Minimal Realization Algorithm for Multidimensional Hybrid Systems

VALERIU PREPELIŢĂ
University Politehnica of Bucharest
Department of Mathematics I
Splaiul Independenţei 313, Bucharest
ROMANIA
vprepelita@mathem.pub.ro

Abstract: A class of multidimensional time-invariant hybrid systems is studied, which depends on q continuous-time variables and on r discrete-time ones. The formula of the input-output map of these systems is given. A multiple (q, r)-hybrid Laplace transformation is defined and it is used to determine the transfer matrices of the considered systems. The minimal realization problem is analysed and an algorithm is given which provides minimal realizations.

Key–Words: multidimensional hybrid systems, multiple hybrid Laplace transformation, minimal realization, controllability, reachability, time-invariant systems.

1 Introduction

The topic of 2D systems (and more generally, of nD systems) has known an important development in the last three decades. Their theory has become a distinct and significant branch of the Systems and Control Theory, due to their richness in theoretical approaches, as well as in potential application fields, such as control and signal processing, circuits, image processing, computer tomography, seismology etc.

A quite new direction in this theory is represented by the 2D hybrid (i.e. continuous-discrete) systems, whose state-space representation includes a differential-difference equation [5], [7], [8], [9]. These hybrid models are motivated by their applications in various areas such as long-wall coal cutting and metal rolling [12], linear repetitive processes [3], [11], pollution modelling [2] or in iterative learning control synthesis [6].

In the present paper a multidimensional hybrid model is studied. The state equation is a partial differential-difference equation, the states, the controls and the outputs being vector functions which depend on $q \geq 1$ continuous-time variables and $r \geq 1$ discrete-time variables. This model is an extension of the Attasi's 2D discrete-time system [1].

In Section 2 the state space representation is introduced for the (q, r) multidimensional hybrid control systems, and the formulas of the state and of the input-output map of these systems are derived from a variation-of-parameters type formula.

In Section 3 a multiple (q, r)-hybrid Laplace transformation is introduced as an operator defined on a class of suitable original functions. Some properties of this transformation are emphasized such as linearity, first and second time-delay theorems, differentiation of the original and differentiation and delay. These properties are used to obtain the transfer matrix of the considered class of systems. It is shown that the transfer matrix is a separable proper rational function, represented by a sum between a strictly proper matrix and a constant one.

Section 4 is devoted to the minimal realization problem. It is shown that a realization of a rational function having the above described structure is minimal if and only if it is a completely reachable and completely observable hybrid system. A criterion of minimality is obtained which is based on the Markov parameters of the strictly proper matrix. A sequence of block Hankel matrices is constructed by recurrence and two families of shift operators are defined. These devices are used to obtain an algorithm which provides a minimal realization. This method is an extension to multivariable hybrid systems of the Ho-Kalman algorithm [4].

The following notations are used in the paper: $q \in \mathbf{N}$ and $r \in \mathbf{N}$ being the number of continuous and discrete variables respectively, a function $x(t_1, \ldots, t_q;\ k_1, \ldots, k_r)$, $t_i \in \mathbf{R}$, $k_i \in \mathbf{Z}$ will be sometimes denoted by $x(t; k)$, where $t = (t_1, \ldots, t_q)$, $k = (k_1, \ldots, k_r)$. By $s \leq t,\ s, t \in \mathbf{R}^q$ we mean $s_i \leq t_i\ \forall i \in \bar{q}$ where $\bar{q} = \{1, 2, \ldots, q\}$ and a similar signification has $l \leq k,\ l, k \in \mathbf{Z}^r$; $(s; l) < (t; k)$ means

$s \le t$, $(l \le k)$ and $(s;l) \neq (t;k)$. For $t^0, t^1 \in \mathbf{R}^q$ and $k^0, k^1 \in \mathbf{Z}^r$, $t^0 < t^1$, $k^0 < k^1$ we denote by $[t^0, t^1]$ and $[k^0, k^1]$ respectively the sets $[t^0, t^1] = \prod_{i=1}^{q} [t_i^0, t_i^1]$ and $[k^0, k^1] = \prod_{i=1}^{r} \{k_j^0, k_j^0 + 1, \ldots, k_j^1\}$.

