f|_{\gamma_j} = f_j \in H^k(0,1) \right\}$$

A function $w(x,t)$ is said to be a solution to (2), it means that, for each $t > 0$, $w(x,t) \in H^4(G_a)$ and $w(x,t)$ is continuously differentiable with respect to t and $w_{tt}(x,t)$ exists in $L^2(G)$.

Denote the space $\mathcal{W}$ by $\left(H^2(0,1) \times L^2[0,1]\right)^5$ and let the state space be

$$\mathcal{H} = \left\{ (f,g) \in \mathcal{W} \;\middle|\; \begin{array}{l} f_1(0) = f_2(0) = f_3(0), \\ f_1(1) = f_2(1) = 0, \\ f_3(1) = f_4(1) = f_5(1), \\ f_4(0) = f_5(0) = 0, \\ f_{1,s}(1) = f_{2,s}(1) = 0, \\ f_{4,s}(0) = f_{5,s}(0) = 0 \end{array} \right\}$$

equipped with inner product

$$\begin{aligned}&\langle (f,g),(u,v)\rangle_{\mathcal{H}}\\&=\sum_{j=1}^{5}\int_0^1[a_jf_{j,ss}(s)\overline{u_{j,ss}(s)}+g_j(s)\overline{v_j(s)}]ds\end{aligned}$$

Obviously, $\mathcal{H}$ is a Hilbert space.

Denote the space $\mathcal{V}$ by $(H^4(0,1)\times H^2(0,1))^5$ and define the operator $\mathcal{A}$ in $\mathcal{H}$ by

$$\mathcal{D}(\mathcal{A})=\left\{\begin{array}{l}(w,z)\in\mathcal{V},\ \sum\limits_{l=1}^{3}a_lw_{l,sss}(0)=0,\\ w_{1,ss}(0)=z_{1,s}(0),\ w_{3,ss}(0)=z_{3,s}(0)\\ w_{2,ss}(0)=0,\ \sum\limits_{l=3}^{5}a_lw_{l,sss}(1)=0,\\ w_{3,ss}(1)=-z_{3,s}(1),\\ w_{4,ss}(1)=w_{5,ss}(1)=0\end{array}\right\}$$

$\mathcal{A}(w,z)=\{(z_j,-a_jw_{j,ssss})\}\in\mathcal{H},\forall(w,z)\in\mathcal{D}(\mathcal{A})$.

With the help of these notations we can rewrite (2) into an evolutionary equation in $\mathcal{H}$

$$\left\{\begin{array}{l}\dfrac{dW(t)}{dt}=\mathcal{A}W(t),\ t>0\\ W(0)=W_0\in\mathcal{H}.\end{array}\right.\tag{3}$$

where $W(t)=\{(w_j(s,t),w_{j,t}(s,t)\}$, $W_0\in\mathcal{H}$ is the initial data given.

In what follows, we shall discuss the well-poseness of the system (3). Firstly, we have the following result.

Theorem 1 *Let $\mathcal{A}$ be defined as above. Then $\mathcal{A}$ is dissipative, $0\in\rho(\mathcal{A})$ and $\mathcal{A}^{-1}$ is compact. Hence the spectrum of $\mathcal{A}$ consists of all isolated eigenvalues of finite multiplicity.*

Proof: For any real $(w,z)\in\mathcal{D}(\mathcal{A})$, we have

$$\begin{aligned}&\langle\mathcal{A}(w,z),(w,z)\rangle_{\mathcal{H}}\\&=\sum_{j=1}^{5}\int_0^1\big[a_jz_{j,ss}(s)w_{j,ss}(s)-a_jw_{j,ssss}(s)z_j(s)\big]ds\\&=\sum_{j=1}^{3}a_jw_{j,sss}(0)z_j(0)-a_1w_{1,ss}(0)z_{1,s}(0)\\&\quad-a_3w_{3,ss}(0)z_{3,s}(0)-\sum_{j=3}^{5}a_jw_{j,sss}(1)z_j(1)\\&\quad+a_3w_{3,ss}(1)z_{3,s}(1)\\&=-a_1z_{1,s}^2(0)-a_3z_{3,s}^2(0)-a_3z_{3,s}^2(1).\end{aligned}$$

The dissipatedness of $\mathcal{A}$ follows.

For any $(f,g)\in\mathcal{H}$, we consider the resolvent equation $\mathcal{A}(w,z)=(f,g)$, i.e.,

$$\begin{array}{l}\left\{\begin{array}{l}z_1(s)=f_1(s)\\a_1w_{1,ssss}(s)=-g_1(s)\\w_1(0)=w_2(0)\\w_{1,ss}(0)=z_{1,s}(0)\\w_1(1)=0\\w_{1,s}(1)=0\end{array}\right.\quad\left\{\begin{array}{l}z_2(s)=f_2(s)\\a_2w_{2,ssss}(s)=-g_2(s)\\w_2(0)=w_3(0)\\w_{2,ss}(0)=0\\w_2(1)=0\\w_{2,s}(1)=0\end{array}\right.\\ \left\{\begin{array}{l}z_3(s)=f_3(s)\\a_3w_{3,ssss}(s)=-g_3(s)\\\sum\limits_{j=1}^{3}a_jw_{j,sss}(0)=0\\w_{3,ss}(0)=z_{3,s}(0)\\\sum\limits_{j=3}^{5}a_jw_{j,sss}(1)=0\\w_{3,ss}(1)=-z_{3,s}(1)\end{array}\right.\\ \left\{\begin{array}{l}z_4(s)=f_4(s)\\a_4w_{4,ssss}(s)=-g_4(s)\\w_4(1)=w_3(1)\\w_{4,ss}(1)=0\\w_4(0)=0\\w_{4,s}(0)=0\end{array}\right.\quad\left\{\begin{array}{l}z_5(s)=f_5(s)\\a_5w_{5,ssss}(s)=-g_5(s)\\w_5(1)=w_4(1)\\w_{5,ss}(1)=0\\w_5(0)=0\\w_{5,s}(0)=0.\end{array}\right.\end{array}\tag{4}$$

Solving the differential equation in (4) yields

$$\begin{aligned}w_j(s)=\ &w_j(0)+sw_{j,s}(0)+\frac{s^2}{2}w_{j,ss}(0)\\&+\tfrac{s^3}{3!}w_{j,sss}(0)-\int_0^s\tfrac{(s-r)^3}{3!a_j}g_j(r)dr\end{aligned}\tag{5}$$

and

$$\begin{aligned}&w_{j,s}(s)=w_{j,s}(0)+sw_{j,ss}(0)+\tfrac{s^2}{2}w_{j,sss}(0)\\&\qquad-\int_0^s\tfrac{(s-r)^2}{2!a_j}g_j(r)dr\\&w_{j,ss}(s)=w_{j,ss}(0)+sw_{j,sss}(0)-\int_0^s\tfrac{s-r}{a_j}g_j(r)dr\\&w_{j,sss}(s)=w_{j,sss}(0)-\int_0^s\tfrac{g_j(r)}{a_j}dr.\end{aligned}$$

