Fuzzy multiobjective optimization modeling with Mathematica

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***Abstract*—In the real situations, decision makers are often faced to a plurality of objectives and constraints in a world of imprecise data about the preferences of agents, the local constraints and the global environment. In a fuzzy environment, fuzzy linear programming (FLP) and fuzzy goal programming (FGP) problems incorporate fuzzy objective functions and constraints, fuzzy parameters and variable sets. Mathematical operators are used to aggregate the fuzzy objective functions and constraints. The optimal solution corresponds to the maximum degree of the membership function in the decision set. The resolution of the multiobjective FLP consists in reducing the vector optimization of objective functions to a single objective. Weighted goal programming problems consider the relative importance of objectives. This contribution surveys essential techniques with numerical applications to simple economic problems. The computations are carried out using the software $Mathematica^{®}$ 7.0.1 and the subpackage $Fuzzy\ Logic$ 2, from which selected primitives are proposed.**

***Index Terms*—fuzzy linear programming problem, membership function, decision set, multiobjective optimization**

I. INTRODUCTION

DECISION makers (DMs) have often to decide in an environment where the multiple objectives and constraints may be uncertain with mostly imprecise data. The fuzzy multiobjective optimization modeling considers these particular circumstances with fuzzy numbers, fuzzy objectives and soft constraints. Basic problems and some extensions will be presented with numerical examples, using the package $Mathematica^{®}$. The basic concepts are firstly considered for single and multiple objectives problems with a simple application to the well-known trade balance problem with two fuzzy objectives. Secondly, the fuzzy multiobjective optimization model is presented in a context, where the fuzzy numbers (FNs) and soft constraints are simply defined. Some extensions are given with application. Thirdly, the presentation is extended to fuzzy goal programming problems. Other shapes for the fuzzy membership functions are introduced and the objectives are weighted. The well-known production-marketing problem will illustrate this more realistic approach.

A. Fuzzy decision sets

In practical situations, DMs may not be able to specify exact objectives and restrictions of a programming problem. Let X be a set of possible actions, $\{\tilde{G}_j\ (j \in \mathbb{N}_n)\}$ a set of fuzzy objectives and $\{\tilde{C}_i\ (i \in \mathbb{N}_m)\}$ a set of fuzzy constraints.

Definition 1: Fuzzy decision. Let the n fuzzy objectives $\tilde{G}_1, \ldots, \tilde{G}_n$ with membership grades $\mu_{\tilde{G}_j}$, $j \in \mathbb{N}_n$ and the

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m fuzzy constraints $\tilde{C}_1, \ldots, \tilde{C}_m$ in a space of alternatives X with membership grades $\mu_{\tilde{C}_i}$, $i \in \mathbb{N}_m$. The decision set is the intersection of the given fuzzy objectives and constraints. We have a fuzzy set $\tilde{D} = \tilde{G}_1 \cap \tilde{G}_2 \cap \ldots \cap \tilde{G}_n \cap \tilde{C}_1 \cap \tilde{C}_2 \cap \ldots \cap \tilde{C}_m$ with $\mu_{\tilde{D}} = \mu_{\tilde{G}_1} \wedge \mu_{\tilde{G}_2} \wedge \ldots \wedge \mu_{\tilde{G}_n} \wedge \mu_{\tilde{C}_1} \wedge \mu_{\tilde{C}_2} \wedge \ldots \wedge \mu_{\tilde{C}_m}$.

Then, according to the Bellman - Zadeh symmetry principle [2], a fuzzy decision set is achieved by an appropriate aggregation of the fuzzy sets, such as with a min-operator [6]. In condensed form, the decision set is defined by $\tilde{D} = \left(\bigcap_{j=1}^n \tilde{G}_j\right) \cap \left(\bigcap_{i=1}^m \tilde{C}_i\right)$, with a membership function (MF) $\mu_{\tilde{D}} : X \mapsto [0,1]$ given by $\mu_{\tilde{D}}(x) = \bigwedge_{j=1}^n \mu_{\tilde{G}_j}(x) \wedge \bigwedge_{i=1}^m \mu_{\tilde{C}_i}(x)$. One DM can determine $x^* \in X$ to be the optimal solution when $\mu_{\tilde{D}}(x^*) = \sup \mu_{\tilde{D}}(x)$ for all $x \in X$. Suppose (as in [2], [22], [24]) that the objective should be "substantially larger" than 10 and that the constraint imposes that x should be "in the vicinity" of 10. The objective will have a sigmoidal shaped MF $\mu_{\tilde{G}}$, which expression is

$$\mu_{\tilde{G}}(x) = \left\{ \begin{array}{ll} 0, & x \leq 10 \\ 1/(1 + (x - 10)^{-2}), & x > 10. \end{array} \right.$$

The constraint will have a bell-shaped MF $\mu_{\tilde{C}}$ which expression is

$$\mu_{\tilde{C}}(x) = \frac{1}{1 + (x - 10)^4}.$$

The MF of the fuzzy decision set $\tilde{D}$ will then be $\mu_{\tilde{D}}(x) = \mu_{\tilde{G}}(x) \wedge \mu_{\tilde{C}}(x)$ such that

$$\mu_{\tilde{D}}(x) = \left\{ \begin{array}{ll} 1/(1 + (x - 10)^4), & x > 11 \\ 1/(1 + (x - 10)^{-2}), & 10 < x \leq 11 \\ 0, & x \leq 10. \end{array} \right.$$

The highest degree of the MF is $x^* = \arg \max_x \left(\min\{\mu_{\tilde{G}}(x), \mu_{\tilde{C}}(x)\} \right)$. A maximum, reached at $(11, \frac{1}{2})$ is shown in Fig.1. The Mathematica primitives of this figure are framed hereafter.

B. Fuzzy vector-maximum problem

The classical maximizing LP problem states : maximize a single objective linear problem over a bounded feasible region $\mathbf{X} = \{\mathbf{x} \mid \mathbf{A}.\mathbf{x} \leq \mathbf{b}\}(\mathbf{x} \in \mathbb{R}_+^n, \mathbf{b} \in \mathbb{R}^m, \mathbf{A} \in \mathbb{R}^{m \times n})$ defined by all the constraints. We have the problem : $\max_{\mathbf{x} \in \mathbf{X}} z = \mathbf{c}^\top.\mathbf{x}$ ($\mathbf{c}, \mathbf{x} \in \mathbb{R}^n$). However, in the practical situations DMs are confronted to uncertainties about the objectives and constraints. Thus, the value $z = \mathbf{c}^\top.\mathbf{x}$ will "exceed at least" a given objective, and the restrictions [1] $\mathbf{A}_i.\mathbf{x} \leq b_i$ $(i \in \mathbb{N}_m)$

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[1]The symbol $\mathbf{A}_i$ $(i \in \mathbb{N}_m)$ denotes the ith row of the matrix $\mathbf{A}_{m \times n}$.

Fig. 1.   Fuzzy decision set

The fuzzy set of objectives [2] is $\tilde{G} = \bigcap_{j=1}^{k} \tilde{G}_j$ and $\mu_{\tilde{G}}(\mathbf{x}) = \bigwedge_{j=1}^{k} \mu_{\tilde{G}_j}(\mathbf{x})$. The decision set is defined by $\tilde{D} = \tilde{G} \cap \mathbf{X}$. The optimal solution $\max_{\mathbf{x} \in \mathbf{X}} \mu_{\tilde{D}}(\mathbf{x})$ is an efficient solution, which is obtained for the greatest degree of satisfaction α.

Theorem 4: Weak potential solution. If the solution $\mathbf{x}^*$ can be obtain from solving:

$$\max_{\mathbf{x},\alpha} \quad \alpha$$
$$\text{subject to}$$
$$\frac{\mathbf{C}_j.\mathbf{x} - L_j^0}{U_j^0 - L_j^0} \geq \alpha, \ \ (j \in \mathbb{N}_k)$$
$$\mathbf{x} \in \mathbf{X}, \ \mathbf{x} \geq 0, \ \alpha \in (0,1],$$

then $\mathbf{x}^* \in \mathbf{X}$ is a weak potential solution of the problem [3] [4].