If $\tau = \{i_1, \ldots, i_l\}$ is a subset of $\overline{m}$, $|\tau| := l$ and $\tilde{\tau} := \overline{m} \setminus \tau$; for $i \in \overline{m}$, $\tilde{i} := \overline{m} \setminus \{i\}$ and $\tilde{\tilde{i}} := \{i+1, \ldots, m\}$. The notation $(\tau, \delta) \subset (\overline{q}, \overline{r})$ means that τ and δ are subsets of $\overline{q}$ and $\overline{r}$ respectively and $(\tau, \delta) \neq (\overline{q}, \overline{r})$. For $\tau = \{i_1, \ldots, i_l\}$ and $\delta = \{j_1, \ldots, j_h\}$ the operators $\frac{\partial}{\partial \tau}$ and σ_δ are defined by

$$\frac{\partial}{\partial \tau} x(t;k) = \frac{\partial^l}{\partial t_{i_1} \ldots \partial t_{i_l}} x(t;k),$$

$$\sigma_\delta x(t;k) = x(t; k + e_\delta)$$

where

$$e_\delta = e_{j_1} + \ldots + e_{j_h}, \; e_j = (\underbrace{0, \ldots, 0}_{j-1}, 1, 0, \ldots, 0) \in \mathbf{R}^r;$$

when $\tau = \overline{q}$ and $\delta = \overline{r}$ we denote $\partial / \partial \tau = \partial / \partial t$ and $\sigma_\delta = \sigma$.

If A_i, $i \in \overline{m}$ is a family of matrices, $\sum_{i \in \emptyset} A_i = 0$ and $\prod_{i \in \emptyset} A_i = I$.

2 State space representation

The time set of the hybrid multidimensional system is $T = \mathbf{R}_+^q \times \mathbf{Z}_+^r$, $q, r \in \mathbf{N}^*$.

Definition 2.1. A (q,r)-D *hybrid system* is a set $\Sigma = (\{A_{ci} | i \in \overline{q}\}, \{A_{dj} | j \in \overline{r}\}, B, C, D)$ with A_{ci}, $i \in \overline{q}$ and A_{dj}, $j \in \overline{r}$ commuting $n \times n$ matrices $\forall t \in \mathbf{R}^q$, $\forall k \in \mathbf{Z}^r$ and B, C, D respectively $n \times m$, $p \times n$ and $p \times m$ real matrices; the state equation is

$$\frac{\partial}{\partial t} \sigma x(t;k) = \sum_{(\tau,\delta) \subset (\overline{q},\overline{r})} (-1)^{q+r-|\tau|-|\delta|-1} \times \times \left(\prod_{i \in \tilde{\tau}} A_{ci} \right) \left(\prod_{j \in \tilde{\delta}} A_{dj} \right) \frac{\partial}{\partial \tau} \sigma_\delta x(t;k) + Bu(t;k) \tag{2.1}$$

and the output equation is

$$y(t;k) = Cx(t;k) + Du(t;k) \tag{2.2}$$

where

$$x(t;k) = x(t_1, \ldots, t_q; k_1, \ldots, k_r) \in \mathbf{R}^n$$

is the *state*, $u(t;k) \in \mathbf{R}^m$ is the *input* and $y(t;k) \in \mathbf{R}^p$ is the *output*. The number n is called the *dimension of the system* Σ and it is denoted dimΣ.

For $\tau = \{i_1, \ldots, i_l\} \subset \overline{q}$, $\delta = \{j_1, \ldots, j_h\} \subset \overline{r}$ and $t_i \in \mathbf{R}$, $i \in \tau$, $k_j \in \mathbf{Z}$, $j \in \delta$, we use the notation $x(t_\tau; k_\delta) := x(s_1, \ldots, s_q; l_1, \ldots, l_r)$., where $s_i = \begin{cases} t_i & \text{if} \quad i \in \tau \\ 0 & \text{if} \quad i \in \tilde{\tau} \end{cases}$ and $l_j = \begin{cases} k_j & \text{if} \quad j \in \delta \\ 0 & \text{if} \quad j \in \tilde{\delta}. \end{cases}$

Definition 2.2. The vector $x^0 \in \mathbf{R}^n$ is called an *initial state* of the system Σ if

$$x(t_\tau; k_\delta) = \left(\prod_{i \in \tau} e^{A_{ci} t_i} \right) \left(\prod_{j \in \delta} A_{dj}^{k_j} \right) x^0 \tag{2.3}$$

for any $(\tau, \delta) \subset (\overline{q}, \overline{r})$; equalities (2.3) are called the *initial conditions of* Σ.

