Minimal Realization Algorithm for Multidimensional Hybrid Systems

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Abstract: A class of multidimensional time-invariant hybrid systems is studied, which depends on \( q \) continuous-time variables and on \( r \) discrete-time ones. The formula of the input-output map of these systems is given. A multiple \((q,r)\)-hybrid Laplace transformation is defined and it is used to determine the transfer matrices of the considered systems. The minimal realization problem is analysed and an algorithm is given which provides minimal realizations.

Key-Words: multidimensional hybrid systems, multiple hybrid Laplace transformation, minimal realization, controllability, reachability, time-invariant systems.

1 Introduction

The topic of 2D systems (and more generally, of \( n \)D systems) has known an important development in the last three decades. Their theory has become a distinct and significant branch of the Systems and Control Theory, due to their richness in theoretical approaches, as well as in potential application fields, such as control and signal processing, circuits, image processing, computer tomography, seismology etc.

A quite new direction in this theory is represented by the 2D hybrid (i.e. continuous-discrete) systems, whose state-space representation includes a differential-difference equation [5], [7], [8], [9]. These hybrid models are motivated by their applications in various areas such as long-wall coal cutting and metal rolling [12], linear repetitive processes [3], [11], pollution modelling [2] or in iterative learning control synthesis [6].

In the present paper a multidimensional hybrid model is studied. The state equation is a partial differential-difference equation, the states, the controls and the outputs being vector functions which depend on \( q \geq 1 \) continuous-time variables and \( r \geq 1 \) discrete-time variables. This model is an extension of the Attasi’s 2D discrete-time system [1].

In Section 2 the state space representation is introduced for the \((q,r)\) multidimensional hybrid control systems, and the formulas of the state and of the input-output map of these systems are derived from a variation-of-parameters type formula.

In Section 3 a multiple \((q,r)\)-hybrid Laplace transformation is introduced as an operator defined on a class of suitable original functions. Some properties of this transformation are emphasized such as linearity, first and second time-delay theorems, differentiation of the original and differentiation and delay. These properties are used to obtain the transfer matrix of the considered class of systems. It is shown that the transfer matrix is a separable proper rational function, represented by a sum between a strictly proper matrix and a constant one.

Section 4 is devoted to the minimal realization problem. It is shown that a realization of a rational function having the above described structure is minimal if and only if it is a completely reachable and completely observable hybrid system. A criterion of minimality is obtained which is based on the Markov parameters of the strictly proper matrix. A sequence of block Hankel matrices is constructed by recurrence and two families of shift operators are defined. These devices are used to obtain an algorithm which provides a minimal realization. This method is an extension to multivariable hybrid systems of the Ho-Kalman algorithm [4].

The following notations are used in the paper: \( q \in \mathbb{N} \) and \( r \in \mathbb{N} \) being the number of continuous and discrete variables respectively, a function \( x(t_1, \ldots, t_q; k_1, \ldots, k_r) \), \( t_i \in \mathbb{R} \), \( k_i \in \mathbb{Z} \) will be sometimes denoted by \( x(t; k) \), where \( t = (t_1, \ldots, t_q) \), \( k = (k_1, \ldots, k_r) \). By \( s \leq t \), \( s, t \in \mathbb{R}^q \) we mean \( s_i \leq t_i \ \forall i \in \bar{q} \) where \( \bar{q} = \{1, 2, \ldots, q\} \) and a similar signification has \( l \leq k \), \( l, k \in \mathbb{Z}^r \); \( s;l) < (t;k) \) means
s \leq t$, $(l \leq k)$ and $(s; l) \neq (t; k)$. For $t^0, t^1 \in \mathbb{R}^q$ and $k^0, k^1 \in \mathbb{Z}$, $t^0 < t^1$, $k^0 < k^1$ we denote by $[t^0, t^1]$ and $[k^0, k^1]$ respectively the sets $\prod_{i=1}^{q} [t^0_i, t^1_i]$ and $\prod_{i=1}^{r} \{k^0_j, k^0_j + 1, \ldots, k^1_j\}$.

If $\tau = \{i_1, \ldots, i_l\}$ is a subset of $\mathbb{m}$, $|\tau| = l$ and $\tilde{\tau} = \mathbb{m} \setminus \tau$, for $i \in \mathbb{m}$, $\tilde{i} := \mathbb{m} \setminus \{i\}$ and $\tilde{t} := \{i + 1, \ldots, m\}$. The notation $(\tau, \delta) \subset (\bar{\eta}, \bar{\tau})$ means that $\tau$ and $\delta$ are subsets of $\bar{\eta}$ and $\bar{\tau}$ respectively and $(\tau, \delta) \neq (\bar{\eta}, \bar{\tau})$. For $\tau = \{i_1, \ldots, i_l\}$ and $\delta = \{j_1, \ldots, j_l\}$ the operators $\frac{\partial}{\partial \tau}$ and $\sigma_{\delta}$ are defined by

$$\frac{\partial}{\partial \tau} x(t; k) = \frac{\partial^l}{\partial t_{i_1} \cdots \partial t_{i_l}} x(t; k),$$

$$\sigma_{\delta} x(t; k) = x(t; k + e_{\delta})$$

where $e_{\delta} = e_{i_1} + \ldots + e_{i_l}$, $e_j = (0, 0, \ldots, 0)$ in $\mathbb{R}^r$;

when $\tau = \bar{\eta}$ and $\delta = \bar{\tau}$ we denote $\partial / \partial \tau = \partial / \partial t$ and $\sigma_{\delta} = \sigma$.

If $A_i, i \in \mathbb{m}$ is a family of matrices, $\sum_{i \in \emptyset} A_i = 0$ and $\prod_{i \in \emptyset} A_i = I$.

### 2 State space representation

The time set of the hybrid multidimensional system is $T = \mathbb{R}^q \times \mathbb{Z}^r$, $q, r \in \mathbb{N}^*$. The operators $\frac{\partial}{\partial t}$ and $\sigma_{\delta}$ are defined by

$$\frac{\partial}{\partial \tau} x(t; k) = \frac{\partial^l}{\partial t_{i_1} \cdots \partial t_{i_l}} x(t; k),$$

$$\sigma_{\delta} x(t; k) = x(t; k + e_{\delta})$$

where $e_{\delta} = e_{i_1} + \ldots + e_{i_l}$, $e_j = (0, 0, \ldots, 0)$ in $\mathbb{R}^r$;

when $\tau = \bar{\eta}$ and $\delta = \bar{\tau}$ we denote $\partial / \partial \tau = \partial / \partial t$ and $\sigma_{\delta} = \sigma$.

If $A_i, i \in \mathbb{m}$ is a family of matrices, $\sum_{i \in \emptyset} A_i = 0$ and $\prod_{i \in \emptyset} A_i = I$.

### 2.1 Definition 2.1

A $(q, r)$-D hybrid system is a set $\Sigma = \{A_{ci}, i \in \bar{\eta}\}, \{A_{dj}, j \in \bar{\tau}\}, B, C, D$ with $A_{ci}, i \in \bar{\eta}$ and $A_{dj}, j \in \bar{\tau}$ commuting $n \times n$ matrices $\forall t \in \mathbb{R}^q$, $\forall k \in \mathbb{Z}^r$ and $B, C, D$ respectively $n \times m$, $p \times n$ and $p \times m$ real matrices; the state equation is

$$\frac{\partial}{\partial t} \sigma x(t; k) = \sum_{(\tau, \delta) \subset (\bar{\eta}, \bar{\tau})} (-1)^{q+r-|\tau|-|\delta|-1} \times$$

$$\times \left( \prod_{i \in \bar{\eta}} A_{ci}, \prod_{j \in \bar{\tau}} A_{dj} \right) \frac{\partial}{\partial \tau} \sigma_{\delta} x(t; k) + Bu(t; k)$$

and the output equation is

$$y(t; k) = Cx(t; k) + Du(t; k)$$

where $x(t; k) = x(t_1, \ldots, t_q; k_1, \ldots, k_r) \in \mathbb{R}^n$ is the state, $u(t; k) \in \mathbb{R}^m$ is the input and $y(t; k) \in \mathbb{R}^p$ is the output. The number $n$ is called the dimension of the system $\Sigma$ and it is denoted $\dim \Sigma$.