C. Trade balance problem with multiple fuzzy objectives

A manufacturer produces two goods A and B (taken from [11], [23], [24]) : A yields a unit profit of 2 and B, a unit profit of 1; A needs imported materials. The DMs have two objectives : maximize the profit and maximize the performance of the trade balance (in terms of net exports). The FLP problem for $\mathbf{x} = (x_1, x_2)^\top$ is

$$\max_{\mathbf{x}} \ \mathbf{C}.\mathbf{x} \ \gtrsim \mathbf{Z}$$
$$\text{subject to}$$
$$\mathbf{A}.\mathbf{x} \leq \mathbf{b},$$
$$\mathbf{x} \geq 0,$$

where

$$C = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}, A = \begin{pmatrix} -1 & 3 \\ 1 & 3 \\ 4 & 3 \\ 3 & 1 \end{pmatrix}, b = \begin{pmatrix} 21 \\ 27 \\ 45 \\ 30 \end{pmatrix}$$

The solution space [5] , is shown in Fig.2 with individual optima in $\mathbf{x}^1$ and $\mathbf{x}^4$: $\mathbf{x}^1$ is optimum w.r.t. the second objective $z_2(\mathbf{x}) = -x_1 + 2x_2$ and $\mathbf{x}^4$ is optimum w.r.t. the first objective $z_1(\mathbf{x}) = 2x_1 + x_2$. The optimal values are $z_1(\mathbf{x}^4) = 21 \quad > z_1(\mathbf{x}^1) = 7$ and $z_2(\mathbf{x}^1) = 14 \quad > z_2(\mathbf{x}^4) = -3$. The Mathematica primitives of this figure are framed hereafter. Two figures are plotted successively : the first plot represents the region corresponding to the constraints, the second plot delimitates the feasible region of the problem. The **A[[1]]** to **A[[4]]** are the rows of the matrix **A** and **x** states for the vector of variables. The nondecreasing ramp-type MFs of the

will be "almost satisfied". The fuzzy LP problem (FLP) may be written : $\widetilde{\max} \ \mathbf{c}^\top.\mathbf{x}$ subject to $\mathbf{A}_i.\mathbf{x} \lesssim b_i$, $\mathbf{x} \geq 0$. With the same MFs, the FLP may also be written : $\widetilde{\max} \ \mathbf{c}^\top.\mathbf{x}$ subject to $\mathbf{A}_i.\mathbf{x} \leq \tilde{b}_i$, $\mathbf{x} \geq 0$, where the linguistic "roughly less than b_i" has been replaced by the "vicinity of b_i". In vector-maximum problems, multiple objectives are maximized over a bounded feasible region of constraints. The multiobjective problem (MOP) [11], [15] is $\max_{\mathbf{x} \in \mathbf{X}} \mathbf{Z}(\mathbf{x}) = \mathbf{C}_{k \times n}.\mathbf{x}$ where $\mathbf{Z}(\mathbf{x})$ states a k-vector valued objective function $(z_1(\mathbf{x}), \ldots, z_k(\mathbf{x}))^\top$.

Definition 2: Pareto optimal solution. Let $\{\mathbf{Z}(\mathbf{x})|\mathbf{x} \in \mathbf{X}\}$ be a vector-maximum problem, $\mathbf{x}^* \in \mathbf{X}$ is an efficient Pareto optimal solution, if and only if, there is no $\mathbf{x}$ such that $z_j(\mathbf{x}) \geq z_j(\mathbf{x}^*)(j \in \mathbb{N}_k)$ and $z_j(\mathbf{x}) > z_j(\mathbf{x}^*)$ for at least one j.

Theorem 3: Scalarization Theorem. A solution $\mathbf{x}^* \in \mathbf{X}$ is efficient, if and only if, there is a vector $\lambda \in \mathbb{R}^k$, $\lambda_j > 0$ ($j \in \mathbb{N}_k$) such that $\mathbf{x}^*$ solves $\max_{\mathbf{x} \in \mathbf{X}} \sum_{j=1}^{k} \lambda_j \mathbf{C}_j \mathbf{x}$.

Given a problem $\{\mathbf{C}.\mathbf{x} \gtrsim \mathbf{Z}$, s.t. $\mathbf{A}.\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq 0\}$ with fuzzy objective functions and crisp constraints, the resolution may consist in solving single objective linear programs using each objective $\mathbf{C}_j.\mathbf{x}$, $(j \in \mathbb{N}_k)$. We can obtain an upper bound U_j^0 and a lower bound L_j^0 of the jth $\mathbf{C}_j.\mathbf{x}$ over the feasible region $\mathbf{X} \subset \mathbb{R}^n$. The jth MF $(j \in \mathbb{N}_k)$ is expressed as

$$\mu_{\tilde{G}_j}(\mathbf{C}_j.\mathbf{x}) = \begin{cases} 1, & \mathbf{C}_j.\mathbf{x} \geq U_j^0 \\ (\mathbf{C}_j.\mathbf{x} - L_j^0)/(U_j^0 - L_j^0), & L_j^0 \leq \mathbf{C}_j.\mathbf{x} \leq U_j^0 \\ 0, & \mathbf{C}_j.\mathbf{x} \leq L_j^0. \end{cases}$$

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[2] Other real-valued functions have been proposed in the literature : a weighted sum of objectives $\sum_{j=1}^{k} \alpha_j (z_j(\mathbf{x}))^{\beta_j}$, $\alpha_j > 0$, $\beta_j > 0$ or a product of objectives $\prod_{j=1}^{k} \alpha_j(z_j)$.This aggregation may also be based on the DM's preferences with utility functions.

[3] The two-phase approach for solving FLP problems [8] does not only achieve the highest membership degree (as with the max-min operator' solution), but also realizes the better utilization of each constraint.

[4] Werners'interactive approach [20] for solving multiple objective FLP problems considers a situation where the DM cannot determine the exact membership function a priori. The system then suggests the functions given the available information, according to interactive changes.

[5] The coordinates of the extreme points in the solution space are $\mathbf{x}^0 = (0,0)$, $\mathbf{x}^1 = (0,7)$, $\mathbf{x}^2 = (3,8)$, $\mathbf{x}^3 = (6,7)$, $\mathbf{x}^4 = (9,3)$, $\mathbf{x}^6 = (3.4,.2)$.
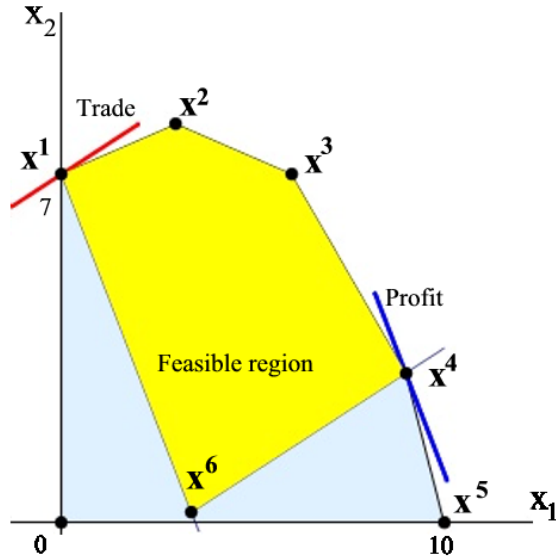
Fig. 2.   Feasible solution region

objectives (Fig.3) are

$$\mu_1(z_1(\mathbf{x})) = \begin{cases} 1, & z_1(\mathbf{x}) > 21 \\ (-7 + z_1(\mathbf{x}))/14, & 7 \leq z_1(\mathbf{x}) \leq 21 \\ 0, & z_1(\mathbf{x}) < 7 \end{cases}$$

$$\mu_2(z_2(\mathbf{x})) = \begin{cases} 1, & z_2(\mathbf{x}) > 14 \\ (3 + z_2(\mathbf{x}))/17, & -3 \leq z_2(\mathbf{x}) \leq 14 \\ 0, & z_2(\mathbf{x}) < -3 \end{cases}$$

The Mathematica primitives for of this figure are framed hereafter. Thus the preference $\mu_1(\mathbf{x})$, w.r.t. the profit, rises from 0 for a profit of 7 to 1 for a profit of 21, and similarly the preference $\mu_2(\mathbf{x})$, w.r.t. the trade balance, rises from 0 for imports of 3 to 1 for exports of 14.