Thus

$$\begin{aligned}&w_j(1)=w_j(0)+w_{j,s}(0)+\tfrac12w_{j,ss}(0)\\&\qquad+\tfrac16w_{j,sss}(0)-\int_0^1\tfrac{(1-r)^3}{3!a_j}g_j(r)dr\\&w_{j,s}(1)=w_{j,s}(0)+w_{j,ss}(0)+\tfrac12w_{j,sss}(0)\\&\qquad-\int_0^1\tfrac{(1-r)^2}{2!a_j}g_j(r)dr\\&w_{j,ss}(1)=w_{j,ss}(0)+w_{j,sss}(0)-\int_0^1\tfrac{1-r}{a_j}g_j(r)dr\\&w_{j,sss}(1)=w_{j,sss}(0)-\int_0^1\tfrac{g_j(r)}{a_j}dr\end{aligned}$$

Substituting above into the boundary conditions in (4), we can determine the $w_j(0),w_{j,s}(0),w_{j,ss}(0)$ and $w_{j,sss}(0)$. Substituting them into (5), we can get unique functions $w_j(s),j=1,2,\cdots,5$. Set $z=f$, then we have $(w,z)\in\mathcal{D}(\mathcal{A})$ and $\mathcal{A}(w,z)=(f,g)$. The Inverse Operator Theorem asserts $0\in\rho(\mathcal{A})$. Note that $\mathcal{D}(\mathcal{A})\subset H^2(G_a)\times H^4(G_a)$. The Sobolev's embedding Theorem ensures that $\mathcal{A}^{-1}$ is compact on $\mathcal{H}$. □

As a direct consequence of Theorem 1 and Lumer-Phillips Theorem (e.g. see,[19]), we have the following result.

Corollary 2 *Let $\mathcal{A}$ be defined as above. Then the system (3) is well-posed in $\mathcal{H}$.*

3 Spectral Analysis

In order to investigate the properties of the semigroup $T(t)$ generated by $\mathcal{A}$, we need to find out some spectral properties of $\mathcal{A}$. From Theorem 1, we know that $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$. In this section, we shall discuss the eigenvalue problem of $\mathcal{A}$ and give its asymptotical distribution.

Let $\lambda \in \mathbb{C}$, we consider the solvability of nonzero solution of the equation $\mathcal{A}(w,z) = \lambda(w,z)$ in $\mathcal{H}$. For the sake of simplicity, we assume $a_i = 1$, $i = 1,2,\cdots,5$. From the representation of $\mathcal{A}$, we know that $\mathcal{A}(w,z) = \lambda(w,z)$ implies that $z = \lambda w$ and w satisfy the differential equations

$$\begin{cases} w_{j,ssss}(s) = -\lambda^2 w_j(s),\ j = 1,2,3,4,5 \\ w_1(0) = w_2(0) = w_3(0) \\ \sum\limits_{i=1}^{3} w_{i,sss}(0) = 0 \\ w_3(1) = w_4(1) = w_5(1) \\ \sum\limits_{i=3}^{5} w_{i,sss}(1) = 0 \\ w_{1,ss}(0) = \lambda w_{1,s}(0),\ w_1(1) = w_{1,s}(1) = 0 \\ w_{2,ss}(0) = w_2(1) = w_{2,s}(1) = 0 \\ w_{3,ss}(0) = \lambda w_{3,s}(0),\ w_{3,ss}(1) = -\lambda w_{3,s}(1) \\ w_{4,ss}(1) = w_4(0) = w_{4,s}(0) = 0 \\ w_{5,ss}(1) = w_5(0) = w_{5,s}(0) = 0 \end{cases} \tag{6}$$

In order to discuss the existence of nonzero solution of (6) and distribution of the eigenvalues of $\mathcal{A}$, we divide the discussion into the following several subsections.

3.1 Fundamental matrix and distribution of eigenvalues

In this subsection we discuss existence of nonzero solution of (6) and distribution of the eigenvalues of $\mathcal{A}$. To this end, we formulate equations (6) into the normalized boundary value problem. Let

$$\begin{cases} W_j = [w_j, w_{j,s}, w_{j,ss}, w_{j,sss}]^T,\ j = 1,2,\ldots,5 \\ W(\cdot) = [W_1, W_2, W_3, W_4, W_5]^T \end{cases} \tag{7}$$

Then (6) can be written as

$$\begin{cases} \dfrac{dW(s)}{ds} = A(\lambda)W(s) \\ B^0(\lambda)W(0) + B^1(\lambda)W(1) = 0 \end{cases} \tag{8}$$

where

$$A(\lambda) = \begin{bmatrix} A_1(\lambda) & 0 & 0 & 0 & 0 \\ 0 & A_2(\lambda) & 0 & 0 & 0 \\ 0 & 0 & A_3(\lambda) & 0 & 0 \\ 0 & 0 & 0 & A_4(\lambda) & 0 \\ 0 & 0 & 0 & 0 & A_5(\lambda) \end{bmatrix}$$

$$B^0(\lambda) = \begin{bmatrix} B_1^0(\lambda) & 0 & C_1 & 0 & 0 \\ 0 & B_2^0(\lambda) & C_1 & 0 & 0 \\ D_1 & D_2 & B_3^0(\lambda) & 0 & 0 \\ 0 & 0 & 0 & B_4^0(\lambda) & 0 \\ 0 & 0 & 0 & 0 & B_5^0(\lambda) \end{bmatrix}$$

$$B^1(\lambda) = \begin{bmatrix} B_1^1(\lambda) & 0 & 0 & 0 & 0 \\ 0 & B_2^1(\lambda) & 0 & 0 & 0 \\ 0 & 0 & B_3^1(\lambda) & D_4 & D_5 \\ 0 & 0 & C_2 & B_4^1(\lambda) & 0 \\ 0 & 0 & C_2 & 0 & B_5^1(\lambda) \end{bmatrix}$$

with

$$A_j(\lambda) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\lambda^2 & 0 & 0 & 0 \end{bmatrix},\ j = 1,2,\cdots,5$$

and

$$B_1^0(\lambda) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ B_1^1(\lambda) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$B_2^0(\lambda) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ B_2^1(\lambda) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$B_3^0(\lambda) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ B_3^1(\lambda) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \lambda & 1 & 0 \end{bmatrix}$$

$$B_4^0(\lambda) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ B_4^1(\lambda) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B_5^0(\lambda) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ B_5^1(\lambda) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ D_i = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ i = 1,2$$

$$C_2 = \begin{bmatrix} 0\,0\,0\,0 \\ 0\,0\,0\,0 \\ -1\,0\,0\,0 \\ 0\,0\,0\,0 \end{bmatrix},\ D_j = \begin{bmatrix} 0\,0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,1 \\ 0\,0\,0\,0 \end{bmatrix},\ j = 4, 5$$