For $\tau = \{i_1, \ldots, i_l\} \subset \bar{\eta}$, $\delta = \{j_1, \ldots, j_l\} \subset \bar{\tau}$ and $t_i \in \mathbb{R}$, $i \in \tau$, $k_j \in \mathbb{Z}$, $j \in \delta$, we use the notation $x(t_{\tau}; k_\delta) := x(s_{i_1}, s_{i_2}; t_1, \ldots, t_l)$, where

$$s_i = \begin{cases} t_i & \text{if } i \in \tau \text{ and } \tilde{t}_j = \begin{cases} k_j & \text{if } j \in \delta \\ 0 & \text{if } j \in \tilde{\delta} \end{cases} \end{cases}$$

### Definition 2.2

The vector $x^0 \in \mathbb{R}^n$ is called an initial state of the system $\Sigma$ if

$$x(t_{\tau}; k_\delta) = \left( \prod_{i \in \tau} e^{A_{ci}t_i} \right) \left( \prod_{j \in \delta} A_{dj}^{k_j} \right) x^0$$

(2.3)

for any $(\tau, \delta) \subset (\bar{\eta}, \bar{\tau})$; equalities (2.3) are called the initial conditions of $\Sigma$.

We can prove (see [9])

### Proposition 2.3

The solution of the initial value problem

$$\frac{\partial}{\partial t} \sigma x(t; k) = \sum_{(\tau, \delta) \subset (\bar{\eta}, \bar{\tau})} (-1)^{q+r-|\tau|-|\delta|-1} \times$$

$$\cdot \left( \prod_{i \in \bar{\eta}} A_{ci} \right) \left( \prod_{j \in \bar{\tau}} A_{dj} \right) \frac{\partial}{\partial \tau} \sigma_{\delta} x(t; k) + f(t; k)$$

with the initial conditions (2.3) is given by the generalized variation-of-parameters formula

$$x(t; k) = \left( \prod_{i=1}^{q} e^{A_{ci}t_i} \right) \left( \prod_{j=1}^{r} A_{dj}^{k_j} \right) x^0 +$$

$$+ \int_{t_0}^{t_1} \cdots \int_{t_q}^{t_1} e^{A_{ci}(t_i-s_i)} \cdot$$

$$\cdot \left( \prod_{i=0}^{k_0} \sum_{l=0}^{k_l} \sum_{j=1}^{r} A_{dj}^{k_j-l_j-1} \right) f(s; l) ds_1 \ldots ds_q;$$

(2.5)

here $s = (s_1, \ldots, s_q)$, $l = (l_1, \ldots, l_r)$; $f : \mathbb{R}^3 \times \mathbb{Z}^r \rightarrow \mathbb{R}^p$ is a continuous function with respect to $t = (t_1, \ldots, t_q)$.

### Theorem 2.4

The state of the system $\Sigma$ (2.1) determined by the initial state $x_0 \in \mathbb{R}^n$ and the control
Multiple \((q, r)\)-hybrid Laplace transformation and transfer matrices

Definition 3.1. A function \(f : \mathbb{R}^q \times \mathbb{Z}^r \to \mathbb{C}\) is said to be a continuous-discrete original function (or simply an original) if \(f\) has the following properties:

(i) \(f(t_1, \ldots, t_q; k_1, \ldots, k_r) = 0\) if \(t_i < 0\) or \(k_j < 0\) for some \(i \in \bar{q}\) or \(j \in \bar{r}\).

(ii) \(f(\cdot; k_1, \ldots, k_r)\) is piecewise smooth on \(\mathbb{R}^q_+\) for any \((k_1, \ldots, k_r) \in \mathbb{Z}^r_+\).

(iii) \(\exists M_f > 0, \sigma_{f_i} \geq 0, i \in \bar{q}, R_{f_j} > 0, j \in \bar{r}\) such that

\[
|f(t_1, \ldots, t_q; k_1, \ldots, k_r)| \leq M_f \exp \left( \sum_{i=1}^{q} \sigma_{f_i} t_i \right) \prod_{j=1}^{r} R_{f_j}^{k_j}
\]

\(\forall t_i > 0, i \in \bar{q}, \forall k_j \geq 0, j \in \bar{r}\).

The constants \(\sigma_{f_i}, R_{f_j}\) will be also denoted by \(\sigma_i, R_j\). The smallest such constants are called respectively the indices of the order of growth and the radii of convergence of the original function \(f\).

We denote by \(\mathcal{O}_{q,r}\), the set of original functions \(f: \mathbb{R}^q \times \mathbb{Z}^r \to \mathbb{C}\). Sometimes we shall denote by \(f(t; k)\) the value of \(f\) at \(t = (t_1, \ldots, t_q), k = (k_1, \ldots, k_r)\).

### Definition 3.2.

For any original function, the function

\[
F(s_1, \ldots, s_q; z_1, \ldots, z_r) = \int_0^\infty \cdots \int_0^\infty \prod_{k=0}^{q} \prod_{r=0}^{r} e^{-s_k t_k - s_r z_r} dt_k \cdots dt_q
\]

is called the \((q, r)\)-hybrid Laplace transform (or the image) of \(f\).

We shall use also the notation which defines the Laplace Transformation \(\mathcal{L}_{q,r}: f(z) = \mathcal{L}_{q,r}[f(t; k)], \) where \(s = (s_1, \ldots, s_q)\) and \(z = (z_1, \ldots, z_r)\).

The following results are proved in [10].

#### Proposition 3.3.

The multiple improper integral and the multivariable Taylor series in (3.2) are absolutely convergent in the domain

\[
D(f) = \{(s_1, \ldots, s_q; z_1, \ldots, z_r) \in \mathbb{C}^{q+r} | \Re s_i > \sigma_{f_i}, i \in \bar{q}, |z_j| > R_{f_j}, j \in \bar{r}\}
\]

and uniformly convergent on any domain

\[
D^f(f) = \{(s_1, \ldots, s_q; z_1, \ldots, z_r) \in \mathbb{C}^{q+r} | \Re s_i \geq \sigma'_i, i \in \bar{q}, |z_j| \geq R'_j, j \in \bar{r}\}
\]

with \(\sigma'_i > \sigma_{f_i}, i \in \bar{q}\) and \(R'_j > R_{f_j}, j \in \bar{r}\).

#### Theorem 3.4 (Linearity).

For any \(f, g \in \mathcal{O}_{q,r}\) and \(\alpha, \beta \in \mathbb{C}\),

\[
\mathcal{L}_{q,r}[(\alpha f + \beta g)] = \alpha \mathcal{L}_{q,r}[f] + \beta \mathcal{L}_{q,r}[g].
\]

#### Theorem 3.5 (First time-delay theorem).

For any \((a_1, \ldots, a_q) \in \mathbb{R}_+^q, (b_1, \ldots, b_r) \in \mathbb{Z}_+^r\),

\[
\mathcal{L}_{q,r}[f(t_1 - a_1, \ldots, t_q - a_q; k_1 - b_1, \ldots, k_r - b_r)] = \exp \left( - \sum_{i=1}^{q} a_i s_i \right) \prod_{j=1}^{r} z_j^{-b_j}.
\]
We shall use the following notations: for some sets $\alpha = \{i_1, \ldots, i_p\} \subset \bar{q}$ and $\beta = \{j_1, \ldots, j_q\} \subset \bar{r}$, $E_{\alpha} = \{\varepsilon \in \alpha \cup \beta \cup \gamma \} = \emptyset$, $E_{\beta} = \{\delta \in \beta \cup \delta \cup \beta \} = \emptyset$

For $a = (a_i)_{i \in \alpha} \in \mathbb{R}_+^{\alpha}$ with $a_i > 0$, $\forall i \in \alpha$ and \( b = (b_j)_{j \in \beta} \in \mathbb{Z}_+^{\beta} \) with $b_j > 0$, $\forall j \in \beta$ and for $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_\gamma) \in E_{\alpha}$ and $\delta = (\delta_1, \ldots, \delta_\mu) \in E_{\beta}$ we denote by $D_{a,\varepsilon}$ and $D_{b,\delta}$ the sets $D_{a,\varepsilon} = \{0, a_i \} \cap \mathbb{Z}_+$ and $D_{b,\delta} = \{0, 1, \ldots, b_j - 1\}$ and by $\int_{D_{a,\varepsilon}}$ and $\sum$ the multiple integral $\int_{e_1}^{e_2} \cdots \int_{e_1}^{e_2}$, respectively the multiple sum $\sum_{k_{j_1} = 0}^{b_{j_1} - 1} \cdots \sum_{k_{j_r} = 0}^{b_{j_r} - 1}$; if $\varepsilon = \emptyset$ or $\delta = \emptyset$ the corresponding multiple integral or sum lack; $f(t + a; k + b)$ denotes $f(t_1, \ldots, t_{i_1 - 1}, t_{i_1} + a_{i_1}, t_{i_1 + 1}, \ldots, t_{i_p - 1}, t_{i_p} + a_{i_p}, t_{i_p + 1}; j_1, \ldots, j_q, k_1, j_1 + 1, \ldots, j_q + 1, k_1 + b_{j_1}, j_1 + 1, \ldots, j_q + 1, k_1 + b_{j_1}; k_1, j_1 + 1, \ldots, k_r, j_r)$.