Using the Bellman-Zadeh's max-min operator, the formal problem which satisfies the constraints and the objective, with
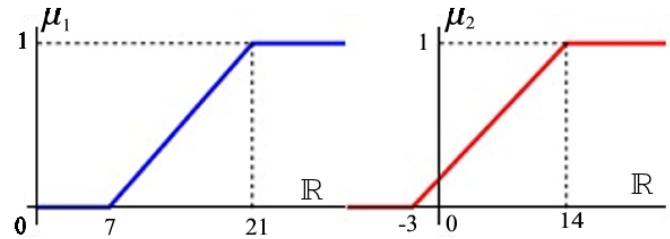

Fig. 3.   MFs of the objectives for a maximizing problem

the maximum degree α, is

$$\max_{\mathbf{x}, \alpha} \quad \alpha$$
$$\text{subject to}$$
$$\mu_1(\mathbf{x}) \equiv 1 - \frac{21 - 2x_1 - x_2}{14} \geq \alpha,$$
$$\mu_2(\mathbf{x}) \equiv 1 - \frac{14 + x_1 - 2x_2}{17} \geq \alpha,$$
$$g_1(\mathbf{x}) \equiv -x_1 + 3x_2 \leq 21,$$
$$g_2(\mathbf{x}) \equiv x_1 + 3x_2 \leq 27,$$
$$g_3(\mathbf{x}) \equiv 4x_1 + 3x_2 \leq 45,$$
$$g_4(\mathbf{x}) \equiv 3x_1 + x_2 \leq 30,$$
$$\mathbf{x} \geq 0.$$

The solution in Fig.4 is $\mathbf{x}^* = \{5.03, 7.32\}$ with a satisfaction level of $\alpha^* = 74$ per cent. The optimal solution yields a profit of 17.38 and a net export of 4.48.

An animation of the 3D figure is obtained with the framed primitives hereafter. The primitives render Fig.5.

II. FUZZY MULTIOBJECTIVE OPTIMIZATION

A. Basic model with application

Let a minimizing problem MOP with all crisp coefficients, k objectives, m rigid constraints, and n variables [7], [12], [21]. We have

$$\min \quad \mathbf{z} = \mathbf{C}.\mathbf{x} \ (\mathbf{C} \in \mathbb{R}^{k \times n}, \mathbf{x} \in \mathbb{R}^n_+)$$
$$\text{subject to}$$
$$\mathbf{B}.\mathbf{x} \geq \mathbf{b}_0 \ (\mathbf{B} \in \mathbb{R}^{m \times n}, \mathbf{b}_0 \in \mathbb{R}^m).$$

In a cost minimizing problem, $\mathbf{C}$ will denote the $k \times n$ matrix of costs, the $m \times n$ matrix $\mathbf{B}$ technical coefficients, $\mathbf{b}_0$ the m demands and $\mathbf{x}$ the n variables.

■ data of the problem
  x := {x1, x2}; A := {{-1, 3}, {1, 3}, {4, 3}, {3, 1}};
    b := {21, 27, 45, 30}; c := {{2, 1}, {-1, 2}};

■ maximization program
  Maximize[{α,
    mu1[x1, x2] >= α &&
      mu2[x1, x2] >= α &&
        A[[1]].x <= b[[1]] && A[[2]].x <= b[[2]] &&
        A[[3]].x <= b[[3]] && A[[4]].x <= b[[4]] &&
          x1 >= 0 && x2 >= 0 && α >= 0},
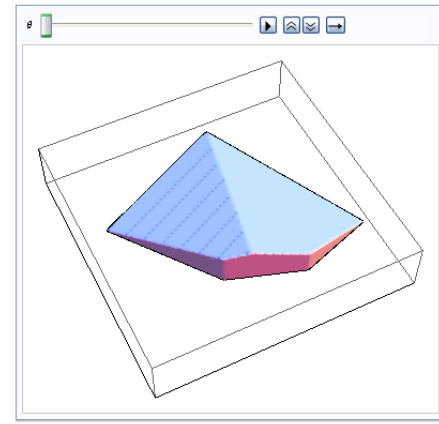   {x1, x2, α}
          ]

{0.741935, {x1 → 5.03226, x2 → 7.32258, α → 0.741935}}

Fig. 5.   Animation of the solution set

■ plot the solution set
  fig1b = RegionPlot3D[
    A[[1]].x < b[[1]] &&
    A[[2]].x < b[[2]] &&
      A[[3]].x < b[[3]] &&
      A[[4]].x < b[[4]] &&
        mu1[x1, x2] >= x3 && mu2[x1, x2] >= x3,
        {x1, 0, 10}, {x2, 0, 10}, {x3, 0, 1},
      BoundaryStyle -> Directive[Thickness[Medium], Black],
      PlotStyle -> Opacity[.8], Mesh -> None, PlotPoints -> 100,
        BoxRatios -> {1, 1, .35}
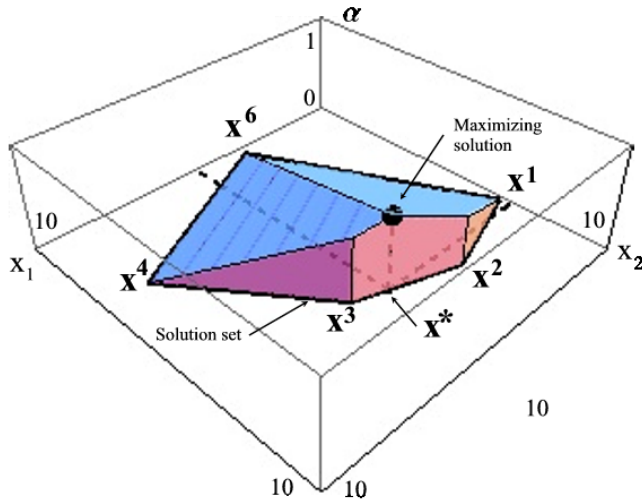   ]

Fig. 4.   Solution set and maximizing solution

■ animate the figure
  Animate[
    With[{v = RotationTransform[θ, {0, 0, 1}][{1, 2, 3}]},
      Show[fig1b, Ticks -> None, ViewPoint -> v,
        PlotRange -> {{0, 10}, {0, 10}, {0, 1}}, BoxRatios -> {1, 1, .20}
        ]
      ], {θ, 0, 2 Pi}, AnimationRunning -> False
      ]

Definition 5: Efficient solution. Let $\mathbf{x}^*$ and $\mathbf{y}^*$ two feasible solutions of a minimizing MOP, $\mathbf{x}^*$ is more efficient than $\mathbf{y}^*$ $(\mathbf{x}^* \succ \mathbf{y}^*)$ if $z_i(\mathbf{x}^*) \leq z_i(\mathbf{y}^*)$ for all k and $z_j(\mathbf{x}^*) < z_j(\mathbf{y}^*)$ for some j.

According to the symmetric method, the solution algorithm will have the following steps:

a) Step 1 : Fuzzify the objectives: The fuzzication of the objectives consists in determining an acceptable level (or aspiration level) for the achievement of the k objectives. For each of the k objectives, one aspiration level may be obtained when solving k LP problems. Each single-objective problem will determine a lower bound z_j^* from $\min\left\{z_j^* = \mathbf{C}_j.\mathbf{x} \mid \mathbf{B}.\mathbf{x} \geq \mathbf{b}_0, \mathbf{x} \geq 0\right\}$. The jth cost objective will be replaced by the fuzzy objective $\mathbf{C}_j.\mathbf{x} \lesssim z_j^*$. The overall objective will then be to find a solution which minimizes the single worst deviation of $\mathbf{Z}$ from the aspiration level $\mathbf{Z}^*$. Let $\mathbf{A} = \begin{pmatrix} \mathbf{C} \\ -\mathbf{B} \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} \mathbf{Z}^* \\ -\mathbf{b}_0 \end{pmatrix}$, we have the FLP problem:

$$\min \{\text{"the single worst deviation of } \mathbf{Z} \text{ from } \mathbf{Z}^*\text{"}\}$$
$$\text{subject to}$$
$$\mathbf{A}.\mathbf{x} \lesssim \mathbf{b},$$
$$\mathbf{x} \geq 0.$$

b) Step 2 : Define one MF for each objective: Let z_j^* be the lower bound (the best possible value) and z_j' the worst possible value (upper bound), the jth fuzzy objective may be characterized by the following linear MF:

$$\mu_j(z_j) = \begin{cases} 1, \ z_j \geq z_j^*, \ (j \in \mathbb{N}_k) \\ 1 - (z_j^* - z_j)/(z_j^* - z_j'), \ z_j' \leq z_j \leq z_j^* \ (j \in \mathbb{N}_k) \\ 0, \ z_j \leq z_j', \ (j \in \mathbb{N}_k). \end{cases}$$