Lemma 3 *Let $\mathcal{A}$ be defined as before, then the spectrum of $\mathcal{A}$ distributes symmetrically with respect to the real axis. Moreover $\lambda \in \sigma(\mathcal{A})$ if and only if there is a non-trivial solution $W(s)$ to (8).*

Thanks to Lemma 3, we consider only λ that are located in the first and second quadrant of the complex plane. Set $\lambda = i\rho^2$, $\rho \in \mathbb{C}$. If $0 \leqslant \arg\lambda < \frac{\pi}{2}$, choosing $\frac{3\pi}{4} \leqslant \arg\rho < \pi$, then we have

$$\Re(\rho) < 0,\ \Re(-\rho) > 0,\ \Re(i\rho) < 0,\ \Re(-i\rho) > 0 \quad (9)$$

and if $\frac{\pi}{2} \leqslant \arg\lambda < \pi$, taking $0 \leqslant \arg\rho < \frac{\pi}{4}$, we have

$$\Re(\rho) > 0,\ \Re(-\rho) < 0,\ \Re(i\rho) \leqslant 0,\ \Re(-i\rho) \geqslant 0 \quad (10)$$

Define invertible matrices by

$$P(\rho) = \begin{bmatrix} P_1(\rho) & 0 & 0 & 0 & 0 \\ 0 & P_2(\rho) & 0 & 0 & 0 \\ 0 & 0 & P_3(\rho) & 0 & 0 \\ 0 & 0 & 0 & P_4(\rho) & 0 \\ 0 & 0 & 0 & 0 & P_5(\rho) \end{bmatrix}$$

where

$$P_j(\rho) = \begin{bmatrix} \rho & \rho & \rho & \rho \\ \rho^2 & -\rho^2 & i\rho^2 & -i\rho^2 \\ \rho^3 & \rho^3 & -\rho^3 & -\rho^3 \\ \rho^4 & -\rho^4 & -i\rho^4 & i\rho^4 \end{bmatrix},\ j = 1, 2, \cdots, 5.$$

Then, we have

$$P_j^{-1}(\rho) A_j(i\rho^2) P_j(\rho) = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & -\rho & 0 & 0 \\ 0 & 0 & i\rho & 0 \\ 0 & 0 & 0 & -i\rho \end{bmatrix}.$$

Lemma 4 *Let $P(\rho)$ be defined as above, $P(\rho)E(s)$ is a fundamental solution matrix to system (8), where*

$$E(s) = \begin{bmatrix} E_1(s) & 0 & 0 & 0 & 0 \\ 0 & E_2(s) & 0 & 0 & 0 \\ 0 & 0 & E_3(s) & 0 & 0 \\ 0 & 0 & 0 & E_4(s) & 0 \\ 0 & 0 & 0 & 0 & E_5(s) \end{bmatrix} \quad (11)$$

with

$$E_j(s) = \begin{bmatrix} e^{\rho s} & 0 & 0 & 0 \\ 0 & e^{-\rho s} & 0 & 0 \\ 0 & 0 & e^{i\rho s} & 0 \\ 0 & 0 & 0 & e^{-i\rho s} \end{bmatrix},\ j = 1, 2, \cdots, 5$$

Proof: The inverse matrix of $P(\rho)$ is given by

$$P^{-1}(\rho) = \begin{bmatrix} P_1^{-1}(\rho) & 0 & 0 & 0 & 0 \\ 0 & P_2^{-1}(\rho) & 0 & 0 & 0 \\ 0 & 0 & P_3^{-1}(\rho) & 0 & 0 \\ 0 & 0 & 0 & P_4^{-1}(\rho) & 0 \\ 0 & 0 & 0 & 0 & P_5^{-1}(\rho) \end{bmatrix}$$

Set

$$\hat{W}(s) = P^{-1}(\rho) W(s) \quad (12)$$

and $\hat{A}(i\rho^2) = P^{-1}(\rho)A(i\rho^2)P^{-1}(\rho)$, then we have

$$\frac{d\hat{W}(s)}{ds} = \hat{A}(i\rho^2)\hat{W}(s). \quad (13)$$

Since

$$\hat{A}(i\rho^2) = P^{-1}(\rho)A(i\rho^2)P(\rho) = \begin{bmatrix} \hat{A}_1(i\rho^2) & 0 & 0 & 0 & 0 \\ 0 & \hat{A}_2(i\rho^2) & 0 & 0 & 0 \\ 0 & 0 & \hat{A}_3(i\rho^2) & 0 & 0 \\ 0 & 0 & 0 & \hat{A}_4(i\rho^2) & 0 \\ 0 & 0 & 0 & 0 & \hat{A}_5(i\rho^2) \end{bmatrix}$$

with

$$\hat{A}_j(i\rho^2) = P_j^{-1}(\rho)A(i\rho^2)P_j(\rho), j = 1, 2, \cdots .5$$

Obviously, $\hat{A}(i\rho^2)$ is a diagonal matrix. Hence, the fundamental matrix to (13) can be written as $E(s)$. And by (12), we can assert that $P(\rho)E(s)$ is a fundamental matrix to system (8). □

Set

$$\Omega(\rho) = B^0(i\rho^2)P(\rho)E(0) + B^1(i\rho^2)P(\rho)E(1) \quad (14)$$

Before calculating the determinant of $\Omega(\rho)$, we put forward some decompositions of $E(1)$ and $E(0)$ as

$$E(0) = E^0 \cdot E^2, \quad E(1) = E^1 \cdot E^2$$

where E^0, E^1, E^2 are diagonal matrices similar to $E(s)$ with their elements E_j^0, E_j^1, E_j^2, $j = 1, 2, \cdots, 5$, they will be defined according to the sign of $\Re(\lambda)$, respectively.