We can prove (see [9])

Proposition 2.3. *The solution of the initial value problem*

$$\frac{\partial}{\partial t} \sigma x(t;k) = \sum_{(\tau,\delta) \subset (\overline{q},\overline{r})} (-1)^{q+r-|\tau|-|\delta|-1} \cdot \cdot \left(\prod_{i \in \tilde{\tau}} A_{ci} \right) \left(\prod_{j \in \tilde{\delta}} A_{dj} \right) \frac{\partial}{\partial \tau} \sigma_\delta x(t;k) + f(t;k) \tag{2.4}$$

with the initial conditions (2.3) *is given by the generalized variation-of-parameters formula*

$$x(t;k) = \left(\prod_{i=1}^{q} e^{A_{ci} t_i} \right) \left(\prod_{j=1}^{r} A_{dj}^{k_j} \right) x^0 + + \int_0^{t_1} \cdots \int_0^{t_q} \left(\prod_{i=1}^{q} e^{A_{ci}(t_i - s_i)} \right) \cdot \cdot \sum_{l_1=0}^{k_1-1} \cdots \sum_{l_r=0}^{k_r-1} \left(\prod_{j=1}^{r} A_{dj}^{k_j - l_j - 1} \right) f(s;l) ds_1 \ldots ds_q; \tag{2.5}$$

here $s = (s_1, \ldots, s_q)$, $l = (l_1, \ldots, l_r)$; $f : \mathbf{R}^q \times \mathbf{Z}^r \to \mathbf{R}^n$ *is a continuous function with respect to* $t = (t_1, \ldots, t_q)$.

Theorem 2.4. *The state of the system* Σ (2.1) *determined by the initial state* $x_0 \in \mathbf{R}^n$ *and the control* *u is*

$$x(t;k) = \left(\prod_{i=1}^{q} e^{A_{ci}t_i}\right)\left(\prod_{j=1}^{r} A_{dj}^{k_j}\right)x^0 + \\ + \int_0^{t_1}\cdots\int_0^{t_q}\left(\prod_{i=1}^{q} e^{A_{ci}(t_i-s_i)}\right)\cdot \\ \cdot\sum_{l_1=0}^{k_1-1}\cdots\sum_{l_r=0}^{k_r-1}\left(\prod_{j=1}^{r} A_{dj}^{k_j-l_j-1}\right) \\ Bu(s,l)ds_1\ldots ds_q. \tag{2.6}$$

Proof. Equation (2.1) has the form (2.4) with $f(t;k) = Bu(t;k)$ and (2.6) results by replacing $f(t;k)$ in (2.5).

If we replace the state $x(t;k)$ (2.6) in the output equation (2.2) we obtain

Theorem 2.5. *The general response of the* (q,r)-D *hybrid system* Σ (2.1), (2.2) *is* given by the formula

$$y(t;k) = C\left(\prod_{i=1}^{q} e^{A_{ci}t_i}\right)\left(\prod_{j=1}^{r} A_{dj}^{k_j}\right)x^0 + \\ + \int_0^{t_1}\cdots\int_0^{t_q} C\left(\prod_{i=1}^{q} e^{A_{ci}(t_i-s_i)}\right)\cdot \\ \cdot\sum_{l_1=0}^{k_1-1}\cdots\sum_{l_r=0}^{k_r-1}\left(\prod_{j=1}^{r} A_{dj}^{k_j-l_j-1}\right) \\ Bu(s,l)ds_1\ldots ds_q + Du(t;k). \tag{2.7}$$

Example 2.6. Let us consider the (1,1)-D hybrid system $\Sigma = (A_c, A_d, B, C, D)$, where $A_c = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$, $A_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -1 \end{bmatrix}$ and $D = 0$. !!!!!!!

3 Multiple (q,r)-hybrid Laplace transformation and transfer matrices

Definition 3.1. A function $f : \mathbf{R}^q \times \mathbf{Z}^r \to \mathbf{C}$ is said to be a *continuous-discrete original function* (or simply an *original*) if f has the following properties:

(i) $f(t_1,\ldots,t_q;k_1,\ldots,k_r) = 0$ if $t_i < 0$ or $k_j < 0$ for some $i \in \bar{q}$ or $j \in \bar{r}$.

(ii) $f(\cdot,\ldots,\cdot;k_1,\ldots,k_r)$ is piecewise smooth on $\mathbf{R}_+^q$ for any $(k_1,\ldots,k_r) \in \mathbf{Z}_+^r$.