Let δ_j be the deviation of an isosceles triangular $\tilde{b}_j$, the MF of the jth objective would be defined by

$$\mu_j(\mathbf{C}_j.\mathbf{x}) = \begin{cases} 1 - |(\mathbf{C}_j.\mathbf{x} - b_j)/\delta_k|, \\ \quad \text{if } \mathbf{C}_j.\mathbf{x} \in [b_j - \delta_j, b_j + \delta_j] \ (j \in \mathbb{N}_k) \\ 0, \ \text{otherwise.} \end{cases}$$

c) Step 3 : Solve using the max-min operator: The problem is solved by maximizing the minimum MF. Introducing $\alpha = \bigcap_{j=1}^{k} \mu_j(\mathbf{C}_j.\mathbf{x})$, the FLP problem with linear MFs is converted to the crisp equivalent LP problem:

$$\max_{\mathbf{x},\alpha} \ \alpha$$
$$\text{subject to}$$
$$\mu_j(\mathbf{C}_j.\mathbf{x}) = 1 - \frac{z_j^* - \mathbf{C}_j.\mathbf{x}}{z_j^* - z_j'}, \ (j \in \mathbb{N}_k)$$
$$\mathbf{B}.\mathbf{x} \geq \mathbf{b}_0,$$
$$\mathbf{x}, \alpha \geq 0.$$

B. Parametric programming in fuzzy optimization

Let a vector-maximum linear problem with k fuzzy objectives of n crisp variables and linear constraints, such as

$$\widetilde{\max}_{\mathbf{x}} \ \mathbf{Z}(\mathbf{x}) = \mathbf{C}_j.\mathbf{x} \ (j \in \mathbb{N}_k, \ \mathbf{x} \in \mathbb{R}_+^n, \ \mathbf{C} \in \mathbb{R}^{k \times n})$$
$$\text{subject to}$$
$$\mathbf{A}_i.\mathbf{x} \leq b_i \ (i \in \mathbb{N}_m).$$

This problem may be transformed to a partially FLP problem with only one of the k objective functions, the remaining $k-1$ fuzzy objectives being placed into the set of constraints (according to [3]). Choosing the first objective and transferring the other objectives, we have to consider:

$$\widetilde{\max}_{\mathbf{x}} \ z_1(\mathbf{x}) = \mathbf{C}_1.\mathbf{x},$$
$$\text{subject to}$$
$$\mathbf{C}_j.\mathbf{x} \gtrsim z_j^*, \ (j \in \mathbb{N}_k \setminus \{1\})$$
$$\mathbf{A}_i.\mathbf{x} \leq b_i, \ (i \in \mathbb{N}_m)$$
$$\mathbf{x} \geq 0,$$

where the aspiration level equals the upper value of z_1^* with a tolerance of $z_1^* - z_1'$, the tolerances in fuzzy constraints being $p_l = z_l^* - z_l'$. The MFs of the objectives are defined by

$$\mu_j(\mathbf{C}_j.\mathbf{x}) = \begin{cases} 1, & \mathbf{C}_j.\mathbf{x} \geq z_j^* \\ 1 - (z_j^* - \mathbf{C}_j.\mathbf{x})/(z_j^* - z_j'), & z_j' \leq \mathbf{C}_j.\mathbf{x} \leq z_j^* \\ 0, & \mathbf{C}_j.\mathbf{x} \leq z_j' \end{cases}$$

Then we have to solve the parametric programming problem:

$$\max_{\mathbf{x}} \ z_1(\mathbf{x}) = \mathbf{C}_1.\mathbf{x},$$
$$\text{subject to}$$
$$\mathbf{C}_j.\mathbf{x} \geq z_j^* - \alpha(z_j^* - z_j'), \ (j \in \mathbb{N}_k \setminus \{1\})$$
$$\mathbf{A}_i.\mathbf{x} \leq b_i, \ (i \in \mathbb{N}_m)$$
$$\mathbf{x} \geq 0.$$

This programming technique will provide a fuzzy decision dependent on the preference parameter α.

C. Reexamination of the trade balance problem

Suppose that the DM decides to transfer the second objective (trade balance performance) into the crisp constraints set of the problem. Since we have calculated the objectives intervals $z_1' = 7$ to $z_1^* = 21$ for profits and $z_2' = -4$ to

$z_2^* = 14$ for the performance of the trade balance, we have to solve the crisp problem

$$\max_{\mathbf{x}} \ 2x_1 + x_2$$
$$\text{subject to}$$
$$-x_1 + 2x_2 \geq 14 - 17\alpha,$$
$$-x_1 + 3x_2 \leq 21,$$
$$x_1 + 3x_2 \leq 27,$$
$$4x_1 + 3x_2 \leq 45,$$
$$3x_1 + x_2 \leq 30,$$
$$x_1, x_2 \geq 0.$$

The α-parameterized solutions $\mathbf{x}_\alpha = (x_{1\alpha}, x_{2\alpha})$ are

$$\mathbf{x}_\alpha = \begin{cases} (\frac{48+51\alpha}{11}, \frac{101-68\alpha}{11}), & \frac{6}{17} < \alpha \leq 1 \\ (\frac{12+51\alpha}{5}, \frac{41-17\alpha}{5}), & \frac{1}{17} < \alpha \leq \frac{6}{17} \\ (51\alpha, 7+17\alpha), & 0 < \alpha \leq \frac{1}{17}. \end{cases}$$
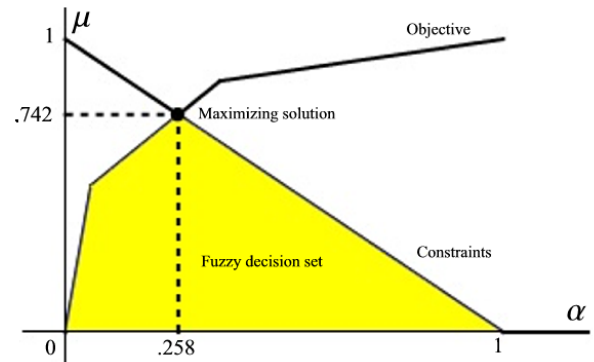

Fig. 6.   Fuzzy decision set

The fuzzy decision set is shown in Fig.6. The Mathematica primitives for this figure are framed hereafter.

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■ frontiers of the fuzzy decision set
  mu01 = 1 - (z1u - c[[1]].x)/(z1u - z1l) /. {x1 -> x1max[[1]], x2 -> x1max[[2]]};
  mu02 = 1 - (z1u - c[[1]].x)/(z1u - z1l) /. {x1 -> x2max[[1]], x2 -> x2max[[2]]};
  mu03 = 1 - (z1u - c[[1]].x)/(z1u - z1l) /. {x1 -> x3max[[1]], x2 -> x3max[[2]]};
  Plot[
    Min[1 - α, Piecewise[{{mu01, 0 < α <= 1/17},
                          {mu02, 1/17 < α <= 6/17},
                          {mu03, 6/17 < α <= 1}}]
    ],
    {α, 0, 2},
    PlotRange -> {{0, 1.1}, {0, 1.1}}, Filling -> Bottom, FillingStyle -> Yellow]
```

The $mu01$, $mu02$ and $mu03$ are the (linear) pieces of a nonlinear objective function with parameter α. The manufacturer's decision is to produce 5.03 of good A and 7.32 of good B, with a satisfaction degree of 74.2 per cent.

III. Fuzzy Goal Programming

A. Goal programming with fuzzy triangular numbers

A fuzzy goal programming (FGP) problem consists [1], [9] in the following problem with fuzzy equality constraints:

$$\text{find} \quad \mathbf{x}$$
$$\text{such that}$$
$$\mathbf{A}_i.\mathbf{x} \cong b_i, \ (i \in \mathbb{N}_m)$$
$$\mathbf{x} \geq 0,$$

where $\mathbf{A}$ states a matrix of unit costs or profit and technical coefficients, and $\mathbf{b}$ a vector of goal and available resources. The MFs $\mu_i(\mathbf{A}_i.\mathbf{x})$ are assumed to be symmetrically triangular such as with

$$\mu_i(\mathbf{A}_i.\mathbf{x}) = \begin{cases} 1 + (\mathbf{A}_i.\mathbf{x} - b_i)/d_i, \ b_i - d_i \leq \mathbf{A}_i.\mathbf{x} < b_i \\ (b_i - \mathbf{A}_i.\mathbf{x})/d_i - 1, \ b_i \leq \mathbf{A}_i.\mathbf{x} \leq b_i + d_i \\ 0, \ \mathbf{A}_i.\mathbf{x} < b_i - d_i \ \text{ or } \ \mathbf{A}_i.\mathbf{x} > b_i + d_i \end{cases}$$

The Mathematica primitives of a triangular fuzzy number (TFN) are framed hereafter. In Fig.7, a TFN is defined by