**Definition 3.6.** For $\alpha = \{i_1, \ldots, i_p\} \subset \bar{q}$ and $\beta = \{j_1, \ldots, j_q\} \subset \bar{r}$, the $(\alpha, \beta)$-partial $(q, r)$-hybrid Laplace transform of the original $f$ is defined by

$$L_{q,r}^{\alpha,\beta}[f(t, k)] = \int_0^{\infty} \cdots \int_0^{\infty} \sum_{k_{j_1} = 0}^{b_{j_1} - 1} \cdots \sum_{k_{j_r} = 0}^{b_{j_r} - 1} \frac{1}{\varepsilon_1, \ldots, \varepsilon_{\gamma_1}} d\varepsilon_1 \cdots d\varepsilon_{\gamma_1} f(t_1, \ldots, t_{i_1 - 1}, t_{i_1} + a_{i_1}, t_{i_1 + 1}, \ldots, t_{i_p - 1}, t_{i_p} + a_{i_p}, t_{i_p + 1}; j_1, \ldots, j_q, k_1, j_1 + 1, \ldots, j_q + 1, k_1 + b_{j_1}, j_1 + 1, \ldots, j_q + 1, k_1 + b_{j_1}; k_1, j_1 + 1, \ldots, k_r, j_r).$$

(6.3)

If $\alpha = \emptyset$ and $\beta = \emptyset$, $\mathbb{L}_{q,r}^{\alpha,\beta} = \mathbb{L}_{q,r}$; if $\alpha = \emptyset$, $\mathbb{L}_{q,r}^{\emptyset,\beta} = \mathbb{L}_{q,r}$ (the multiple Laplace transformation); if $\alpha = \emptyset$, $\mathbb{L}_{q,r}^{\phi,\beta} = \mathbb{L}_{q,r}$ (the multiple $z$-transformation); if $\alpha = \emptyset$, $\mathbb{L}_{q,r}^{\phi,\emptyset}[f] = f$.

**Theorem 3.7 (Second delay theorem).** For any $a = (a_i)_{i \in \alpha} \in \mathbb{R}_+^{\alpha}$ and $b = (b_j)_{j \in \beta} \in \mathbb{Z}_+^{\beta}$

$$L_{n,m}^{\alpha,\beta}[f(t, k)] = \exp \left( \sum_{i \in \alpha} a_i s_i \right) \left( \prod_{j \in \beta} z_j^{b_j} \right) \cdot F(s; z) + \sum_{\varepsilon \in \alpha} \sum_{\delta \in \beta} a_s s_i (-1)^{|\varepsilon| + |\delta|} \int_{D_{a,\varepsilon}} \sum_{D_{b,\delta}} \frac{1}{\varepsilon_1, \ldots, \varepsilon_{\gamma_1}} d\varepsilon_1 \cdots d\varepsilon_{\gamma_1}$$

(7.3)

We introduce the following notations: given $\alpha = \{i_1, \ldots, i_p\} \subset \bar{q}$, a $p$-tuple $(\gamma_1, \ldots, \gamma_p) \in \mathbb{N}_p$ is denoted by $\gamma_\alpha$ or simply by $\gamma_\alpha$ and $\frac{\partial f}{\partial \gamma_\alpha} = \frac{\partial f}{\partial \gamma_1} + \cdots + \frac{\partial f}{\partial \gamma_p}$, $\gamma = \gamma_1 \cdots \gamma_p$. The family of all undefined subsets $\varepsilon$ of $\alpha$ is denoted by $E_{\gamma_\alpha}$ or $E_{\gamma_\alpha}$. For $\varepsilon \in E_{\gamma_\alpha}$, $\varepsilon = \alpha \setminus \varepsilon$, $\gamma_\varepsilon = \prod_{i \in \varepsilon} \gamma_i$ and $s_{\varepsilon_1} = 1$ if $\varepsilon = \emptyset$;

if $\varepsilon = \{i_1, \ldots, i_p\}$ and $\eta_\varepsilon = (\eta_{i_1}, \ldots, \eta_{i_p}) \in \mathbb{N}_p$, $\eta_\varepsilon \leq \gamma_\varepsilon$ means $\eta_i \leq \gamma_i$, $\forall i \in \varepsilon$; $f(0^+; k)$ denotes the limit from the right

$$f(t_1, \ldots, t_{i_1 - 1}, 0 + t_{i_1 + 1}, \ldots, t_{i_p - 1}, 0 + t_{i_p + 1}, \ldots, t_{i_p} + 1, \ldots, t_{i_p + 1}, k_1, \ldots, k_r).$$

If $\varepsilon = \{i\}$ then $f(0^+; k)$ is denoted by $f(0^+; k)$. Similarly, $f(t; k_1, \ldots, k_{j_1 - 1}, 0, k_1, \ldots, k_r)$ is denoted $f(t; 0_j)$ and we can use the notation $f(0^+; 0_j)$ which combines these notations.

**Theorem 3.8 (Differentiation of the original).** For any $i \in \bar{q}$

$$\mathbb{L}_{q,r}^{\alpha,\emptyset}\left[ \frac{\partial f}{\partial t_i}(t; k) \right] = s_i F(s; z) - \mathbb{L}_{q,r}^{\gamma,\phi}[f(0^+; k)] \quad (8.3i)$$

$$\mathbb{L}_{q,r}^{\alpha,\emptyset}\left[ \frac{\partial f}{\partial \gamma_\alpha}(t; k) \right] = s_\gamma F(s; z) + \sum_{\varepsilon \in E_{\gamma_\alpha}} (-1)^{|\varepsilon|} s_{\varepsilon_1} \quad (8.3ii)$$

$$\sum_{\eta_\varepsilon \leq \gamma_\varepsilon - 1} s_{\varepsilon_1} - \eta_\varepsilon - 1 \mathbb{L}_{q,r}^{\gamma,\emptyset}\left[ \frac{\partial f}{\partial \gamma_\varepsilon}(0^+; k) \right] .$$

**Theorem 3.9 (Differentiation and delay).** For any $i \in \bar{q}, j \in \bar{r}$

$$\mathbb{L}_{q,r}^{\alpha,\emptyset}\left[ \frac{\partial f}{\partial t_i}(t_1, \ldots, t_q; k_1, \ldots, k_{j_1 - 1}, k_1 + 1, \ldots, k_r) \right] = s_\gamma z_\gamma F(s; z) + s_\gamma z_\gamma F(s; z) + s_\gamma z_\gamma F(s; z) + s_\gamma z_\gamma F(s; z) + s_\gamma z_\gamma F(s; z) + s_\gamma z_\gamma F(s; z) + s_\gamma z_\gamma F(s; z) + s_\gamma z_\gamma F(s; z) + s_\gamma z_\gamma F(s; z) + 1 \mathbb{L}_{q,r}^{\alpha,\emptyset}(f(0^+; k)) .$$

(9.3ii)
Let us consider the time-invariant system $\Sigma$, i.e., the system with constant matrices $A_{ci}, A_{dj}, B, C$ and $D$. Obviously, we can extend the multiple hybrid Laplace to vector functions

$$x(t; k) = [x_1(t; k), x_2(t; k), \ldots, x_n(t; k)]^T$$

by

$$X(s; z) = L[x(t; k)] = \left[L[x_1(t; k)], L[x_2(t; k)], \ldots, L[x_n(t; k)]\right].$$