(iii) $\exists M_f > 0$, $\sigma_{fi} \geq 0$, $i \in \bar{q}$, $R_{fj} > 0$, $j \in \bar{r}$ such that

$$|f(t_1,\ldots,t_q;k_1,\ldots,k_r)| \leq \\ \leq M_f \exp\left(\sum_{i=1}^{q}\sigma_{fi}t_i\right)\prod_{j=1}^{r} R_{fj}^{k_j} \tag{3.1}$$

$\forall t_i > 0$, $i \in \bar{q}$, $\forall k_j \geq 0$, $j \in \bar{r}$.

The constants σ_{fi}, R_{fj} will be also denoted by σ_i, R_j. The smallest such constants are called respectively *the indices of the order of growth* and *the radii of convergence* of the original function f.

We denote by $\mathcal{O}_{q,r}$ the set of original functions $f : \mathbf{R}^q \times \mathbf{Z}^r \to \mathbf{C}$. Sometimes we shall denote by $f(t;k)$ the value of f at $t = (t_1,\ldots,t_q)$, $k = (k_1,\ldots,k_r)$.

Definition 3.2. For any original f, the function

$$F(s_1,\ldots,s_q;z_1,\ldots,z_r) = \\ = \int_0^{\infty}\cdots\int_0^{\infty}\sum_{k_1=0}^{\infty}\cdots\sum_{k_r=0}^{\infty} \\ f(t_1,\ldots,t_q;k_1,\ldots,k_r)\cdot \\ \cdot e^{-s_1t_1}\ldots e^{-s_qt_q}z_1^{-k_1}\ldots z_q^{-k_r}dt_1\ldots dt_q \tag{3.2}$$

is called the (q,r)-*hybrid Laplace transform* (or the *image*) of f.

We shall use also the notation which defines the Laplace Transformation $\mathcal{L}_{q,r}$: $F(s;z) = \mathcal{L}_{q,r}[f(t;k)]$, where $s = (s_1,\ldots,s_q)$ and $z = (z_1,\ldots,z_r)$.

The following results are proved in [10].

Proposition 3.3. *The multiple improper integral and the multivariable Taylor series in* (3.2) *are absolutely convergent in the domain*

$$D(f) = \{(s_1,\ldots,s_q;z_1,\ldots,z_r) \in \mathbf{C}^{q+r} | \text{Re } s_i > \\ > \sigma_{fi},\ i \in \bar{q};\ |z_j| > R_{fj},\ j \in \bar{r}\} \tag{3.3}$$

and uniformly convergent on any domain

$$D'(f) = \{(s_1,\ldots,s_q;z_1,\ldots,z_r) \in \mathbf{C}^{q+r} | \text{Re } s_i \geq \sigma_i', \\ i \in \bar{q};\ |z_j| \geq R_j',\ j \in \bar{r}\}$$

with $\sigma_i' > \sigma_{fi}$, $i \in \bar{q}$ *and* $R_j' > R_{fj}$, $j \in \bar{r}$.

Theorem 3.4 (Linearity). *For any* $f, g \in \mathcal{O}_{q,r}$ *and* $\alpha, \beta \in \mathbf{C}$,

$$\mathcal{L}_{q,r}[\alpha f + \beta g] = \alpha\mathcal{L}_{q,r}[f] + \beta\mathcal{L}_{q,r}[g]. \tag{3.4}$$

Theorem 3.5 (First time-delay theorem). *For any* $(a_1,\ldots,a_q) \in \mathbf{R}_+^q$, $(b_1,\ldots,b_r) \in \mathbf{Z}_+^r$,

$$\mathcal{L}_{q,r}[f(t_1-a_1,\ldots,t_q-a_q;k_1-b_1,\ldots,k_r-b_r)] = \\ = \exp\left(-\sum_{i=1}^{q}a_is_i\right)\left(\prod_{j=1}^{r} z_j^{-b_j}\right)\cdot \\ \cdot F(s_1,\ldots,s_q;z_1,\ldots,z_r). \tag{3.5}$$

We shall use the following notations: for some sets $\alpha = \{i_1, \dots, i_p\} \subset \bar{q}$ and $\beta = \{j_1, \dots, j_q\} \subset \bar{r}$, $E_\alpha = \{\varepsilon | \varepsilon \subset \alpha \text{ or } \varepsilon = \emptyset\}$, $E'_\beta = \{\delta | \delta \subset \beta \text{ or } \delta = \emptyset\}$