```
■ data of the triangular fuzzy number TFN = (b, a, c)
  a := 5; b := 10; c := 20;

■ membership function
  mu[x_] = Max[
            Min[(x - a)/(b - a), (c - x)/(c - b)],
            0];
  Graphics[{Dashed, {Line[{{0, 1}, {10, 1}}], Line[{{10, 0}, {10, 1}}]}}];
  Plot[mu[x], {x, 0, 25}, PlotStyle -> {Thick, Blue}]
```

$\tilde{u} = (u, \underline{u}, \bar{u})$ where $\underline{u} \neq \bar{u}$. The crisp number u is the most preferred value, $u - \underline{u}$ is the lower limit pessimistic evaluation and $u + \bar{u}$ is the upper limit optimistic evaluation [6]. According to the MF for some i, the optimization problem is : $\max_{\mathbf{x},\alpha} \left\{ \alpha = \min \left\{ (\mathbf{A}_i.\mathbf{x} - b_i)/d_i - 1 \right\} \right\}$ such that $b_i - d_i \leq \mathbf{A}_i.\mathbf{x} < b_i, \mathbf{x} \geq 0$. The complete auxiliary program
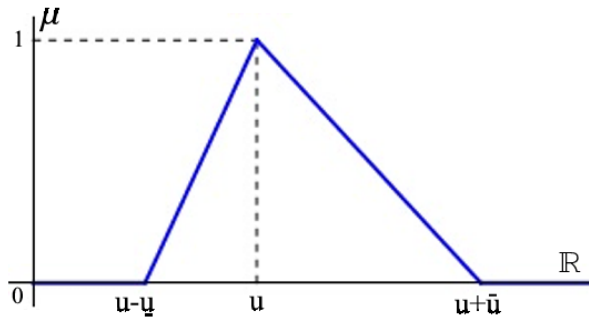

Fig. 7.   TFN $\tilde{u} = (u, \underline{u}, \bar{u})$

[6]The Fuzzy Logic subpackage provides a number of functions to create particular fuzzy sets. Creating a triangular MF is achieved by **FuzzyTrapezoid[a,b,c,d]**. Hence, we will have here **FuzzyTrapezoid[5,10,10,20]** for the TFN $\tilde{u} = (10, 5, 10)$.

is

$$\max_{\mathbf{x},\alpha} \quad \alpha$$
$$\text{subject to}$$
$$1 + (\mathbf{A}_i.\mathbf{x} - b_i)/d_i \geq \alpha,$$
$$(b_i - \mathbf{A}_i.\mathbf{x})/d_i - 1 \geq \alpha,$$
$$b_i - d_i \leq \mathbf{A}_i.\mathbf{x} \leq b_i + d_i,$$
$$\alpha \in (0, 1], \ \mathbf{x} \geq 0.$$

B. Fuzzy weighted goal programming problem

For the FGP problem "find the variables $\mathbf{x}$ to satisfy the goals $g_i(\mathbf{x}) \gtrsim g_i$, subject to the constraints $\mathbf{A}.\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0$", we may have the weighted additive model [8], [13], [19]

$$\max_{\mathbf{x}} V(\mu) = \sum_{i=1}^{k} w_i \mu_i \text{ such that } \sum_{i=1}^{k} w_i = 1$$
$$\text{subject to}$$
$$\mu_i = (g_i(\mathbf{x}) - L_i^0)/(g_i - L_i^0), \ (i \in \mathbb{N}_k)$$
$$\mathbf{A}.\mathbf{x} \leq \mathbf{b},$$
$$\mathbf{x} \geq 0, \mu_i \in [0, 1] \ (i \in \mathbb{N}_k),$$

where $V(\mu)$ is a weighted decision function, L_i^0 the lower tolerance limit of the fuzzy goal $g_i(\mathbf{x}) \lesssim g_i$. DMs may be able to provide relative weights w_i for the fuzzy goals. Their importance will be reflected in the composite MF $\mu_{w_i}(x)$, $x \in \mathbb{R}$. The weighted contribution of the ith goal is denoted by $\mu_{w_i}(\mu_i(x))$. Using the min-operator, the decision set's MF will be $\mu_{\tilde{D}}(x) = \bigwedge_{i=1}^{m} \mu_{w_i}(\mu_i(x))$, $x \in \mathbb{R}$. The maximal decision $\mathbf{x}^*$ is achieved for $\max_{\mathbf{x} \geq 0} \{ \min_i \{ (\mu_{w_i} \circ \mu_i)(\mathbf{A}_i.\mathbf{x}) \} \}$.

C. Production - Marketing Problem with fuzzy weights

A manufacturer (taken from [12]) produces two goods A and B, whose unit profit are respectively 80 and 40. According to a marketing survey, the sales for A and B would be about 6 for A and about 4 for B, with an identical maximum deviation of 2. Moreover, the DMs anticipate a profit about 630 with a maximum deviation of 10. The FGP problem of this example is

$$\text{find} \quad \mathbf{x}$$
$$\text{such that}$$
$$g_1(\mathbf{x}) \equiv A_1.\mathbf{x} = 80x_1 + 40x_2 = \widetilde{630},$$
$$g_2(\mathbf{x}) \equiv A_2.\mathbf{x} = x_1 = \widetilde{6},$$
$$g_3(\mathbf{x}) \equiv A_3.\mathbf{x} = x_2 = \widetilde{4},$$
$$\mathbf{x} \geq 0,$$

where g_1 is the profit, g_2 and g_3 the sales expectations. Let the MFs of the fuzzy priorities be defined by:

$$\mu_{w_1}(x) = \begin{cases} 5\mu_1(x) - 4, \ .8 \leq \mu_1(x) \leq 1 \\ 0, \ \text{otherwise.} \end{cases}$$

$$\mu_{w_i}(x) = \begin{cases} 0, \ \mu_{w_i} < .6, \ i = 2, 3 \\ 5\mu_i(x) - 3, \ .6 \leq \mu_i(x) \leq .8, \ i = 2, 3 \\ 1, \ \mu_i(x) > .8, \ i = 2, 3 \end{cases}$$

With the triangular MFs $\mu(g_1) = (630, 10, 10)$, $\mu(g_2) = (6, 2, 2)$ and $\mu(g_3) = (4, 2, 2)$, we then obtain the composite

MFs of the Fig.8. The Mathematica primitives of this figure are framed hereafter.