By linearity (Theorem 3.4), if we apply the multiple $(q, r)$-hybrid Laplace transform $L_{q,r}$ to the state equation (2.1) we get

$$L_{q,r}\left[\frac{\partial}{\partial t}\sigma x(t; k)\right] =$$

$$= \sum_{(\tau,\delta) \in (q,r)} (-1)^{q+r-|r|-|\delta|-1} \left(\prod_{i \in \tau} A_{ci}\right) \left(\prod_{j \in \delta} A_{dj}\right) \cdot$$

$$\cdot L_{q,r}\left[\frac{\partial}{\partial t}\sigma x(t, h)\right] + B L_{q,r}[u(t; h)].$$

By Theorem 3.9, using formula (3.9ii) for $a = (1,1,\ldots,1) \in \mathbb{N}^q$ and $b = (1,1,\ldots,1) \in \mathbb{N}^r$ this equality becomes:

$$s_1 s_2 \ldots s_q z_1 z_2 \ldots z_r X(s; z) + T_1 =$$

$$= \sum_{(\tau,\delta) \in (q,r)} (-1)^{q+r-|r|-|\delta|-1} \left(\prod_{i \in \tau} A_{ci}\right) \left(\prod_{j \in \delta} A_{dj}\right) \cdot$$

$$\cdot \left(\prod_{i \in \tau} s_i\right) \left(\prod_{j \in \delta} z_j\right) X(s; z) + T_2 + BU(s; z)$$

where

$$T_1 = z_1 z_2 \ldots z_r \sum_{\varepsilon \in E_{\gamma}} \sum_{\delta \in E_{\delta}} (-1)^{|\varepsilon|+|\delta|} s_\varepsilon^{\gamma_{\varepsilon}}$$

$$\sum_{n_\varepsilon \leq n_\delta - 1} s_\varepsilon^{\gamma_{\varepsilon}} - n_\delta - 1 \left(\prod_{j \in \delta} z_j^{-k_j}\right) \cdot$$

$$\cdot \sum_{D_{\varepsilon,\delta}} L_{q,r}^{\varepsilon,\delta} \left[ L \left[ \frac{\partial \eta_{\varepsilon}}{\partial \eta_{\varepsilon}}(0^+; 0_\delta) \right] \right]$$

and

$$T_2 = z^\delta \sum_{\varepsilon \in E_{\gamma}} \sum_{\lambda \in E_{\delta}} (-1)^{|\varepsilon|+|\lambda|} s_\varepsilon^{\gamma_{\varepsilon}} \sum_{n_\varepsilon \leq n_\delta - 1} s_\varepsilon^{\gamma_{\varepsilon}} - n_\delta - 1.$$
where
\[
T_\Sigma(s; z) = C \left( \prod_{i=1}^{q} (s I - A_{ci})^{-1} \right) 
\times \left( \prod_{j=1}^{r} (z_j I - A_{dj})^{-1} \right) B + D.
\] (3.16)
The matrix \(T_\Sigma(s; z)\) (3.16) is called the transfer matrix of the system \(\Sigma\).

A rational matrix \(T(s; z)\) is said to be proper if its elements have the form
\[
t_{ij}(s; z) = \frac{a_{ij}(s_1, \ldots, s_q; z_1, \ldots, z_r)}{b_{ij}(s_1, \ldots, s_q; z_1, \ldots, z_r)}
\]
and \(\deg a_{ij} \leq \deg b_{ij}, \forall k \in \vec{q}, \deg z_{ij}, \forall l \in \vec{r}, \forall i \in \vec{p}, \forall j \in \vec{m}\), where \(\deg a_{ij}\) denotes the degree of the polynomial \(a_{ij}\) w.r.t. the variable \(s_k\).

If all these inequalities are strict ones, \(T(s; z)\) is said to be strictly proper. If \(T(s; z) = \frac{1}{\pi(s)\theta(z)} M(s, z)\) where \(\pi(s)\) and \(\theta(z)\) are polynomials of the form \(\pi(s) = \pi_1(s_1) \ldots \pi_q(s_q), \theta(z) = \theta_1(z_1) \ldots \theta_r(z_r)\) and \(M(s; z)\) is a polynomial matrix, then \(T(s; z)\) is said to be with separable denominator.

The following characterization of \(T_\Sigma(s; z)\) is a direct consequence of (3.16):

**Proposition 3.11.** The transfer matrix of a \((q, r)\)-D hybrid system is a rational proper \(p \times m\) matrix with separable denominator. If \(D = 0\) (the \(p \times m\) null matrix) then \(T_\Sigma(s; z)\) is strictly proper.

We shall denote by \(T_\Sigma(s; z)\) the set of the proper rational matrices \(T(s; z)\) with separable denominator which can be decomposed as a sum between a strictly proper matrix and a constant one.

**Example 3.12.** The system \(\Sigma = (A_c, A_d, B, C, D)\) considered in Example 2.6 has the strictly proper separable transfer matrix
\[
T_\Sigma(s; z) = C(s I - A_c)^{-1}(z I - A_d)^{-1}B + D
\]
hence
\[
T_\Sigma(s; z) = \begin{bmatrix}
1 & -1 \\
\frac{1}{s^2 + 2s + 1} & \frac{1}{(s + 1)^2(z - 1)} \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\]
\[
= 4
\]
\[
((s + 1)^2(z - 1)).
\]

4 **Minimal realizations**

In this section we shall give an algorithm which provides minimal realizations for proper matrices with separable denominator.

**Definition 4.1.** Given a rational matrix \(T(s; z) \in T_\Sigma(s; z)\), a system \(\Sigma = (\{A_{ci}|i \in \vec{q}\}, \{A_{dj}|j \in \vec{r}\}, B, C, D)\) is said to be a realization of \(T(s; z)\) if \(T(s; z) = T_\Sigma(s; z)\), i.e.
\[
T(s; z) = C \left( \prod_{i=1}^{q} (s_i I - A_{ci})^{-1} \right) 
\times \left( \prod_{j=1}^{r} (z_j I - A_{dj})^{-1} \right) B + D.
\] (4.1)

The realization \(\Sigma\) is said to be minimal if \(\dim \Sigma \leq \dim \Sigma^{\text{min}}\) for any realization \(\Sigma\) of \(T(s; z)\).

Since the matrix \(D\) can be determined by (3.17), the realization problem will be formulated as follows: given a strictly proper \(p \times m\) matrix \(T(s; z) \in T_\Sigma(s; z)\), determine the system \(\Sigma = (\{A_{ci}|i \in \vec{q}\}, \{A_{dj}|j \in \vec{r}\}, B, C)\) such that
\[
T(s; z) = C \left( \prod_{i=1}^{q} (s_i I - A_{ci})^{-1} \right) 
\times \left( \prod_{j=1}^{r} (z_j I - A_{dj})^{-1} \right) B.
\] (4.2)

Following the lines of the proof in [7, Theorem 5.4], we obtain (see [8] and [9]):

**Theorem 4.2.** A system \(\Sigma\) is a minimal realization of a strictly proper matrix \(T(s; z) \in T_\Sigma(s; z)\) if and only if \(\Sigma\) is completely reachable and completely observable.

Now, let us expand \(T(s; z)\) in Laurent series about infinity:
\[
T(s_1, \ldots, s_q; z_1, \ldots, z_r) =
\sum_{i_1=0}^{\infty} \cdots \sum_{i_q=0}^{\infty} \sum_{j_1=0}^{\infty} \cdots \sum_{j_r=0}^{\infty} M_{i_1, \ldots, i_q, j_1, \ldots, j_r}.
\] (4.3)

The constant \(p \times m\) matrices \(M_{i_1, \ldots, i_q, j_1, \ldots, j_r}\) are called the Markov parameters of the matrix \(T(s; z)\).