For $a = (a_i)_{i\in\alpha} \in \mathbf{R}_+^{|\alpha|}$ with $a_i > 0$, $\forall i \in \alpha$ and $b = (b_j)_{j\in\beta} \in \mathbf{Z}_+^{|\beta|}$, with $b_j > 0$, $\forall j \in \beta$ and for $\varepsilon = (\varepsilon_1, \dots, \varepsilon_\gamma) \in E_\alpha$ and $\delta = (\delta_1, \dots \delta_\mu) \in E'_\beta$ we denote by $D_{a,\varepsilon}$ and $D'_{b,\delta}$ the sets $D_{a,\varepsilon} = \prod_{i\in\varepsilon}[0, a_i]$ and $D'_{b,\delta} = \prod_{j\in\delta}\{0, 1, \dots, b_j - 1\}$ and by $\int_{D_{a,\varepsilon}}$ and $\sum_{D'_{b,\delta}}$ the multiple integral $\int_0^{\varepsilon_1} \dots \int_0^{\varepsilon_\gamma}$, respectively the multiple sum $\sum_{k_{\delta_1}=0}^{b_{\delta_1}-1} \dots \sum_{k_{\delta_\mu}=0}^{b_{\delta_\mu}-1}$; if $\varepsilon = \emptyset$ or $\delta = \emptyset$ the corresponding multiple integral or sum lack; $f(t+a; k+b)$ denotes

$$f(t_1, \dots t_{i_1-1}, t_{i_1} + a_{i_1}, t_{i_1+1}, \dots, t_{i_p-1}, t_{i_p}+$$
$$+a_{i_p}, t_{i_p+1}, \dots, t_q; k_1, \dots, k_{j_1-1}, k_{j_1}+$$
$$+b_{j_1}, k_{j_1+1} \dots k_{j_q-1}, k_{j_q} + b_{j_q}, k_{j_q+1}, \dots, k_r).$$

Definition 3.6. For $\alpha = \{i_1, \dots, i_p\} \subset \bar{q}$ and $\beta = \{j_1, \dots, j_q\} \subset \bar{r}$, the (α, β)-*partial* (q, r)-*hybrid Laplace transform* of the original f is defined by

$$\mathcal{L}_{q,r}^{\alpha,\beta}[f(t,k)] = \int_0^\infty \dots \int_0^\infty \sum_{k_{j_1}=0}^{\infty} \dots \sum_{k_{j_r}=0}^{\infty}$$
$$f(t_1, \dots, t_q; k_1, \dots, k_r) \cdot \tag{3.6}$$
$$\cdot \exp\left(-\sum_{i\in\alpha} s_i t_i\right)\left(\prod_{j\in\beta} z_j^{-k_j}\right) dt_{i_1} \dots dt_{i_p}.$$

If $\alpha = \bar{q}$ and $\beta = \bar{r}$, $\mathcal{L}_{q,r}^{\alpha,\beta} = \mathcal{L}_{q,r}$; if $\beta = \phi$, $\mathcal{L}_{q,r}^{\alpha,\phi} = \mathcal{L}_p$ (the multiple Laplace transformation); if $\alpha = \phi$, $\mathcal{L}_{q,r}^{\phi,\beta} = \mathcal{Z}_q$ (the multiple z-transformation); if $\alpha = \beta = \phi$, $\mathcal{L}_{q,r}^{\phi,\phi}[f] = f$.

Theorem 3.7 (Second delay theorem). *For any* $a = (a_i)_{i\in\alpha} \in \mathbf{R}_+^{|\alpha|}$ *and* $b = (b_j)_{j\in\beta} \in \mathbf{Z}_+^{|\beta|}$

$$\mathcal{L}_{n,m}[f(t+a; k+b)] = \exp\left(\sum_{i\in\alpha} a_i s_i\right)\left(\prod_{j\in\beta} z_j^{b_j}\right) \cdot$$
$$\cdot[F(s;z) + \sum_{\varepsilon\in E_\alpha}\sum_{\delta\in E_\beta} a_i s_i (-1)^{|\varepsilon|+|\delta|}\int_{D_{a,\varepsilon}}\sum_{D'_{b,\gamma}}$$
$$\mathcal{L}_{n,m}^{\bar{\varepsilon},\bar{\delta}}[f(t,k)]\exp\left(-\sum_{i\in\varepsilon} s_i t_i\right)\left(\prod_{j\in\delta} z_j^{-k_j}\right)\prod_{i\in\varepsilon} dt_i. \tag{3.7}$$