```
■ data of the triangular fuzzy number
  Clear[a, b, c]; a := 620; b := 630; c := 640;

■ membership function
  mu1[x_] = Max[Min[(x - a)/(b - a), (c - x)/(c - b)], 0];

■ membership function of the weight
  muw1[x_] = Piecewise[{
                        {(x - .8)/.2, .8 <= x <= 1},
                        {0, 0 <= x < .8}
                      }];

■ composition of the membership functions
  Composition[muw1, mu1][x] // Simplify
```

```
⎡ 316. - 0.5 x    630. ≤ x ≤ 632.
⎢ -314. + 0.5 x   628. ≤ x < 630.
⎣ 0               True
```

$$(\mu_{w_1} \circ \mu_1)(\mathbf{A}_1.\mathbf{x}) = \begin{cases} 316 - 40x_1 - 20x_2, \\ \quad\quad 630 < 80x_1 + 40x_2 \le 632 \\ -314 + 40x_1 + 20x_2, \\ \quad\quad 628 < 80x_1 + 40x_2 \le 630 \\ 0, \text{ elsewhere.} \end{cases}$$

$$(\mu_{w_2} \circ \mu_2)(\mathbf{A}_2.\mathbf{x}) = \begin{cases} 1, \ 5.6 < x_1 \le 6.4 \\ 17 - 2.5x_1, \ 6.4 \le x_1 \le 6.8 \\ -13 + 2.5x_1, \ 5.2 \le x_1 \le 5.6 \\ 0, \text{ elsewhere.} \end{cases}$$

$$(\mu_{w_3} \circ \mu_3)(\mathbf{A}_3.\mathbf{x}) = \begin{cases} 1, \ 3.6 < x_2 \le 4.4 \\ 12 - 2.5x_2, \ 4.4 \le x_2 \le 4.8 \\ -8 + 2.5x_2, \ 3.2 \le x_2 \le 3.6 \\ 0, \text{ elsewhere.} \end{cases}$$
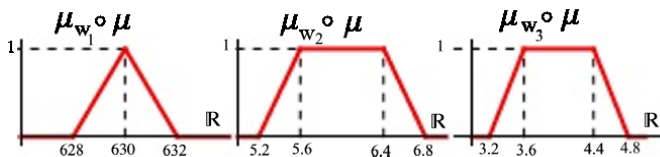
The auxiliary crisp goal programming is

Fig. 8. Composite MFs $\mu_w \circ \mu$

$$\max_{\mathbf{x},\alpha} \quad \alpha$$
$$\text{subject to}$$
$$316 - 40x_1 - 20x_2 \ge \alpha,$$
$$-314 + 40x_1 + 20x_2 \ge \alpha,$$
$$17 - 2.5x_1 \ge \alpha,$$
$$-13 + 2.5x_1 \ge \alpha,$$
$$12 - 2.5x_2 \ge \alpha,$$
$$-8 + 2.5x_2 \ge \alpha,$$
$$0 < \alpha \le 1, \ x_1 \ge 0, \ x_2 \ge 0.$$

The company will then produce $x_1^* = 6.1$ of good A and $x_2^* = 3.6$ of good B with a plenty satisfactory level ($\alpha^* = 1$). This company earns a profit of 630.

IV. CONCLUSION

This introductive self-contained presentation is deliberately application-oriented. It aims to be an useful and attractive support for academicians and practitioners. The basic techniques with some extensions and simple examples introduce to a vast and various domain of Operations Research. This presentation may be easily transposed to other examples and larger concrete situations of the real world. The mathematical software $Mathematica^{®}$ with its $Fuzzy\ Logic$ 2 package, will be adequate for these practical developments. Some primitives of the software are given for illustrating the power of such a software.

APPENDIX

A. Single objective reference model

1) Elementary single-objective FLP problem: Let the maximizing LP problem with fuzzy (imprecise) resources be

$$\max_{\mathbf{x}} \quad \mathbf{c}^\top.\mathbf{x}, \quad (\mathbf{c}, \mathbf{x} \in \mathbb{R}^n)$$
$$\text{subject to}$$
$$\mathbf{A}_i.\mathbf{x} \le \tilde{b}_i, \ (i \in \mathbb{N}_m)$$
$$\mathbf{x} \ge 0.$$

In a production scheduling problem, $\mathbf{c}$ will denote the n costs, the $m \times n$ matrix $\mathbf{A}$ technical coefficients, $\mathbf{b}$ the m resources and $\mathbf{x}$ the n variables.

2) Solution algorithm: The algorithm of the symmetric method consists in the following steps

a) Step 1 : Define the MFs and determine the fuzzy feasible set: Let the ith resource b_i being defined by the interval $[b_i, b_i + p_i]$ with tolerance p_i. The MFs of the fuzzy $\tilde{b}_i$ are of the ramp-type ($x \in \mathbb{R}$):

$$\mu_i(x) = \begin{cases} 1, x \ \le b_i \\ 1 - (x - b_i)/p_i, \ b_i \le x \le b_i + p_i \\ 0, \ x \ge b_i + p_i. \end{cases}$$

The degree $D_i(\mathbf{x})$ to which $\mathbf{x}$ satisfies the ith non rigid $\tilde{C}_i$ constraint is then $\mu_i(\mathbf{A}_i.\mathbf{x})$. All the μ_i's define fuzzy sets on $\mathbb{R}^n$ and the MF of fuzzy feasible set is $\bigwedge_{i=1}^{m} \mu_i(\mathbf{A}_i.\mathbf{x})$.

b) Step 2 : Define the fuzzy set of the optimal values:
The objective admits a lower and an upper bound respectively
equal to: $\max \left\{ z' = \mathbf{c}^\top . \mathbf{x} \mid \mathbf{A}_i . \mathbf{x} \le b_i \ (i \in \mathbb{N}_m), \ \mathbf{x} \ge 0 \right\}$ and
$\max \left\{ z^* = \mathbf{c}^\top . \mathbf{x} \mid \mathbf{A}_i . \mathbf{x} \le b_i + p_i \ (i \in \mathbb{N}_m), \ \mathbf{x} \ge 0 \right\}$. The
MF of the single fuzzy objective $\tilde{G}$ is then defined $(x \in \mathbb{R})$
by

$$
\mu_{\tilde{G}}(\mathbf{c}^\top . \mathbf{x}) = \begin{cases} 1, & \mathbf{c}^\top . \mathbf{x} \ge b_i \\ (\mathbf{c}^\top . \mathbf{x} - z')/(z^* - z'), & z' < \mathbf{c}^\top . \mathbf{x} < z^* \\ 0, & \text{otherwise.} \end{cases}
$$

c) Step 3: Solve using the max-min operator: The prob-
lem $\max \left(\bigcap_{i=1}^m C_i \right) \cap \tilde{G}$ is described by the equivalent crisp
LP [4], [10]:

$$\max_{\mathbf{x},\alpha} \alpha$$
$$\text{subject to}$$
$$\mu_{\tilde{G}}(\mathbf{c}^\top \mathbf{x}) = (\mathbf{c}^\top . \mathbf{x} - z')/(z^* - z') \ge \alpha \,,$$
$$\mu_i(\mathbf{A}_i \mathbf{x}) = 1 - (\mathbf{A}_i . \mathbf{x} - b_i)/p_i \ge \alpha, \ (i \in \mathbb{N}_m)$$
$$\mathbf{x}, \alpha \ge 0.$$

3) Extension to nonsymmetric triangular fuzzy numbers:
Suppose that all the coefficients of the constraints are nonsym-
metric TFNs $\tilde{a}_{ij} = (a_{ij}, a_{ij} - \underline{a}_{ij}, a_{ij} + \bar{a}_{ij})$, and $\tilde{b}_i = (b_i, b_i - \underline{b}_{ij}, b_i + \bar{b}_i)$. According to the operations on the TFNs (using
a simple partial order such that $\tilde{u} \le \tilde{v} \Leftrightarrow \max\{\tilde{u}, \tilde{v}\} = \tilde{v}$), we
have to solve the following crisp LP:

$$\max_{\mathbf{x}} \mathbf{c}^\top . \mathbf{x}$$
$$\text{subject to}$$
$$\mathbf{A}_i . \mathbf{x} \le b_i, \ (i \in \mathbb{N}_m)$$
$$(\mathbf{A}_i - \underline{\mathbf{A}}_i) . \mathbf{x} \le b_i - \underline{b}_i, \ (i \in \mathbb{N}_m)$$
$$(\mathbf{A}_i + \bar{\mathbf{A}}_i) . \mathbf{x} \le b_i + \bar{b}_i, \ (i \in \mathbb{N}_m)$$
$$\mathbf{x} \ge 0.$$

4) Extension to soft constraints: Let a maximizing problem
with triangular coefficients (excluding the objective function)
and soft constraints. We have

$$\max_{\mathbf{x}} \mathbf{c}^\top . \mathbf{x} \ (\mathbf{c}, \mathbf{x} \in \mathbb{R}^n)$$
$$\text{subject to}$$