**Theorem 4.3.** A system \(\Sigma = (\{A_{ci}|i \in \vec{q}\}, \{A_{dj}|j \in \vec{r}\}, B, C)\) is a realization of the strictly proper matrix \(T(s; z)\) (4.3) if and only if
\[
M_{i_1, \ldots, i_q, j_1, \ldots, j_r} = C \left( \prod_{k=1}^{q} A_{ck}^{i_k} \right) \left( \prod_{l=1}^{r} A_{dl}^{j_l} \right) B.
\] (4.4)

**Proof.** For any square matrix \(A\) and \(|s| > \max_{\lambda \in \sigma(A)} |\lambda|\), the following Laurent series expansion
holds: \((sI - A)^{-1} = \sum_{i=0}^{\infty} A^i s^{-i+1}\). Then (4.2) gives for any realization \(\Sigma\) of \(T(s; z)\):

\[
T(s; z) = \sum_{i_1=0}^\infty \ldots \sum_{i_q=0}^\infty \sum_{j_1=0}^\infty \ldots \sum_{j_r=0}^\infty C \left( \prod_{k=1}^q A_{ck}^{i_k} \right) \cdot \left( \prod_{l=1}^r A_{dl}^{j_l} \right) B \left( \prod_{k=1}^q s_k^{-i_k-1} \right) \left( \prod_{l=1}^r z_l^{-j_l-1} \right) .
\]

(4.5)

Therefore, \(\Sigma\) is a realization of \(T(s; z)\) if and only if the Laurent series in (4.3) and (4.5) are equal, and this condition is equivalent to the equality of all their corresponding coefficients, i.e. with (4.4). ∎

Now we shall use the following notations: \(k\) and \(j\) denotes respectively \(k_1, \ldots, k_q\) and \(j_1, \ldots, j_r\), while \(i_a, i_{a+1}, \ldots, i_q\) and \(j_b, j_{b+1}, \ldots, j_r\).

We associate to the strictly proper matrix \(T(s; z)\) the following sequence of block Hankel matrices, for \(i_\alpha \geq 0, k_\alpha \geq 1, \alpha \in q, \delta_3 \geq 0, l_\beta \geq 1, \beta \in r\): 

\[
H_{k_1, \ldots, k_q, j_1, \ldots, j_r}^{i_1, \ldots, i_q, j_1, \ldots, j_r} = \begin{bmatrix}
M_{0, i_2, j_1} & M_{1, i_2, j_1} & \ldots & M_{k_1-1, i_2, j_1} \\
M_{1, i_2, j_1} & M_{2, i_2, j_1} & \ldots & M_{k_1, i_2, j_1} \\
\vdots & \vdots & \ddots & \vdots \\
M_{k_1-1, i_2, j_1} & M_{k_1, i_2, j_1} & \ldots & M_{2k_1-2, i_2, j_1}
\end{bmatrix}
\]

(a \(pk_1 \times mk_1 \) matrix).

\[
H_{k_1, k_2}^{i_3, \ldots, i_q, j_1, \ldots, j_r} = \begin{bmatrix}
H_{k_1}^{0, i_3, j_1} & H_{k_1}^{1, i_3, j_1} & \ldots & H_{k_1}^{k_2-1, i_3, j_1} \\
H_{k_1}^{1, i_3, j_1} & H_{k_1}^{2, i_3, j_1} & \ldots & H_{k_1}^{k_2, i_3, j_1} \\
\vdots & \vdots & \ddots & \vdots \\
H_{k_1}^{k_2-1, i_3, j_1} & H_{k_1}^{k_2, i_3, j_1} & \ldots & H_{k_1}^{2k_2-2, i_3, j_1}
\end{bmatrix}
\]

(a \(pk_1 k_2 \times mk_1 k_2 \) matrix).

Generally, being determined the block Hankel matrices \(H_{k_1, \ldots, k_q, j_1, \ldots, j_r}^{i_1, \ldots, i_q, j_1, \ldots, j_r}\), we define the \(pk_1 k_2 \times k_1 k_2 \times \ldots \times k_q k_2\) block Hankel matrix 

\[
H_{k_1, \ldots, k_q, j_1, \ldots, j_r}^{i_1, \ldots, i_q, j_1, \ldots, j_r} = \begin{bmatrix}
H_{k_1, \ldots, k_q, j_1, \ldots, j_r}^{0, i_1, \ldots, i_q, j_1, \ldots, j_r} & \ldots & H_{k_1, \ldots, k_q, j_1, \ldots, j_r}^{i_1, \ldots, i_q, j_1, \ldots, j_r} \\
\vdots & \ddots & \ddots & \vdots \\
H_{k_1, \ldots, k_q, j_1, \ldots, j_r}^{i_1, \ldots, i_q, j_1, \ldots, j_r} & \ldots & H_{k_1, \ldots, k_q, j_1, \ldots, j_r}^{k_1-1, i_1, \ldots, i_q, j_1, \ldots, j_r} \\
H_{k_1, \ldots, k_q, j_1, \ldots, j_r}^{i_1, \ldots, i_q, j_1, \ldots, j_r} & \ldots & H_{k_1, \ldots, k_q, j_1, \ldots, j_r}^{k_1, i_1, \ldots, i_q, j_1, \ldots, j_r} \\
\vdots & \ddots & \ddots & \vdots \\
H_{k_1, \ldots, k_q, j_1, \ldots, j_r}^{i_1, \ldots, i_q, j_1, \ldots, j_r} & \ldots & H_{k_1, \ldots, k_q, j_1, \ldots, j_r}^{k_1-1, i_1, \ldots, i_q, j_1, \ldots, j_r}
\end{bmatrix}
\]

(4.6)

Sometimes the matrix (4.6) will be denoted \(H_{k_1}\).

**Proposition 4.4.** For any realization \(\Sigma\) of \(T(s; z) \in T_d(s; z)\) and any \(k_\alpha \geq 1, k_b \geq 1, a \in q, b \in r\),

\[
\text{rank} H_{k_1, \ldots, k_q, l_1, \ldots, l_r} \leq \text{dim} \Sigma .
\]

**Proof.** Let us consider a realization as in Definition 2.1, hence \(n = \text{dim} \Sigma\). We shall define by recurrence the following controllability-type block matrices:

\[
C(A_{c_1}; B; k_1) = [B A_{c_1} B A_{c_1}^2 B \ldots A_{c_1}^{k_1-1} B]
\]

\[
C(A_{c_1}; A_{c_2}; B; k_1, k_2) = [C(A_{c_1}; B; k_1)
\]

\[
C(A_{c_1}; A_{c_2}; B; k_1) \ldots C(A_{c_1}; A_{c_2} \ldots A_{c_1}^{k_2-1} B; k_1)]
\]

\[
C(A_{c_1}, A_{c_2}, \ldots, A_{c_q-1}, A_{c_1}; k_1) = C(A_{c_1}; A_{c_2}, \ldots, A_{c_q-1}, A_{c_1}; k_1, k_2, \ldots, k_q-1, k_q) = [C(A_{c_1}, A_{c_2}, \ldots, A_{c_q-1}, A_{c_1}; k_1, k_2, \ldots, k_q-1, k_q) = \ldots C(A_{c_1}, A_{c_2}, \ldots, A_{c_q-1}, A_{c_1}^{k_q-1} B; k_1, k_2, \ldots, k_q-1, k_q)]
\]

\[
\forall i, 2 \leq i \leq q.\text{ For } A_c = (A_{c_1}, A_{c_2}, \ldots, A_{c_q-1}, A_{c_q})
\]

and \(k = (k_1, k_2, \ldots, k_{q-1}, k_q)\) we denote

\[
C(A_c; B; k) = C(A_{c_1}, A_{c_2}, \ldots, A_{c_q-1}, A_{c_q}; B; k_1, k_2, \ldots, k_{q-1}, k_q)
\]

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Then
\[ C(A_c; A_{d_1}; B; k; l_1) = [C(A_c; B; k) C(A_c; A_{d_1} B; k) \ldots C(A_c; A_{d_{l_1}}^{-1} B; k)] \]
\[ C(A_c; A_{d_1}, A_{d_2}; B; k; l_1, l_2) = [C(A_c; A_{d_1} B; k; l_1) \ldots C(A_c; A_{d_{l_1}}^{-1} B; k; l_1)] \]
\[ \vdots \]
\[ C(A_c; A_{d_1}, A_{d_2}, \ldots, A_{d_{j-1}}, A_{d_j}; B; k; l_1, l_2, \ldots, l_{j-1}, l_j) = [C(A_c; A_{d_1} B; k; l_1, l_2, \ldots, l_{j-1}) \ldots C(A_c; A_{d_{l_1}}^{-1} B; k; l_1, l_2, \ldots, l_{j-1})] \]
\[ \forall j, 2 \leq j \leq r. \]
Finally, we denote, for \( A_d = (A_{d_1}, A_{d_2}, \ldots, A_{d_{r-1}}, A_{d_r}) \) and \( l = (l_1, l_2, \ldots, l_{r-1}, l_r) \)
\[ C(A_c; A_d B; k; l) := C(A_c; A_{d_1}, A_{d_2}, \ldots, A_{d_{r-1}}, A_{d_r}; B; k; l_1, l_2, \ldots, l_{r-1}, l_r). \] (4.8)