When $\Re(\lambda) > 0$, according to (9), we have

$$|e^{\rho}| \to 0, |e^{i\rho}| \to 0 \quad \text{as } \Re(\lambda) \to +\infty.$$

the entries $E_j^i, i = 0,1,2, j = 1,2,\cdots,5$ are defined as

$$E_j^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-\rho} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-i\rho} \end{bmatrix},$$

$$E_j^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\rho} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\rho} \end{bmatrix},$$

$$E_j^1 = \begin{bmatrix} e^{\rho} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\rho} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad j = 1,2,\cdots,5.$$

When $\Re(\lambda) < 0$, we have

$$|e^{-\rho}| \to 0, |e^{i\rho}| \to 0 \quad \text{as } \Re(\lambda) \to -\infty.$$

we define

$$E_j^2 = \begin{bmatrix} e^{\rho} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-i\rho} \end{bmatrix},$$

$$E_j^0 = \begin{bmatrix} e^{-\rho} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\rho} \end{bmatrix},$$

$$E_j^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-\rho} & 0 & 0 \\ 0 & 0 & e^{i\rho} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad j = 1,2,\cdots,5.$$

Thus we get

$$\Omega(\rho) = [B^0(i\rho^2)P(\rho)E^0 + B^1(i\rho^2)P(\rho)E^1]E^2 \quad (15)$$

Obviously,(8) has a non-zero solution if and only if $\det \Omega(\rho) = 0$.

Now we set

$$\begin{aligned} \Delta(\rho) &= \det[\Omega(\rho)]/\det[E^2] \\ &= \det[B^0(i\rho^2)P(\rho)E^0 + B^1(i\rho^2)P(\rho)E^1]. \end{aligned}$$

In order to calculate $\Delta(\rho)$, we need some transformations. Let

$$\hat{B}^0(i\rho^2) = \begin{bmatrix} B^0(i\rho^2) & 0 \\ 0 & I^0 \end{bmatrix}, \hat{B}^1(i\rho^2) = \begin{bmatrix} B^1(i\rho^2) & 0 \\ 0 & I^1 \end{bmatrix}$$
$$\hat{E}^0 = \begin{bmatrix} E^0 & 0 \\ 0 & E_3^0 \end{bmatrix}, \qquad \hat{E}^1 = \begin{bmatrix} E^1 & 0 \\ 0 & E_3^1 \end{bmatrix}$$
$$\hat{P}(\rho) = \begin{bmatrix} P(\rho) & 0 \\ 0 & P_3(\rho) \end{bmatrix}$$

and

$$\hat{\Omega}(\rho) = \hat{B}^0(i\rho^2)\hat{P}(\rho)\hat{E}^0 + \hat{B}^1(i\rho^2)\hat{P}(\rho)\hat{E}^1 \quad (16)$$

where

$$I^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, I^1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Set $\hat{\Delta}(\rho) = \det[\hat{\Omega}(\rho)]$, obviously

$$\begin{aligned} \Delta(\rho) &= \hat{\Delta}(\rho)/\det[I^0 P_3(\rho)E_3^0 + I^1 P_3(\rho)E_3^1] \\ &= \hat{\Delta}(\rho)/[2i\rho^{10} + O(\rho^9)]. \end{aligned}$$

After some elementary transformation to $\hat{\Omega}(\rho)$, we get

$$\hat{\Delta}(\rho) = \begin{vmatrix} M_1 & 0 & 0 & 0 & 0 & L_1 \\ 0 & M_2 & 0 & 0 & 0 & L_2 \\ 0 & 0 & M_3 & 0 & 0 & L_3 \\ 0 & 0 & 0 & M_4 & 0 & L_4 \\ 0 & 0 & 0 & 0 & M_5 & L_5 \\ N_1 & N_2 & N_3 & N_4 & N_5 & Q \end{vmatrix}$$

where

$$M_j = B_j^0(i\rho^2)P_j(\rho)E_j^0 + B_j^1(i\rho^2)P_j(\rho)E_j^1,\ j = 1,2,4,5$$

$$M_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -i\rho^2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} P_3(\rho)E_3^0 + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & i\rho^2 & 1 & 0 \end{bmatrix} P_3(\rho)E_3^1$$

$$N_j = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} P_j(\rho)E_j^0 + 0 \cdot P_j(\rho)E_j^1, j = 1,2$$

$$N_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} P_3(\rho)E_3^0 + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} P_3(\rho)E_3^1$$

$$N_j = 0 \cdot P_j(\rho)E_j^0 + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} P_j(\rho)E_j^1, j = 4,5$$

$$L_j = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} P_3(\rho)E_3^0 + 0\cdot P_3(\rho)E_3^1,\ j=1,2$$

$$L_3 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} P_3(\rho)E_3^0 + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} P_3(\rho)E_3^1$$

$$L_j = 0\cdot P_3(\rho)E_3^0 + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} P_3(\rho)E_3^1,\ j=4,5$$

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} P_3(\rho)E_3^0 + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} P_3(\rho)E_3^1.$$

Furthermore,

$$\hat{\Delta}(\rho) = \prod_{j=1}^{5}\det(M_j)\hat{\Delta}(\rho)\begin{vmatrix} M_1^{-1} & 0 & \cdots & 0 & 0 \\ 0 & M_2^{-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & M_5^{-1} & 0 \\ 0 & 0 & \cdots & 0 & I \end{vmatrix}$$
$$= \prod_{j=1}^{5}\det(M_j)\begin{vmatrix} I & 0 & \cdots & 0 & L_1 \\ 0 & I & \cdots & 0 & L_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots 0 & I & L_5 \\ N_1M_1^{-1} & N_2M_2^{-1} & \cdots & N_5M_5^{-1} & Q \end{vmatrix}$$
$$= \prod_{j=1}^{5}\det(M_j)\begin{vmatrix} I & 0 & \cdots & 0 & L_1 \\ 0 & I & \cdots & 0 & L_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & L_5 \\ 0 & 0 & \cdots & 0 & Q-\sum_{j=1}^{5}N_jM_j^{-1}L_j \end{vmatrix}$$
$$= \prod_{j=1}^{5}\det(M_j)\cdot\det[Q-\sum_{j=1}^{5}N_jM_j^{-1}L_j]$$

A direct computation shows

$$\begin{aligned}\hat{\Delta}(\rho) &= \prod_{j=1}^{5}\det(M_j)\cdot\det[Q-\sum_{j=1}^{5}N_jM_j^{-1}L_j] \\ &= \begin{cases} -1280(1-i)\rho^{55}+O(\rho^{54}), & \Re(\lambda)>0, \\ -256(1+i)\rho^{55}+O(\rho^{54}), & \Re(\lambda)<0. \end{cases}\end{aligned}$$