We introduce the following notations: given $\alpha = \{i_1, \dots, i_p\} \subset \bar{q}$, a p-tuple $(\gamma_{i_1}, \dots \gamma_{i_p}) \in \mathbf{N}^p$ is denoted by γ_α or simply by γ and $\dfrac{\partial^\gamma f}{\partial t^\gamma} = \dfrac{\partial^{\gamma_{i_1}+\dots+\gamma_{i_p}}}{\partial t_{i_1}^{\gamma_{i_1}} + \dots + \partial t_{i_p}^{\gamma_{i_p}}}$, $s^\gamma = s_{i_1}^{\gamma_{i_1}} \dots s_{i_p}^{\gamma_{i_p}}$. The family of all unvoid subsets ε of α is denoted by E_γ^α or E_γ. For $\varepsilon \in E_\gamma^\alpha$, $\bar{\varepsilon} = \alpha \setminus \varepsilon$, $s_\varepsilon^{\gamma_\varepsilon} = \prod_{i\in\varepsilon} s_i^{\gamma_i}$ and $s_{\bar{\varepsilon}}^{\gamma_{\bar{\varepsilon}}} = 1$ if $\varepsilon = \alpha$; if $\varepsilon = \{\iota_1, \dots, \iota_\rho\}$ and $\eta_\varepsilon = (\eta_{\iota_1}, \dots, \eta_{\iota_\rho}) \in \mathbf{N}^\rho$, $\eta_\varepsilon \le \gamma_\varepsilon$ means $\eta_i \le \gamma_i$, $\forall i \in \varepsilon$; $f(0_\varepsilon^+; k)$ denotes the limit from the right

$$f(t_1, \dots t_{\iota_1-1}, 0 + t_{\iota_1+1}, \dots, t_{\iota_\rho-1},$$
$$0+, t_{\iota_\rho+1}, \dots, t_q; k_1, \dots, k_r);$$

if $\varepsilon = \{i\}$ then $f(0_\varepsilon^+; k)$ is denoted by $f(0_i^+; k)$. Similarly $f(t; k_1, \dots, k_{j-1}, 0, k_j, \dots, k_r)$ is denoted $f(t; 0_j)$ and we can use the notation $f(0_i^+; 0_j)$ which combines these notations.

Theorem 3.8 (Differentiation of the original). *For any* $i \in \bar{q}$

$$\mathcal{L}_{q,r}\left[\frac{\partial f}{\partial t_i}(t;k)\right] = s_i F(s;z) - \mathcal{L}_{q,r}^{\tilde{i},\bar{r}}[f(0_i^+;k)] \tag{3.8i}$$

$$\mathcal{L}_{q,r}\left[\frac{\partial^\gamma f}{\partial t^\gamma}(t;k)\right] = s^\gamma F(s;z) +$$
$$+\sum_{\varepsilon\in E_\gamma}(-1)^{|\varepsilon|} s_{\bar{\varepsilon}}^{\gamma_{\bar{\varepsilon}}} \tag{3.8ii}$$
$$\sum_{\eta_\varepsilon \le \gamma_\varepsilon - 1} s_\varepsilon^{\gamma_\varepsilon - \eta_\varepsilon - 1}\mathcal{L}_{q,r}^{\tilde{\varepsilon},\bar{r}}\left[\frac{\partial^{\gamma_\varepsilon} f}{\partial t^{\gamma_\varepsilon}}(0_\varepsilon^+;k)\right].$$

Theorem 3.9 (Differentiation and delay). *For any* $i \in \bar{q}$, $j \in \bar{r}$,

$$\mathcal{L}_{q,r}\left[\frac{\partial f}{\partial t_i}(t_1, \dots, t_q; k_1, \dots, k_{j-1}, k_j + 1, k_{j+1}, \dots,\right.$$
$$k_r)] = s_i z_j F(s,z) - s_i z_j \mathcal{L}_{q,r}^{\bar{q},\tilde{j}}[f(t;0_j)] - \tag{3.9i}$$
$$-z_j \mathcal{L}_{q,r}^{\tilde{i},\bar{r}}[f(0_i^+;k)] + z_j f(0_i^+;0_j)].$$