Similarly, we define by recurrence the observability type matrices:
\[ \mathcal{O}(A_c; C; k_1) = \begin{bmatrix} C & C A_{c_1} & \cdots & C A_{c_1}^{k_1-1} \end{bmatrix} \]
\[ \mathcal{O}(A_c; C; k_2) = \begin{bmatrix} \mathcal{O}(A_c; C; k_1) \\ \mathcal{O}(A_c; C; A_{c_2}; k_1) \\ \vdots \\ \mathcal{O}(A_c; C; A_{c_{k_2}}^{-1}; k_1) \end{bmatrix} \]
and so on. Finally, we obtain the matrix (as in (4.7)):
\[ \mathcal{O}(A_c; A_d; C; k; l) := \mathcal{O}(A_c; A_{d_1}, A_{d_2}, \ldots, A_{d_{r-1}}; C; k; l_1, l_2, \ldots, l_{r-1}, l_r). \] (4.9)

Using (4.6), (4.8) and (4.9), we can prove that
\[ H_{k_1, \ldots, k_q; l_1, \ldots, l_r} = \mathcal{O}(A_c; A_d; C; k; l) \mathcal{O}(A_c; A_d; B; k; l) \] (4.10)

Now we shall employ Sylvester’s inequalities. If \( P \) is a \( p \times n \) matrix and \( M \) is an \( n \times m \) matrix, then
\[ \text{rank} P + \text{rank} M - n \leq \text{rank} PM \leq \min(\text{rank} P, \text{rank} M). \] (4.11)

Obviously, \( \mathcal{O}(A_c; A_d; B; k; l) \) is a \( p k l \times n \) matrix and \( \mathcal{O}(A_c; A_d; B; k; l) \) is an \( n \times m k l \) matrix, where \( k = k_1 k_2 \ldots k_q \) and \( l = l_1 l_2 \ldots l_r \).

By the second inequality (4.11) and by (4.10) we get
\[ \text{rank} H_{k_1, \ldots, k_q; l_1, \ldots, l_r} \leq \min(\text{rank} \mathcal{O}(A_c; A_d; C; k; l)), \]
\[ \text{rank} \mathcal{O}(A_c; A_d; C; k; l) \leq n, \]
i.e. (4.7), since \( n = \text{rank} \Sigma \).

Now let us assume that
\[ T(s; z) = \left( \prod_{a=1}^{q} \pi_a(s_a) \right)^{-1} \left( \prod_{b=1}^{r} \theta_b(z_b) \right)^{-1} M(s; z) \]
where \( M(s; z) = M(s_1, \ldots, s_q; z_1, \ldots, z_r) \) is a polynomial matrix and \( \pi_a(s), \theta_b(z) \) are polynomials of degree \( k_a, l_b \) respectively, \( a \in \bar{q}, b \in \bar{r} \).

We define the first level shift operators \( \sigma^\alpha_a, a \in \bar{q}, \alpha \geq 1 \) by
\[ \sigma^\alpha_a H_{k_1, \ldots, k_q; l_1, \ldots, l_r} = \] (4.12)
\[ \begin{bmatrix} H_{k_1, \ldots, k_q; a_1}^{\alpha+1} & H_{k_1, \ldots, k_q; a_1}^{\alpha+2} & \cdots & H_{k_1, \ldots, k_q; a_1}^{\alpha+k_a-1} \\ \vdots & \ddots & \ddots & \vdots \\ H_{k_1, \ldots, k_q; a_1}^{\alpha-k_a+1} & H_{k_1, \ldots, k_q; a_1}^{\alpha-k_a+2} & \cdots & H_{k_1, \ldots, k_q; a_1}^{\alpha-1} \end{bmatrix} \]

Similarly, the first level operators \( \tilde{\delta}^\beta_a, b \in \bar{r}, \beta \geq 1 \) are defined by
\[ \tilde{\delta}^\beta_a H_{k_1, \ldots, k_q; l_1, \ldots, l_r} = \] (4.13)
\[ \begin{bmatrix} H_{k_1, \ldots, k_q; b_1}^{\beta+1} & H_{k_1, \ldots, k_q; b_1}^{\beta+2} & \cdots & H_{k_1, \ldots, k_q; b_1}^{\beta+l_b-1} \\ \vdots & \ddots & \ddots & \vdots \\ H_{k_1, \ldots, k_q; b_1}^{\beta-1} & H_{k_1, \ldots, k_q; b_1}^{\beta} & \cdots & H_{k_1, \ldots, k_q; b_1}^{\beta+l_b-2} \end{bmatrix} \]

The second level shift operators \( \sigma^\alpha_a \) and \( \tilde{\delta}^\beta_b, a \in \bar{q}, b \in \bar{r}, \alpha, \beta \geq 1 \) acting on the block Hankel matrix \( H_{k; l} = H_{k_1, \ldots, k_q; l_1, \ldots, l_r} \) (4.6) are defined as follows: \( \sigma^\alpha_a H_{k; l} \) is the matrix obtained by recurrence as \( H_{k; l} \) (4.6) by replacing \( H_{k_1, \ldots, k_q; j_1, \ldots, j_r}^{\alpha+1} \) by \( \sigma^\alpha_a H_{k_1, \ldots, k_q; a_1-k_a+1}^{\alpha+1} \); \( \tilde{\delta}^\beta_b H_{k; l} \) is the matrix obtained by recurrence as \( H_{k; l} \) (4.6) by replacing \( H_{k_1, \ldots, k_q; b_1}^{\beta+1} \) by \( \tilde{\delta}^\beta_b H_{k_1, \ldots, k_q; b_1}^{\beta+1} \).

We shall denote \( \sigma^1_a \) and \( \delta^1_b \) by \( \sigma_a \) and \( \delta_b \) respectively.
Assume that the polynomials in the denominator of $T(s; z)$ are
\[
\pi_a(s_a) = s_a^k + \alpha_{a,k} - 1 s_a^{k-1} + \ldots + \alpha_{a,1} s_a + \alpha_{a,0}, \quad a \in \{\bar{q}\},
\]

and
\[
\theta_b(z_b) = z_b^l + \beta_{b,l} - 1 z_b^{l-1} + \ldots + \beta_{b,1} z_b + \beta_{b,0}, \quad b \in \{\bar{r}\},
\]

We associate to the polynomials $\pi_a$ and $\theta_b$ the companion cells $K_a = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1 \\
-\alpha_{a,0} & -\alpha_{a,1} & -\alpha_{a,2} & \ldots & -\alpha_{a,k_a-2} & -\alpha_{a,k_a-1}
\end{bmatrix}
\]

and $L_b = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1 \\
-\beta_{b,0} & -\beta_{b,1} & -\beta_{b,2} & \ldots & -\beta_{b,l_b-2} & -\beta_{b,l_b-1}
\end{bmatrix}
\]

We consider (for $a \in \{\bar{q}\}$ and $b \in \{\bar{r}\}$) the matrices
\[
\tilde{F}_a = \left( \begin{array}{c}
q \\
j = a+1
\end{array} \right) I_k_j \otimes \left( \begin{array}{c}
r \\
j = 1
\end{array} \right) \otimes K_a \otimes \left( \begin{array}{c}
a-1 \\
j = i+1
\end{array} \right) I_k_i \otimes I_p
\]
\[
\tilde{F}_a = \left( \begin{array}{c}
q \\
j = a+1
\end{array} \right) I_k_j \otimes \left( \begin{array}{c}
r \\
j = 1
\end{array} \right) \otimes K_a^T \otimes \left( \begin{array}{c}
a-1 \\
j = i+1
\end{array} \right) I_k_i \otimes I_m
\]
\[
\tilde{G}_b = \left( \begin{array}{c}
r \\
j = b+1
\end{array} \right) I_l_j \otimes L_b \otimes \left( \begin{array}{c}
b-1 \\
j = 1
\end{array} \right) I_l_i \otimes \left( \begin{array}{c}
a \\
j = 1
\end{array} \right) I_k_i \otimes I_p
\]
\[
\tilde{G}_b = \left( \begin{array}{c}
r \\
j = b+1
\end{array} \right) I_l_j \otimes L_b^T \otimes \left( \begin{array}{c}
b-1 \\
j = 1
\end{array} \right) I_l_i \otimes \left( \begin{array}{c}
a \\
j = 1
\end{array} \right) I_k_i \otimes I_m
\]

where $\otimes$ denotes the Kronecker product of matrices,

where $\otimes$ denotes the Kronecker product of matrices,

\[
A \otimes B = \begin{bmatrix}
A_{11} B & \ldots & A_{1n} B \\
\vdots & \ddots & \vdots \\
A_{m1} B & \ldots & A_{mn} B
\end{bmatrix}
\]

if $A = \begin{bmatrix}
a_{11} & \ldots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{m1} & \ldots & a_{mn}
\end{bmatrix}$

(hence $\tilde{F}_a = I_{k_{a+1}} \ldots I_k_{l_b} \otimes K_a \otimes I_{k_1} \ldots I_{k_p}$ and $\tilde{F}_a, \tilde{G}_b, \tilde{G}_b$ have similar representations).