So

$$\begin{aligned}\Delta(\rho) &= \hat{\Delta}(\rho)/[2i\rho^{10}+O(\rho^9)] \\ &= \begin{cases} 640(1+i)\rho^{45}+O(\rho^{44}), & \Re(\lambda)>0, \\ -128(1-i)\rho^{45}+O(\rho^{44}), & \Re(\lambda)<0. \end{cases}\end{aligned}$$

Thus

$$\det[\Omega(\rho)] = \Delta(\rho)\cdot\det(E^2) = \Delta(\rho)\cdot\prod_{j=1}^{5}\det(E_j^2)$$
$$= \begin{cases} [640(1+i)\rho^{45}+O(\rho^{44})]e^{-5(1+i)\rho}, & \Re(\lambda)>0, \\ [-128(1-i)\rho^{45}+O(\rho^{44})]e^{5(1-i)\rho}, & \Re(\lambda)<0. \end{cases} \quad (17)$$

Therefore, we have

$$\lim_{\Re(\lambda)\to+\infty}\frac{\det[\Omega(\rho)]}{\rho^{45}\cdot e^{-5(1+i)\rho}} = 640(1+i)\neq 0 \quad (18)$$

and

$$\lim_{\Re(\lambda)\to-\infty}\frac{\det[\Omega(\rho)]}{\rho^{45}\cdot e^{5(1-i)\rho}} = -128(1-i)\neq 0. \quad (19)$$

Hence, there are positive constants c_1, c_2 and h such that

$$c_1|\rho^{45}e^{-5(i\pm1)\rho}| \leqslant |\det[\Omega(\rho)]| \leqslant c_2|\rho^{45}e^{-5(i\pm1)\rho}| \quad (20)$$

where $\pm = \mathrm{sign}\Re(\lambda)$, $\quad |\Re(\lambda)|\geqslant h$.

Eqs.(20) shows that $|\det[\Omega(\rho)]|$ is a sine type function on $\mathbb{C}$ (see [16, Definition II, 1.27, pp61]). Levin theorem (see [16, Proposition II, 1.28]) asserts that the set of zeros of $\det[\Omega(\rho)]$ is a union of finitely many separable sets. From the above analysis, we have the following result.

Theorem 5 *Let $\mathcal{A}$ be defined as before, then the spectrum of $\mathcal{A}$ distributes in a strip parallel to the imaginary, that is, there is a positive constant h such that*

$$\sigma(\mathcal{A}) \subset \{\lambda\in\mathbb{C} \big| -h\leqslant\Re(\lambda)\leqslant 0\}$$

In particular, $\sigma(\mathcal{A})$ is an union of finite many separated sets.

Proof: Under the assumption, we always have inequality (20). This together with dissipatedness of $\mathcal{A}$ asserts that the spectrum of $\mathcal{A}$ distributes in a strip parallel to the imaginary axis. In particular, the Levin's Theorem says that the set of zero of $\det[\Omega(\rho)]$ is an union of finite separated sets. □

3.2 Completeness and basis property of eigenvectors

In this subsection, we shall discuss the completeness and Riesz basis property of (generalized) eigenvectors of $\mathcal{A}$, Firstly, we establish the completeness of (generalized) eigenvectors of operator $\mathcal{A}$ and then use the spectral distribution of $\mathcal{A}$ to obtain the Riesz property. We need the following lemma

Lemma 6 *Let $\mathcal{A}$ be the generator of a C_0-semigroup in a Hilbert space $\mathcal{H}$. Assume that $\mathcal{A}$ is discrete and for $\lambda\in\rho(\mathcal{A}^*)$, $R(\lambda,\mathcal{A}^*)$ is of the form*

$$R(\lambda,\mathcal{A}^*)x = \frac{G(\lambda)x}{F(\lambda)},\ \forall x\in\mathcal{H}$$

where $G(\lambda)x$ for each $x \in \mathcal{H}$ is a H-valued entire function with order less than or equal to ρ_1 and $F(\lambda)$ is a scalar entire function of order ρ_2. Let $\rho = max\{\rho_1, \rho_2\} < \infty$ and an integer n so that $n-1 \leqslant \rho < n$. If there are $n+1$ rays $\gamma_j, j = 0, 1, \ldots, n$, on the complex plane

$$arg\gamma_0 = \frac{\pi}{2} < arg\gamma_1 \leqslant arg\gamma_2 \leqslant \ldots \leqslant arg\gamma_n = \frac{3\pi}{2}$$

with $\arg\gamma_{j+1} - arg\gamma_j \leqslant \frac{\pi}{n}$, $0 \leqslant j \leqslant n-1$ so that $R(\lambda, \mathcal{A}^*)x$ is bounded on each ray $arg\gamma_j$, $0 \leqslant j \leqslant n$ as $|\lambda| \to \infty$ for any $x \in \mathcal{H}$, then $Sp(\mathcal{A}) = Sp(\mathcal{A}^*) = \mathcal{H}$ where $Sp(\mathcal{A})$ be the closed subspace spanned by all generalized eigenvectors of $\mathcal{A}$.

Now we can prove the completeness of generalized eigenvectors of $\mathcal{A}$.

Theorem 7 *Let $\mathcal{A}$ be defined as before. Then the generalized eigenvectors of $\mathcal{A}$ is complete in $\mathcal{H}$, i.e, $Sp(\mathcal{A}) = \mathcal{H}$.*

Proof: we prove the assertion by the following four steps:
Step 1. The adjoint operator of $\mathcal{A}$, $\mathcal{A}^*$, has the form:

$$\mathcal{A}^*(w,z) = -\{(z_j, -a_j w_{j,ssss})\}, \forall (w,z) \in \mathcal{D}(\mathcal{A}^*)$$

$$\mathcal{D}(\mathcal{A}^*) = \left\{\begin{array}{l} (w,z) \in \mathcal{V},\ \sum\limits_{l=1}^{3} a_l w_{l,sss}(0) = 0, \\ w_{1,ss}(0) = -z_{1,s}(0), w_{2,ss}(0) = 0, \\ w_{3,ss}(0) = -z_{3,s}(0), \\ \sum\limits_{l=3}^{5} a_l w_{l,sss}(1) = 0, \\ w_{3,ss}(1) = z_{3,s}(1), \\ w_{4,ss}(1) = w_{5,ss}(1) = 0 \end{array}\right\}. \quad (21)$$