$$\tilde{\mathbf{A}}_i . \mathbf{x} \lesssim \tilde{b}_i, \ (i \in \mathbb{N}_m)$$
$$\mathbf{x} \ge 0.$$

A DM will admit that the constraints may not be completely
satisfied. The FNs $\tilde{t}_i$ will represent the allowed maximum
violation of the *i*th constraint. Let $<_\Re$ be a ranking relation
between FNs, the auxiliary parametric LP problem is

$$\max_{\mathbf{x}} \mathbf{c}^\top . \mathbf{x} \ (\mathbf{c}, \mathbf{x} \in \mathbb{R}^n)$$
$$\text{subject to}$$
$$\tilde{\mathbf{A}}_i . \mathbf{x} <_\Re \tilde{b}_i + \tilde{t}_i(1 - \alpha), \ (i \in \mathbb{N}_m)$$
$$\mathbf{x} \ge 0, \ \alpha \in (0, 1].$$

The DM may choose different rules to compare FNs [6], [16],
[17]. For ordering the FNs taken from $\mathfrak{F}(\mathbb{R})$, a ranking function
$\Re$ is defined, which maps each FN into the real line such as

$\Re : \mathfrak{F}(\mathbb{R}) \mapsto \mathbb{R}$. We can then define some orders on $\mathfrak{F}(\mathbb{R})$ by
the following rules $\Re_1$ and $\Re_2$: $\tilde{x} <_{\Re_1} \tilde{y} \Leftrightarrow x \le y$ (rule 1)
or $\tilde{x} <_{\Re_2} \tilde{y} \Leftrightarrow \bar{x} \le \underline{y}$ (rule 2). Different solutions will then
be obtained according to each rule.

5) Numerical example: This numerical example is due to
Delgado et al.[5]. The FLP problem is

$$\max_{x_1,x_2} \quad z = 5x_1 + 6x_2$$
$$\text{subject to}$$
$$\tilde{3}x_1 + \tilde{4}x_2 \lesssim \widetilde{18},$$
$$\tilde{2}x_1 + \tilde{1}x_2 \lesssim \tilde{7},$$
$$x_1, x_2 \ge 0.$$

The data may take the form of tensors, such as in Mathematica
in the frame hereafter. According to the rule that the DM

will choose, two different auxiliary problems and solutions
are obtained. We have
rule 1 : $\tilde{x} <_{\Re_1} \tilde{y} \Leftrightarrow x \le y$.

$$\max_{x_1,x_2} \quad z = 5x_1 + 6x_2$$
$$\text{subject to}$$
$$3x_1 + 4x_2 \le 18 + 3(1 - \alpha),$$
$$2x_1 + x_2 \le 7 + (1 - \alpha),$$
$$x_1, x_2 \ge 0, \ \alpha \in (0, 1]$$

The parametrized solution with rule 1 given by Mathematica
is shown hereafter.
rule 2 : $\tilde{x} <_{\Re_2} \tilde{y} \Leftrightarrow \bar{x} \le \underline{y}$.

$$\max_{x_1,x_2} \quad z = 5x_1 + 6x_2$$
$$\text{subject to}$$
$$4x_1 + 5.5x_2 \le 16 + 2.5(1 - \alpha),$$
$$3x_1 + 2x_2 \le 6 + .5(1 - \alpha),$$
$$x_1, x_2 \ge 0, \ \alpha \in (0, 1]$$

The parametrized solution with rule 2 given by Mathematica
is shown hereafter.

Fig. 10. Different types A to F of fuzzy sets

B. Extensions of the basic multiobjective model

Any S-shaped piecewise linear MF can be expressed uniquely as a union of only concave subfunctions [14], [21]. A piecewise linear concave MF of type I (union of concave subfunctions) is represented in Fig.9A. This nonlinear MF is the intersection of two ramp-type MFs in Fig.9B and Fig.9C. Let $b'_{1k} = b_{1k} - \delta_{1k}$, $b'_{2k} = b_{2k} - \delta_{2k}$, where δ_{1k} and δ_{2k}
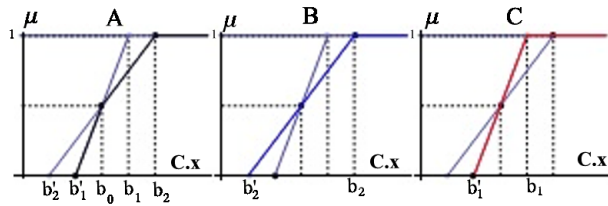

Fig. 9. Concave piecewise linear MF decomposition

denote deviations. The nonlinear concave MFs will then be written

$$\mu_k(\mathbf{C}_k.\mathbf{x}) = \begin{cases} 1, & \mathbf{C}_k.\mathbf{x} \geq b_{2k} \\ 1 - (b_{2k} - \mathbf{C}_k.\mathbf{x})/\delta_{2k}, & \mathbf{C}_k.\mathbf{x} \in [b_{0k}, b_{2k}), \\ 1 - (b_{1k} - \mathbf{C}_k.\mathbf{x})/\delta_{1k}, & \mathbf{C}_k.\mathbf{x} \in [b_{1k} - \delta_{1k}, b_k), \\ 0, & \text{otherwise.} \end{cases}$$

C. Fuzzy Logic package in Mathematica

The *Fuzzy Logic 2* package [18] allows the Mathematica software to work with fuzzy sets and fuzzy relations. This introductive presentation will be limited to our examples. It will be centered on the following two classes of questions : How to create, to plot and to intersect fuzzy sets ? How to create, to plot and to intersect fuzzy relations ?

1) Fuzzy sets: The Fuzzy Logic package provides a number of functions to create different types of fuzzy sets. A fuzzy set object is of the form : FuzzySet$[\{\{x_1, \mu(x_1)\}, \ldots, \{x_n, \mu(x_n)\}\}]$, where $x_1, \ldots, x_n$ are the elements of the universal space and $\mu(x_1, \ldots, \mu(x_n)$ the membership grades.

The framed primitives for plotting such fuzzy sets are shown thereafter. The types A to F of fuzzy sets are shown in Fig.10. The fuzzy set A corresponds to a user-defined ramp-type MF. The fuzzy set B has a sigmoidal MF with parameters c and s, where s controls the slope at the crossover point c. The fuzzy set C shows a bell-shaped MF centered at c, with crossover points $c \pm w$ and slopes of $\pm s/(2w)$ at these points. The fuzzy set D has a trapezoidal MF with parameters a,b,c,d in increasing order, and h in the interval $[0,1]$. The fuzzy sets E and F have a triangular MF with two presentations since the plot may be joined or unjoined. The primitives corresponding
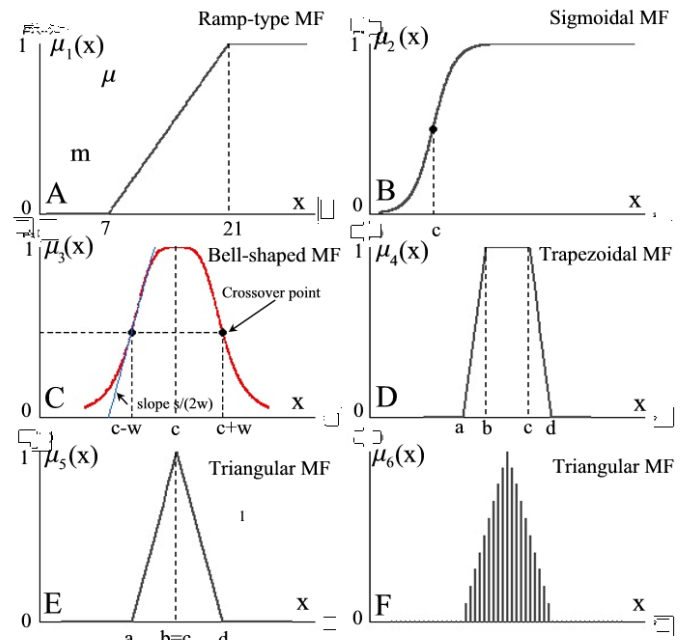

to such cases are framed hereafter. The intersection of the fuzzy sets FS2 and FS3 (as in Fig.1) is shown in Fig.11, using the standard type Min [7]. The Mathematica Epilog function shows the plot of the original fuzzy sets in Fig.11.

2) Fuzzy relations: A fuzzy relation R is defined in the Cartesian product of two fuzzy sets V and W. The product $V \times W$ takes values from the closed interval $[0,1]$. A fuzzy relation can be represented by pairs of elements and membership pairs $R = \left\{ \{\{v_i, w_j\}, R(v_i, w_j)\}, i \in \mathbb{N}_n, j \in \mathbb{N}_m \right\}$. The elements are ordered pairs $\{v_1, w_1\}, \{v_1, w_2\}, \ldots, \{v_n, w_m\}$. These elements are again grouped with their membership grades $R_{11}, R_{12}, \ldots, R_{nm}$. The MFs of the trade balance problem may illustrate these fuzzy relations. For the first objective, a linear piecewise function is defined and a first fuzzy relation R_1 is created as with the framed primitive hereafter. The surface plot of the first fuzzy relation is shown in the 3D Fig.12. For the second objective, a linear piecewise function is also defined and a second fuzzy relation R_2 is

---

[7]Other intersection types are given in Fuzzy Logic : **Hamacher[v]**, **Frank[s]**, **Yager[w]**, **DuboisPrade[a]** or **Dombi[alpha]**.