**Proposition 4.5.** The matrices $\tilde{F}_a, a \in \{\bar{q}\}$, $\tilde{G}_b, b \in \{\bar{r}\}$ are commutative; $\tilde{F}_a, a \in \{\bar{q}\}$ and $\tilde{G}_b, b \in \{\bar{r}\}$ are commutative matrices.

**Proof.** For $a, c \in \{\bar{q}\}, a < c$, we have
\[
\tilde{F}_a \tilde{F}_c = \left( \begin{array}{c}
r \\
j = c+1
\end{array} \right) I_k_j \otimes \left( \begin{array}{c}
r \\
j = j+1
\end{array} \right) \otimes K_a \otimes \left( \begin{array}{c}
a-1 \\
j = i+1
\end{array} \right) I_k_i \otimes \left( \begin{array}{c}
a-1 \\
j = i+1
\end{array} \right) I_p
\]

\[
= \left[ \begin{array}{c}
r \\
j = j+1
\end{array} \right] I_k_j \otimes \left[ \begin{array}{c}
r \\
j = i+1
\end{array} \right] I_k_i \otimes \left[ \begin{array}{c}
a-1 \\
j = i+1
\end{array} \right] I_k_i \otimes \left[ \begin{array}{c}
a-1 \\
j = i+1
\end{array} \right] I_p
\]

\[
\otimes K_a \otimes \left[ \begin{array}{c}
a-1 \\
j = i+1
\end{array} \right] I_k_i \otimes I_p
\]

\[
= \left[ \begin{array}{c}
r \\
j = j+1
\end{array} \right] I_k_j \otimes \left[ \begin{array}{c}
r \\
j = i+1
\end{array} \right] I_k_i \otimes \left[ \begin{array}{c}
a-1 \\
j = i+1
\end{array} \right] I_k_i \otimes \left[ \begin{array}{c}
a-1 \\
j = i+1
\end{array} \right] I_p
\]

\[
\otimes K_a \otimes \left[ \begin{array}{c}
a-1 \\
j = i+1
\end{array} \right] I_k_i \otimes I_p
\]

\[
= \left[ \begin{array}{c}
r \\
j = j+1
\end{array} \right] I_k_j \otimes \left[ \begin{array}{c}
r \\
j = i+1
\end{array} \right] I_k_i \otimes \left[ \begin{array}{c}
a-1 \\
j = i+1
\end{array} \right] I_k_i \otimes \left[ \begin{array}{c}
a-1 \\
j = i+1
\end{array} \right] I_p
\]

\[
\otimes K_a \otimes \left[ \begin{array}{c}
a-1 \\
j = i+1
\end{array} \right] I_k_i \otimes I_p
\]

\[
= \tilde{F}_c \tilde{F}_a
\]

We used the properties of the Kronecker product: $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ for matrices $A, B, C, D$ with suitable dimensions; $I_n \otimes I_m = I_{nm} = I_m \otimes I_n$. Similarly, one obtains $\tilde{F}_a \tilde{G}_b = \tilde{G}_b \tilde{F}_a, \forall a \in \{\bar{q}\}, b \in \{\bar{r}\}, \tilde{G}_b \tilde{G}_d = \tilde{G}_d \tilde{G}_b, \forall b, d \in \{\bar{r}\}, \tilde{F}_a \tilde{F}_c = \tilde{F}_c \tilde{F}_a, \tilde{F}_a \tilde{G}_b = \tilde{G}_b \tilde{F}_a, \tilde{G}_b \tilde{G}_d = \tilde{G}_d \tilde{G}_b$.

**Proposition 4.6.** The second level shift operators $\delta_a$ and $\delta_b$ verify the equalities
\[
\sigma_a H_{k;j} = \tilde{F}_a H_{k;j} = H_{k;i} \tilde{F}_a, \quad a \in \{\bar{q}\},
\]
\[
\delta_b H_{k;j} = \tilde{G}_b H_{k;j} = H_{k;i} \tilde{G}_b, \quad b \in \{\bar{r}\}.
\]

**Proof.** The main idea of the proof is the fact that the product
\[
\pi_a(s_a)T(s; z) = \pi_a(s_a)T(s_1, \ldots, s_a, \ldots, s_q; z_1, \ldots, z_r)
\]

\[
= \prod_{i=1}^{q} \pi_i(s_i) \left( \prod_{j=1}^{r} \theta_j(z_j) \right)^{-1} M(s; z)
\]
is a polynomial matrix with respect to $s_a$, $a \in \bar{q}$, hence the coefficient of the negative powers of $s_a$ vanish. This gives recurrence formulas for the Markov parameters of $T(s; z)$, taking into account (4.14), for instance

$$M_{i_1, \ldots, i_a-1, i_a + k_a, i_a + 1, \ldots, i_q; j_1, \ldots, j_r} = \sum_{c=0}^{k_a-1} \alpha_{a,c} M_{i_1, \ldots, i_a-1, i_a + c, i_a + 1, \ldots, i_q; j_1, \ldots, j_r}$$

$$\forall i_1, \ldots, i_a, \ldots, i_q, j_1, \ldots, j_r \geq 0, \forall a \in \bar{q}.$$

Then a long calculus which is omitted verifies (4.20) and (4.21).

We obtain by induction

**Corollary 4.7.** For any $\alpha, \beta \geq 1, a \in \bar{q}, b \in \bar{r}$,

$$\sigma_a^\alpha H_{k;l} = \bar{E}_a^\alpha H_{k;l} = H_{k;l} \bar{E}_a^\alpha, \quad (4.22)$$

$$\delta_b^\beta H_{k;l} = \bar{G}_b^\beta H_{k;l} = H_{k;l} \bar{G}_b^\beta. \quad (4.23)$$

We shall use the following notations: $0_p^m$ is the null matrix with $p$ rows and $m$ columns; $I_p$ is the unit matrix of order $p$; $E_p^m$ is the $p \times m$ matrix defined by

$$E_p^m = \begin{cases} [I_p \ 0_p^{m-p}] & \text{if } p < m \\ I_p & \text{if } p = m \\ [I_m \ 0_p^{p-m}] & \text{if } p > m. \end{cases}$$

Obviously, these matrices have the following properties:

i) If $p < m$ and $M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$ is a matrix with $M_1$ $p \times q$ and $M_2$ $(m - p) \times q$, then

$$E_p^m M = M_1 \quad (4.24)$$

ii) If $p > m$ and $M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$ is a matrix with $M_1$ $q \times m$ and $M_2$ $q \times (p - m)$ then

$$M E_p^m = M_1 \quad (4.25)$$

iii) If $n \leq p, n \leq m$ then

$$E_p^n E_p^m = I_n \quad \text{and} \quad E_p^n E_p^m = \begin{bmatrix} I_n & 0_n^{m-n} \\ 0_p^n & 0_p^{m-n} \end{bmatrix}. \quad (4.26)$$

**Algorithm 4.8.** (of minimal realization). Let $T(s; z) = T(s_1, \ldots, s_q; z_1, \ldots, z_r)$ be a strictly proper matrix, $T(s; z) \in \mathcal{T}_S(s; z)$.