For any $\gamma = (\gamma_{i_1}, \dots, \gamma_{i_p}) \in \mathbf{N}^p$, $b = (b_1, \dots, b_r) \in \mathbf{N}^r$,

$$\mathcal{L}_{q,r}\left[\frac{\partial^\gamma f}{\partial t^\gamma}(t;k+b)\right] = s^\gamma z^b F(s,z) +$$
$$+z^b \sum_{\varepsilon\in E_\gamma}\sum_{\delta\in E_\beta}(-1)^{|\varepsilon|+|\delta|} s_{\tilde{\varepsilon}}^{\gamma_{\tilde{\varepsilon}}} \sum_{\eta_\varepsilon\le\gamma_\varepsilon-1} s_\varepsilon^{\gamma_\varepsilon-\eta_\varepsilon-1} \cdot \tag{3.9ii}$$
$$\cdot \sum_{D'_{b,\delta}} \mathcal{L}_{q,r}^{\tilde{\varepsilon},\tilde{\delta}}\left[\frac{\partial^{\eta_\varepsilon} f}{\partial t^{\eta_\varepsilon}}(0_\varepsilon^+;0_\delta)\right]\left(\prod_{j\in\delta} z_j^{-k_j}\right).$$

Let us consider the time-invariant system Σ, i.e. the system with constant matrices A_{ci}, A_{dj}, B, C and D. Obviously, we can extend the multiple hybrid Laplace to vector functions

$$x(t;k) = [x_1(t;k)\; x_2(t;k)\; \dots\; x_n(t;k)]^T$$

by

$$X(s;z) = \mathcal{L}[x(t;k)] = $$
$$= [\mathcal{L}[x_1(t;k)]\; \mathcal{L}[x_2(t;k)]\; \dots\; \mathcal{L}[x_n(t;k)]].$$

By linearity (Theorem 3.4), if we apply the multiple (q,r)-hybrid Laplace transform $\mathcal{L}_{q,r}$ to the state equation (2.1) we get

$$\mathcal{L}_{q,r}\left[\frac{\partial}{\partial t}\sigma x(t;k)\right] =$$
$$= \sum_{(\tau,\delta)\subset(\bar{q},\bar{r})} (-1)^{q+r-|\tau|-|\delta|-1}\left(\prod_{i\in\tilde{\tau}} A_{ci}\right)\left(\prod_{j\in\tilde{\delta}} A_{dj}\right)\cdot$$
$$\cdot\mathcal{L}_{q,r}\left[\frac{\partial}{\partial \tau}\sigma_\delta x(t,h)\right] + B\mathcal{L}_{q,r}[u(t;h)].$$

By Theorem 3.9, using formula (3.9ii) for $a = (1,1,\dots,1) \in \mathbf{N}^q$ and $b = (1,1,\dots,1) \in \mathbf{N}^r$ this equality becomes:

$$s_1 s_2 \dots s_q z_1 z_2 \dots z_r X(s;z) + T_1 =$$
$$= \sum_{(\tau,\delta)\subset(\bar{q},\bar{r})} (-1)^{q+r-|\tau|-|\delta|-1}\left(\prod_{i\in\tilde{\tau}} A_{ci}\right)\left(\prod_{j\in\tilde{\delta}} A_{dj}\right)\cdot$$
$$\cdot\left(\prod_{i\in\tau} s_i\right)\left(\prod_{j\in\delta} z_j\right) X(s;z) + T_2 + BU(s;z)$$

where

$$T_1 = z_1 z_2 \dots z_r \sum_{\varepsilon\in E_\gamma}\sum_{\delta\in E_\beta}(-1)^{|\varepsilon|+|\delta|} s_{\tilde{\varepsilon}}^{\gamma_{\tilde{\varepsilon}}}$$
$$\sum_{n_\varepsilon\le\gamma_\varepsilon-1} s_\varepsilon^{r_\varepsilon-\eta_\varepsilon-1}\left(\prod_{j\in\delta} z_j^{-k_j}\right)\cdot$$
$$\cdot\sum_{D'_{b,\delta}}\mathcal{L}_{q,r}^{\tilde{\varepsilon},\tilde{\delta}}\left[\mathcal{L}\left[\frac{\partial^{\eta_\varepsilon} x}{\partial t^{\eta_\varepsilon}}(0_\varepsilon^+;0_\delta)\right]\right]$$