This is a direct verification, we omit the detail. Due to $\sigma(\mathcal{A}^*) = \overline{\sigma(\mathcal{A})}$ and the symmetry of $\sigma(\mathcal{A})$ with respect to the real axis, we have $\sigma(\mathcal{A}^*) = \sigma(\mathcal{A})$.
Step 2. Let $\mathcal{A}_0$ be the operator defined by

$$\mathcal{A}_0(w,z) = \{(z_j, -a_j w_{j,ssss})\}, \forall (w,z) \in \mathcal{D}(\mathcal{A}_0)$$

$$\mathcal{D}(\mathcal{A}_0) = \left\{\begin{array}{l} (w,z) \in \mathcal{V},\ \sum\limits_{l=1}^{3} a_l w_{l,sss}(0) = 0 \\ w_{1,ss}(0) = w_{2,ss}(0) = w_{3,ss}(0) = 0 \\ \sum\limits_{l=3}^{5} a_l w_{l,sss}(1) = 0 \\ w_{3,ss}(1) = w_{4,ss}(1) = w_{5,ss}(1) = 0 \end{array}\right\}. \quad (22)$$

Then $\mathcal{A}_0$ is a skew adjoint operator, i.e., $\mathcal{A}_0^* = -\mathcal{A}_0$.
Step 3. For $\lambda \in \rho(\mathcal{A})$ and $F \in \mathcal{H}$, $R(\lambda, \mathcal{A})F$ and $R(\lambda, \mathcal{A}^*)F$ are meromorphic function of finite exponential type, i.e., $\rho_1 \leqslant 1, \rho_2 \leqslant 1$ since $R(\lambda, \mathcal{A}^*)F$, $R(\lambda, \mathcal{A})F$ consist of functions $\sinh\sqrt{i\lambda}\,\omega_j$, $\cosh\sqrt{i\lambda}\,\omega_j$, $\sin\sqrt{i\lambda}\,\omega_j$, $\cos\sqrt{i\lambda}\,\omega_j$ as well as their integrations.

Step 4. The root vectors of $\mathcal{A}$ is complete in $\mathcal{H}$.
In fact, we can assume without loss of generality that $\mathbb{R}_- \in \rho(\mathcal{A})$. Let $\lambda \in \mathbb{R}_-$, for any given $F \in \mathcal{H}$, set $Y_1 = R(\lambda, \mathcal{A}^*)F$, $Y_2 = R(\lambda, \mathcal{A}_0^*)F$ and $\Phi = Y_1 - Y_2$. Denote $\Phi = (w,z)$, $Y_2 = (u,v)$. Then $\Phi + Y_2$ satisfies (21), and $\Phi = (w,z)$ satisfies $z = \lambda w$ and w satisfying

$$\left\{\begin{array}{l} w_{j,ssss}(s) = -\lambda^2 w_j(s),\ j = 1, 2, \cdots, 5 \\ w_1(0) = w_2(0) = w_3(0) \\ \sum\limits_{i=1}^{3} w_{i,sss}(0) = 0 \\ w_3(1) = w_4(1) = w_5(1) \\ \sum\limits_{i=3}^{5} w_{i,sss}(1) = 0 \\ w_{1,ss}(0) - \lambda w_{1,s}(0) = -v_{1,s}(0) \\ w_1(1) = w_{1,s}(1) = 0 \\ w_{2,ss}(0) = w_2(1) = w_{2,s}(1) = 0 \\ w_{3,ss}(0) - \lambda w_{3,s}(0) = -v_{3,s}(0) \\ w_{3,ss}(1) + \lambda w_{3,s}(1) = v_{3,s}(1) \\ w_{4,ss}(1) = w_4(0) = w_{4,s}(0) = 0 \\ w_{5,ss}(1) = w_5(0) = w_{5,s}(0) = 0 \end{array}\right. \quad (23)$$

Therefore, we have

$$\begin{aligned} \|\Phi\| &= \sum_{j=1}^{5} \int_0^1 \left[w_{j,ss}^2(s) + z_j^2(s)\right] \mathrm{d}s \\ &= \sum_{j=1}^{5} \int_0^1 \left[w_{j,ss}^2(s) + \lambda^2 w_j^2(s)\right] \mathrm{d}s \\ &= -w_{1,ss}(0) w_{1,s}(0) - w_{3,ss}(0) w_{3,s}(0) \\ &\quad + w_{3,ss}(1) w_{3,s}(1) \\ &= \lambda w_{1,s}^2(0) + w_{1,s}(0) v_{1,s}(0) + \lambda w_{3,s}^2(0) \\ &\quad + w_{3,s}(0) v_{3,s}(0) + \lambda w_{3,s}^2(1) + w_{3,s}(1) v_{3,s}(1) \end{aligned}$$

To calculate $w_{1,s}(0), w_{3.s}(0)$ and $w_{3,s}(1)$, we assume that $P(\rho)E(s)$ is the fundamental matrix, then $\Phi = P(\rho)E(s)C$. Substitute it into the boundary conditions of (6) and use (14), we get

$$\left[B^0(i\rho^2)P(\rho)E^0 + B^1(i\rho^2)P(\rho)E^1\right]E^2 C = F_0$$

with $F_0 = [V_1, V_2, V_3, V_4, V_5]^T$, in which $V_1 = [0, -v_{1,s}(0), 0, 0]$,$V_3 = [0, -v_{3,s}(0), 0, v_{3,s}(1)]$ and $V_2 = V_4 = V_5 = [0, 0, 0, 0]$. Thus

$$C = [E^2]^{-1}[B^0(i\rho^2)P(\rho)E^0 + B^1(i\rho^2)P(\rho)E^1]^{-1} F_0.$$

Set $\rho = a + bi$, if λ are located in the first quadrant of the complex plane, ρ satisfies that $\frac{3\pi}{4} \leqslant \arg\rho < \pi$. It's obvious that $a < 0$ and $b > 0$. Therefore,

$$|e^{\rho}| \to 0, |e^{i\rho}| \to 0 \quad \text{as } |\lambda| \to +\infty$$

and

$$|e^{-5(1+i)\rho}| = |e^{5(b-a)-5(a+b)i}| = |e^{5(|b|+|a|)}| \geqslant e^{5|\rho|}.$$

If λ are located in the second quadrant of the complex plane, ρ satisfies that $0 \leqslant \arg\rho < \frac{\pi}{4}$. We have

$$|e^{-\rho}| \to 0, |e^{i\rho}| \to 0 \quad \text{as } |\lambda| \to +\infty$$

and

$$|e^{5(1-i)\rho}| = |e^{5(a+b)-5(a-b)i}| = |e^{5(|b|+|a|)}| \geqslant e^{5|\rho|}.$$