Stage I. Expand $T(s; z)$ in Laurent series (4.3) about infinity:

$$T(s; z) = \sum_{i_1=0}^{\infty} \ldots \sum_{i_q=0}^{\infty} \sum_{j_1=0}^{\infty} \ldots \sum_{j_r=0}^{\infty} M_{i_1, \ldots, i_q; j_1, \ldots, j_r} \cdot \left( \prod_{k=1}^{q} s_k^{-i_k-1} \right) \left( \prod_{l=1}^{r} z_l^{-j_k-1} \right).$$

Stage II. Determine the degrees $k_i, i \in \bar{q}$ and $l_j, j \in \bar{r}$ of the polynomials $\pi_i(s_i)$ and $\theta_j(z_j)$ respectively in the l.c.d of the entries of $T(s; z)$.

Stage III. Using the Markov parameters $M_{i_1, \ldots, i_q; j_1, \ldots, j_r}$ write the block Hankel matrices $H_{k;l}$, $\sigma_a H_{k;l}, a \in \bar{q}, \delta_b H_{k;l}, b \in \bar{r}$ for $k = (k_1, \ldots, k_q)$ and $l = (l_1, \ldots, l_r)$ and write the array

$$\mathcal{A} = \begin{bmatrix} I_{\tilde{p}} & H_{k;l} \\ I_{\tilde{m}} & M \end{bmatrix}$$

where $\tilde{p} = k_1 \ldots k_q l_1 \ldots l_r p$ and $\tilde{m} = k_1 \ldots k_q l_1 \ldots l_r m$.

Stage IV. By applying elementary rows operations (ERO) on the first block rows of $\mathcal{A}$ (i.e. on $[I_{\tilde{p}} \ H_{k;r}]$) and elementary column operations (ECO) on the second block column of $\mathcal{A}$ (i.e. on $[H_{k;r} \ I_{\tilde{m}}]$) transform $\mathcal{A}$ into the array

$$\tilde{\mathcal{A}} = \begin{bmatrix} P & \tilde{H} \\ M \end{bmatrix}$$

where $\tilde{H} = \begin{bmatrix} I_n & 0_n^{\tilde{m}-n} \\ 0_{\tilde{p}-n} \ 0_{\tilde{m}-n}^{\tilde{p}-n} \end{bmatrix}. \quad (4.27)$

Stage V. Determine the minimal realization $\Sigma = (\{A_{ci}, i \in \bar{q}\}, \{A_{dj}, j \in \bar{r}\}; B, C)$ by the following formulas:

$$A_{ci} = E_{n}^{\tilde{p}} P[\sigma_a H_{k;l}] M E_{n}^m, \quad a \in \bar{q}. \quad (4.28)$$

$$A_{db} = E_{n}^{\tilde{p}} P[\delta_b H_{k;l}] M E_{m}^n, \quad b \in \bar{r}. \quad (4.29)$$

$$B = E_{n}^{\tilde{p}} H_{k;l} E_{m}^n, \quad (4.30)$$

$$C = E_{n}^{\tilde{p}} H_{k;l} M E_{m}^n. \quad (4.31)$$

**Proof.** The matrices $P$ and $M$ being the results of ERO and respectively of ECO on the unit matrix, they are products of the corresponding elementary matrices, which are nonsingular, hence $P$ and $M$ are nonsingular too. Moreover, by (4.26) and (4.27) we get

$$PH_{k;l} M = \tilde{H} = E_{n}^{\tilde{p}} E_{m}^n \quad (4.32)$$

The matrix $Q = M E_{m}^n E_{n}^{\tilde{p}} P$ is the pseudoinverse of $H_{k;l}$, i.e.

$$H_{k;l} Q H_{k;l} = H_{k;l}. \quad (4.33)$$
Indeed
\[ \begin{align*}
P H_{k;l} Q H_{k;l} M &= PH_{k;l} M E_{a_n}^n F_p^p PH_{k;l} M \quad (4.32) \\
E_p^p E_{m}^m E_p^p &\quad (4.26) \\
E_p^p E_{m}^m &= PH_{k;l} M \
\end{align*} \]
By premultiplying and postmultiplying this equality by \( P^{-1} \) and \( M^{-1} \) respectively, we get (4.33).

Now let us show that \( A_{ca} \) and \( A_{db} \) are commutative matrices. By (4.20), (4.28) and (4.33) we have, for \( a_1, a_2 \in \bar{q} \): \( A_{ca} A_{ca} = \),

\[ \begin{align*}
&= \left( \prod_{a=1}^{q} A_{ca}^{a} \right) \left( \prod_{b=1}^{r} A_{db}^{b} \right) = \left( \prod_{a=1}^{q} \left( A_{ca}^{a} \right)^{b} \right) \left( \prod_{b=1}^{r} \left( A_{db}^{b} \right)^{a} \right) \\
&= \left( \prod_{a=1}^{q} A_{ca}^{a} \right) \left( \prod_{b=1}^{r} A_{db}^{b} \right) \
\end{align*} \]

Indeed (4.34), (4.22) and (4.23) we obtain

\[ \begin{align*}
\text{for} \quad n; \quad \text{we have} \quad \forall \sigma, \delta \quad \text{and by induction we get} \\
v_i \geq 0, \quad a \in \bar{q}, \quad \forall b \geq 0, \quad b \in \bar{r}. \
\end{align*} \]

Now, let us prove that (4.4) holds. Firstly, by (4.30), (4.31), (4.33), (4.24) and (4.25) and by the definition of \( H(k;l) \) we have

\[ \begin{align*}
C &\quad = \left( \prod_{a=1}^{q} A_{ca}^{a} \right) \left( \prod_{b=1}^{r} A_{db}^{b} \right) \\
&\quad = \left( \prod_{a=1}^{q} \left( A_{ca}^{a} \right)^{b} \right) \left( \prod_{b=1}^{r} \left( A_{db}^{b} \right)^{a} \right) \
&\quad = \left( \prod_{a=1}^{q} A_{ca}^{a} \right) \left( \prod_{b=1}^{r} A_{db}^{b} \right) \
&\quad \quad \quad \quad \quad \quad \quad (4.34) \quad \quad \quad (4.34) \
\end{align*} \]

\( \forall i_a \geq 0, \quad a \in \bar{q}, \quad \forall j_b \geq 0, \quad b \in \bar{r}. \)

For \( i_1, i_2, j_1, \ldots, j_r \geq 0 \), by (4.34), (4.30), (4.31), (4.33), (4.22) and (4.23) we obtain

\[ \begin{align*}
C &\quad = \left( \prod_{a=1}^{q} A_{ca}^{a} \right) \left( \prod_{b=1}^{r} A_{db}^{b} \right) \\
&\quad = \left( \prod_{a=1}^{q} A_{ca}^{a} \right) \left( \prod_{b=1}^{r} A_{db}^{b} \right) \\
&\quad = \left( \prod_{a=1}^{q} \left( A_{ca}^{a} \right)^{b} \right) \left( \prod_{b=1}^{r} \left( A_{db}^{b} \right)^{a} \right) \\
&\quad \quad \quad \quad \quad \quad \quad (4.34) \quad \quad \quad (4.34) \
\end{align*} \]
We have the array

\[ A = \begin{bmatrix} I_2 & H_{2,1} \\ I_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 4 & -8 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} . \]

By the permutation of the first two rows and then by the addition of the second row multiplied by 2 to the first row we transform \( A = \begin{bmatrix} 1/2 & 1/4 \\ 1/4 & 0 \\ 1/4 & 0 \\ 1/2 & 1/4 \end{bmatrix} \), hence \( \hat{H} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \), \( P = \begin{bmatrix} 0 & 4 & -8 & 12 \\ -8 & 4 & -8 & 12 \end{bmatrix} \), \( M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \). By (4.28) - (4.31) we get the minimal realization

\[ A_c = E_2^2P|\sigma_1H_{2,1}|ME_2^2 = \begin{bmatrix} 1/2 & 1/4 \\ 1/4 & 0 \end{bmatrix} \begin{bmatrix} 4 & -8 \\ -8 & 12 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} , \]

\[ A_d = E_2^2P|\sigma_2H_{2,1}|ME_2^2 = \begin{bmatrix} 1/2 & 1/4 \\ 1/4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 4 & -8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} , \]

\[ B = E_2^2PH_{2,1}E_2^2 = \begin{bmatrix} 1/2 & 1/4 \\ 1/4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 4 & -8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} , \]

\[ C = E_1^2H_{2,1}ME_2^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 4 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 4 & -8 \end{bmatrix} \]

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**References:**