```
■ Plotting the fuzzy set with
  user - defined membership grades
  plot1 = FuzzyPlot[FS1, ColorOutput → GrayLevel,
       Ticks → None, PlotJoined → True,
             AxesLabel → {x, μ₁(x)}];
```

```
■ Plotting the first fuzzy relation
  FuzzySurfacePlot[rel1,Ticks→None,ColorOutput→RGBColor,
       MeshStyle→{Thickness[.0008],CMYKColor[1,1,1,1]},PlotPoints→300,
       ViewPoint->{1.240, -2.280, 1.750},Boxed→True,Axes→True,
           BoxRatios→{1,1,.5}]CreateFuzzyRelation[mu1,
             {{1,10,.25},{1,10,.25}},
               UniversalSpace->{{0,12,.25},{0,12,.25}}]
```

```
■ Creating a sigmoidal fuzzy set
  FS2=FuzzySigmoid[
          11,1,UniversalSpace→{6,30,.01}];

■ Creating a bell-shaped fuzzy set
  FS3=FuzzyBell[
          10,1,2,UniversalSpace→{8,12,.01}];

■ Creating a trapezoidal fuzzy set
  FS4=FuzzyTrapezoid[
          5.2,5.6,6.4,6.8,1,UniversalSpace→{4.5,7.5,.1}];

■ Creating a triangular fuzzy set
  FS5=FuzzyTrapezoid[
          628,630,630,632,1,UniversalSpace→{625,635,1}];
```
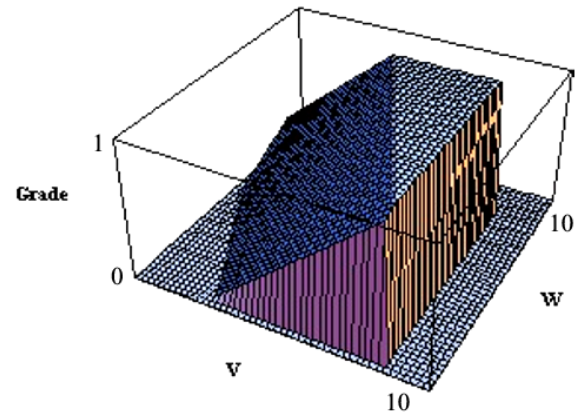

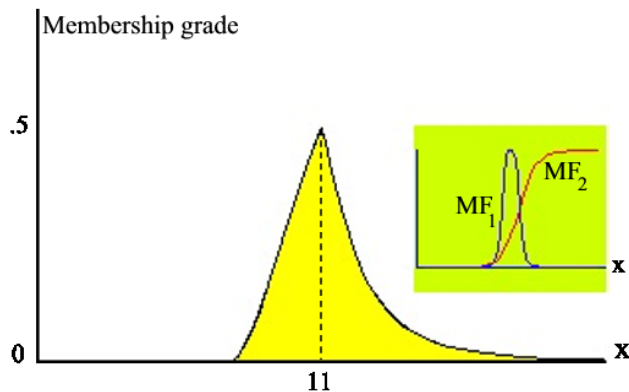Fig. 12.   Surface plot of the first fuzzy relation

Fig. 11.   Min intersection of fuzzy sets

created as with the framed primitive hereafter. The surface

```
■ Defining a second membership function
  mu2[x1_,x2_]=Piecewise[[
          {0,- x1+2x2< -3},
          {(3- x1+2 x2)/17,-3≤ - x1+2 x2≤ 14},
          {1,- x1+2 x2>14}
            }]//Simplify
```

$$\begin{cases} 0 & x1 > 3 + 2\,x2 \\ \frac{1}{17}\,(3 - x1 + 2\,x2) & -14 \le x1 - 2\,x2 \le 3 \\ 1 & \text{True} \end{cases}$$

```
■ Creating a second fuzzy relation
  rel2=CreateFuzzyRelation[mu2,
          {{1,10,.25},{1,10,.25}},
              UniversalSpace->{{0,12,.25},{0,12,.25}}]
  FuzzyRelation[{{{1,1},0.235294},{{1,1.25},0.264706},...
                    (first two elements of the output)
```

```
■ Defining a first membership function
  mu1[x1_,x2_]=Piecewise[[
          {0,2 x1+x2< 7},
          {(-7+2 x1+x2)/14,7≤ 2 x1+x2≤ 21},
          {1,2 x1+x2>21}
            }]//Simplify
```

$$\begin{cases} 0 & 2\,x1 + x2 < 7 \\ \frac{1}{14}\,(-7 + 2\,x1 + x2) & 7 \le 2\,x1 + x2 \le 21 \\ 1 & \text{True} \end{cases}$$

```
■ Creating a first fuzzy relation
  rel1=CreateFuzzyRelation[mu1,
          {{1,10,.25},{1,10,.25}},
              UniversalSpace->{{0,12,.25},{0,12,.25}}]
  FuzzyRelation[{{{1,5.25},0.0178571},{{1,5.5},0.0357143},...
                    (first two elements of the output)
```

plot of the second fuzzy relation is shown in the 3D Fig.13. The intersection of the two relations in $V \times W$ is a fuzzy relation such that for all (u, v), we have $(R_1 \cap R_2)(v, w) = \min\{R_1(v, w), R_2(v, w)\}$. The surface plot of the intersection is shown in the 3D Fig.14.

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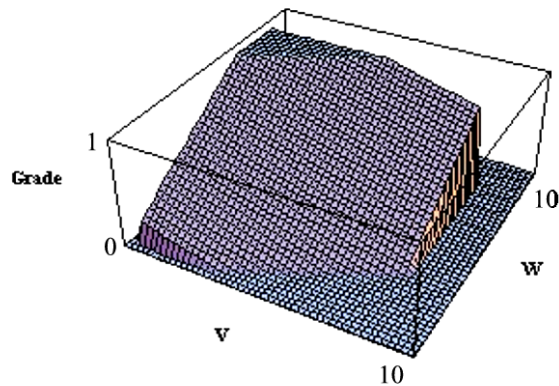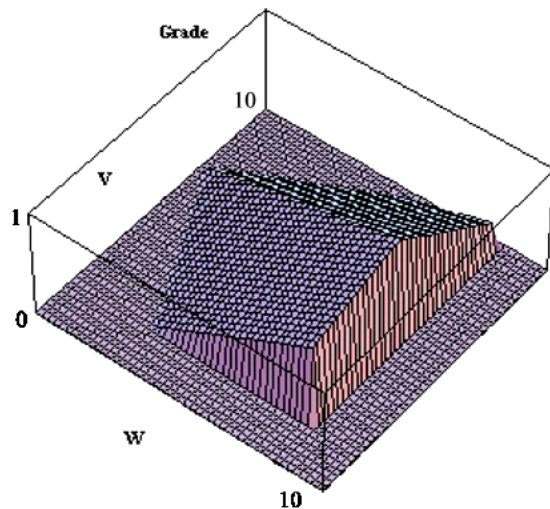
Fig. 13.   Surface plot of the second fuzzy relation

Fig. 14.   Intersection of the two fuzzy relations

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