Similar to (17), there exists positive constants h and M such that $\|\Omega(\rho)\| \geqslant M \cdot |\rho^{45}| \cdot e^{5|\rho|}$ where $|\lambda| > h$. Hence,

$$\begin{aligned}\|\Phi\| &\leqslant \|P(\rho)\| \cdot \|E(s)\| \cdot \|\Omega(\rho)\|^{-1} \cdot \|F_0\| \\ &\leqslant \frac{|16^5 \rho^{50} i|}{M \cdot |\rho^{45}| \cdot e^{5|\rho|}} \cdot \|F_0\| \leqslant M_1 \cdot \|F_0\|\end{aligned}$$

where $|\lambda| > h$ and M_1 is a positive constant. Since the operator V defined by

$$V : F \to F_0 = V(R(\lambda, \mathcal{A}^*)F) \in \mathbb{C}^{20}$$

is a bounded linear operator on $\mathcal{H}$, there exists a constant M_2 such that $\|F_0\| \leqslant M_2\|F\|$. So we have $\|\Phi\| \leqslant M_1 M_2 \|F\|$. Therefore,

$$\begin{aligned}\|R(\lambda, \mathcal{A}^*)F\| &= \|Y_1\| \leqslant \|Y_2\| + \|\Phi\| \\ &\leqslant \|R(\lambda, \mathcal{A}^*{}_0)F\| + M_1 M_2 \|F\| \\ &\leqslant \big(\frac{1}{|\lambda|} + M_1 M_2\big)\|F\|, \quad |\lambda| > h\end{aligned}$$

This means that $R(\lambda, \mathcal{A}^*)$ is bounded on the negative real axis. By now all conditions in Lemma 6 are verified. The desired result follows from Lemma 6. □

To obtain the basis property of generalized eigenvectors of $\mathcal{A}$, we need the following lemma, which from [17] and [18].

Lemma 8 *Let $\mathcal{A}$ be the generator of a C_0-semigroup in a Hilbert space $\mathcal{H}$. Suppose that the following conditions are satisfied:*
1). The spectrum of $\mathcal{A}$ has the decomposition

$$\sigma(\mathcal{A}) = \sigma_1(\mathcal{A}) \bigcup \sigma_2(\mathcal{A});$$

2). There exists a real number $\alpha \in \mathbb{R}$ such that

$$\sup\{\Re\lambda; \lambda \in \sigma_1(\mathcal{A})\} \le \alpha \le \inf\{\Re\lambda; \lambda \in \sigma_2(\mathcal{A})\}$$

3). The set $\sigma_2(\mathcal{A}) = \{\lambda_k\}_{k\in\mathbb{N}}$ consists of eigenvalues of $\mathcal{A}$ and is essential space finite separated (or equivalent saying that it is an union of finitely many separated sets).
Then there exists two $T(t)$-invariant closed subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$

$$\mathcal{H}_1 = \{f \in \mathcal{H} : E(\lambda, \mathcal{A})f = 0, \forall \lambda \in \sigma_2(\mathcal{A})\}$$

$$\mathcal{H}_2 = \overline{\operatorname{span}\{\sum_{k=1}^{m} E(\lambda_k, \mathcal{A})\mathcal{H}, \forall m \in \mathbb{N}\}}$$

and $\mathcal{H}_1 \bigcap \mathcal{H}_2 = \{0\}$ with property that $\sigma(\mathcal{A}|_{\mathcal{H}_1}) = \sigma_1(\mathcal{A})$ and $\sigma(\mathcal{A}|_{\mathcal{H}_2}) = \sigma_2(\mathcal{A})$. Moreover, there exists a finite collection Ω_k of elements in $\sigma_2(\mathcal{A})$, the corresponding Riesz projector

$$E(\Omega_k, \mathcal{A}) = \sum_{\lambda\in\Omega_k} E(\Omega, \mathcal{A})$$

such that $\{E(\Omega_k, \mathcal{A})\mathcal{H}_2\}_{k\in\mathbb{N}}$ forms a subspaces Riesz basis for $\mathcal{H}_2$.

As a direct application of Lemma 8, we have the following result.

Theorem 9 *Let $\mathcal{A}$ be defined as above, then there exist a collection of generalized eigenvectors of $\mathcal{A}$ such that it forms a Riesz basis with parentheses for $\mathcal{H}$.*

Proof: Set $\sigma_2(\mathcal{A}) = \sigma(\mathcal{A})$ and $\sigma_1(\mathcal{A}) = \{\infty\}$, Theorem 5 ensures that all conditions in Lemma 8 are satisfied. By Lemma 8, there is a finite collection Ω_k of elements in $\sigma(\mathcal{A})$ such that $\{E(\Omega_k, \mathcal{A})\mathcal{H}_2\}_{k\in\mathbb{N}}$ forms a subspaces Riesz basis for $\mathcal{H}_2$. Theorem 7 reads that $\mathcal{H} = \mathcal{H}_2$. Therefore $\{E(\Omega_k, \mathcal{A})\mathcal{H}\}_{k\in\mathbb{N}}$ is a subspace Riesz basis. □

4 Stability Analysis

In this section, we discuss the stability of the system. Based on Theorem 9, we need only to determine the location of the eigenvalues of $\mathcal{A}$. Firstly we study eigenvalues of $\mathcal{A}$ on the imaginary axis.

Let $\lambda \in i\mathbb{R}, \lambda \neq 0$. Since for any $(w, z) \in \mathcal{D}(\mathcal{A})$, we have

$$\Re(\mathcal{A}(w, z), (w, z)) = -a_1 z_{1,s}^2(0) - a_3 z_{3,s}^2(0) - a_3 z_{3,s}^2(1)$$

this implies that $\mathcal{A}(w, z) = \lambda(w, z)$ has a nonzero solution if and only if $z_{1,s}(0) = z_{3,s}(0) = z_{3,s}(1) = 0$, and $z(x) = \lambda w(x)$, and $w(x)$ satisfy

$$\begin{cases} a_j\, w_{j,ssss}(s) = -\lambda^2 w_j(s),\ j = 1, 2, \cdots, 5 \\ w_1(0) = w_2(0) = w_3(0) \\ \sum_{i=1}^{3} a_i\, w_{i,sss}(0) = 0 \\ w_3(1) = w_4(1) = w_5(1) \\ \sum_{i=3}^{5} a_i\, w_{i,sss}(1) = 0 \\ w_{1,ss}(0) = w_{1,s}(0) = w_1(1) = w_{1,s}(1) = 0 \\ w_{2,ss}(0) = w_2(1) = w_{2,s}(1) = 0 \\ w_{3,ss}(0) = w_{3,s}(0) = w_{3,ss}(1) = w_{3,s}(1) = 0 \\ w_{4,ss}(1) = w_4(0) = w_{4,s}(0) = 0 \\ w_{5,ss}(1) = w_5(0) = w_{5,s}(0) = 0 \end{cases} \tag{24}$$

Set $\omega_j = \sqrt[4]{1/a_j}$, $\lambda = i\rho^2$, define functions:

$$\begin{aligned}
v_{j1}(s) &= \frac{1}{2}[\cosh\omega_j\rho s + \cos\omega_j\rho s],\\
v_{j2}(s) &= \frac{1}{2}[\sinh\omega_j\rho s + \sin\omega_j\rho s],\\
v_{j3}(s) &= \frac{1}{2}[\cosh\omega_j\rho s - \cos\omega_j\rho s],\\
v_{j4}(s) &= \frac{1}{2}[\sinh\omega_j\rho s - \sin\omega_j\rho s].
\end{aligned}$$

Then the general solution to the differential equation in (24) has the form

$$w_j(s) = b_{j1}v_{j1}(s)+b_{j2}v_{j2}(s)+b_{j3}v_{j3}(s)+b_{j4}v_{j4}(s) \quad (25)$$

Substituting (25) into the boundary condition of $w_1(s)$ yields

$$w_1(s) = \begin{cases} b_{11}[v_{14}(1)v_{11}(s) - v_{11}(1)v_{14}(s)], \\ \quad \text{as } \sinh\omega_1\rho\sin\omega_1\rho = 0; \\ 0, \quad \text{otherwise} \end{cases} \quad (26)$$

substituting (25) into the boundary condition of $w_3(s)$ yields

$$w_3(s) = \begin{cases} b_{31}[v_{33}(1)v_{31}(s) - v_{34}(1)v_{34}(s)], \\ \quad \text{as } \cosh\omega_3\rho\cos\omega_3\rho = 1; \\ 0, \quad \text{otherwise.} \end{cases} \quad (27)$$

Similarly, we can get

$$\begin{aligned}
w_2(s) &= b_{23}[v_{22}(1)v_{23}(1-s) - v_{21}(1)v_{24}(1-s)]\\
w_4(s) &= b_{43}[v_{42}(1)v_{43}(s) - v_{41}(1)v_{44}(s)]\\
w_5(s) &= b_{53}[v_{52}(1)v_{53}(s) - v_{51}(1)v_{54}(s)].
\end{aligned}$$

We divide them into the following cases:

Case I. $w_1(s) = 0$ (or $w_3(s) = 0$). From the connective condition

$$w_1(0) = w_2(0) = w_3(0), \; \sum_{j=1}^{3} a_j w_{j,sss}(0) = 0$$

we can get

$$\begin{aligned} b_{11}v_{14}(1) &= b_{31}v_{34}(1) \\ &= b_{23}[v_{22}(1)v_{23}(1) - v_{21}(1)v_{24}(1)] \end{aligned} \quad (28)$$

and

$$-b_{11}v_{11}(1)/\omega_1+b_{23}v_{21}(1)/\omega_2-b_{31}v_{34}(1)/\omega_3 = 0 \quad (29)$$

If $w_1(s) = 0$, then $w_3(0) = w_1(0) = 0$. According to (28), $b_{31} = 0$, that is $w_3(s) = 0$; Similarly, if $w_3(s) = 0$, $w_1(s) = 0$ holds. And if $w_1(s) = 0$ or $w_3(s) = 0$, using (29), we have $b_{23} = 0$, that is $w_2(s) = 0$. Further, using the connective conditions

$$w_3(1) = w_4(1) = w_5(1), \; \sum_{j=3}^{5} a_j w_{j,sss}(1) = 0$$

we get

$$b_{43}[v_{42}(1)v_{43}(1) - v_{41}(1)v_{44}(1)] = 0 \quad (30)$$

$$b_{53}[v_{52}(1)v_{53}(1) - v_{51}(1)v_{54}(1)] = 0 \quad (31)$$

and

$$\begin{aligned} b_{43}[\, v_{42}(1)v_{44}(1) - v_{41}(1)v_{41}(1)]/\omega_4 \\ +b_{53}[v_{52}(1)v_{54}(1) - v_{51}(1)v_{51}(1)]/\omega_5 = 0 \end{aligned} \quad (32)$$

Obviously, if $b_{43} = 0$ or $b_{53} = 0$, then $w_4(s) = w_5(s) = 0$. Hence, if

$$\begin{aligned} &v_{j2}(1)v_{j3}(1) - v_{j1}(1)v_{j4}(1) \\ &= \frac{1}{2}[\sin\omega_j\rho\cosh\omega_j\rho - \cos\omega_j\rho\sinh\omega_j\rho] = 0, \quad j = 4, 5 \end{aligned}$$

, there exist nonzero solutions. Define the solution set $\mathcal{D} = \{\theta_i |\tan\theta_i = \tanh\theta_i, \; i = 1, 2, \cdots\}$. If there are no θ_m and θ_n such that $\theta_m/\theta_n = \omega_4/\omega_5$, eqs. (24) has only zero solution.

Case II. $\rho \in \mathbb{R}$ satisfies

$$\sinh\omega_1\rho\sin\omega_1\rho = 0, \; \cosh\omega_3\rho\cos\omega_3\rho = 1.$$

In this case, one may have that $w_1(s) \neq 0$ and $w_3(s) \neq 0$. Then there exists a $k \in \mathbb{N}$ such that $\omega_1\rho = k\pi$ and

$$\cosh\frac{\omega_3}{\omega_1}k\pi\cos\frac{\omega_3}{\omega_1}k\pi = 1.$$

From discussion we see that the following assertion holds true.

Theorem 10 *Let $\mathcal{A}$ and ω_j, $\mathcal{D}$ be defined as before, if the following conditions are satisfied:*
1). There are no θ_m and θ_n such that $\theta_m/\theta_n = \omega_4/\omega_5$
2). There is no $k \in \mathbb{N}$ such that

$$\cosh\frac{\omega_3}{\omega_1}k\pi\cos\frac{\omega_3}{\omega_1}k\pi = 1$$

Then there exist no eigenvalues on the imaginary axis, and hence the system is asymptotically stable.

5 Conclusion

In this paper, we design the feedback controllers for a complex network of Euler-Bernoulli beams and then proved that the closed loop system is well posed. We show further that the root vectors of the system operator $\mathcal{A}$ are completeness, and there exists a sequence of the root vectors that forms a Riesz basis for the state space $\mathcal{H}$. By the detailed analysis we get the asymptotic stability of the system. Note that the Riesz basis property implies that the system satisfies the spectrum determined growth condition. Hence the system is exponentially stable if and only if the imaginary axis is not the asymptote of the eigenvalues. In further work we shall analyze the exponential stability of the system.

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