and

$$T_2 = z^\delta \sum_{\varepsilon\in E_\tau}\sum_{\lambda\in E_\delta}(-1)^{|\varepsilon|+|\lambda|} s_{\tilde{\varepsilon}}^{\gamma_{\tilde{\varepsilon}}}\sum_{\eta_\varepsilon\le\delta_\varepsilon-1} s_\varepsilon^{\delta_\varepsilon-\eta_\varepsilon-1}\cdot$$
$$\cdot\sum_{D'_{\delta,\lambda}}\mathcal{L}_{q,r}^{\tilde{\varepsilon},\tilde{\lambda}}\left[\frac{\partial^{\eta_\varepsilon} x}{\partial t^{\eta_\varepsilon}}(0_\varepsilon^+;0_\lambda)\right]\left(\prod_{j\in\lambda} z_j^{-k_j}\right).$$

This equation can be written as

$$\left(\prod_{i=1}^{q}(s_i I - A_{ci})\right)\left(\prod_{j=1}^{r}(z_j I - A_{dj})\right) X(s;z) =$$
$$= BU(s;z) + T_3(s;z)$$

where $T_3(s;z) = T_2 - T_1$.

For $s_i \in \mathbf{C}\setminus\sigma(A_{ci})$, $\forall i\in\bar{q}$, $z_j\in\mathbf{C}\setminus\sigma(A_{dj})$, $\forall j\in\bar{r}$ (where $\sigma(A)$) denotes the spectrum of a matrix A) we premultiply this equation by the products of the matrices $(s_i I - A_{ci})^{-1}$ and $(z_j I - A_{dj})^{-1}$ and we obtain the formula of the state of the system in the frequency domain

$$X(s;z) = \left(\prod_{i=1}^{q}(s_i I - A_{ci})^{-1}\right)\cdot$$
$$\cdot\left(\prod_{j=1}^{r}(z_j I - A_{dj})^{-1}\right) BU(s;z) + T_4(s;z) \tag{3.10}$$

where

$$T_4(s;z) = \left(\prod_{i=1}^{q}(s_i I - A_{ci})^{-1}\right)\cdot$$
$$\cdot\left(\prod_{j=1}^{r}(z_j I - A_{dj})^{-1}\right) T_3(s;z). \tag{3.11}$$

Again by Theorem 3.4, by applying the operator $\mathcal{L}_{q,r}$ to the output equation (2.2), one obtains

$$Y(s;z) = CX(s;z) + DU(s;z). \tag{3.12}$$

By replacing the state $X(s;z)$ given by (3.10) in (3.12), we get the input-output map of the system Σ in the frequency domain:

$$Y(s;z) = \left[C\left(\prod_{i=1}^{q}(s_i I - A_{ci})^{-1}\right)\cdot\right.$$
$$\left.\cdot\left(\prod_{j=1}^{r}(z_j I - A_{dj})^{-1}\right) B + D\right] U(s,z) + CT_4(s,z) \tag{3.13}$$

Now we consider null boundary conditions

$$\frac{\partial^{\eta_\varepsilon} x}{\partial t^{\eta_\varepsilon}}(0_\varepsilon^+;0_\delta) = 0 \tag{3.14}$$

$\forall\varepsilon\in E_\gamma$, $\forall\delta\in E_b$, $\gamma = (1,1,\dots,1)\in\mathbf{N}^q$, $b = (1,1,\dots,1)\in\mathbf{N}^r$. We obtain:

Theorem 3.10. *For null boundary conditions* (3.14), *the input-output map of the system* Σ *is*

$$Y(s;z) = T(s;z)U(s;z) \tag{3.15}$$

where

$$T_\Sigma(s;z) = C\left(\prod_{i=1}^{q}(s_iI - A_{ci})^{-1}\right)\cdot$$

$$\cdot\left(\prod_{j=1}^{r}(z_jI - A_{dj})^{-1}\right)B + D. \qquad (3.16)$$

The matrix $T_\Sigma(s;z)$ (3.16) *is called the transfer matrix of the system* Σ.

A rational matrix $T(s;z)$ is said to be *proper* if its elements have the form

$$t_{ij}(s;z) = \frac{a_{ij}(s_1,\ldots,s_q;z_1,\ldots,z_r)}{b_{ij}(s_1,\ldots,s_q;z_1,\ldots,z_r)}$$

and $deg_{s_k}a_{ij} \le deg_{s_k}b_{ij}$, $\forall k \in \bar{q}$, $deg_{z_l}b_{ij}$, $\forall l \in \bar{r}$, $\forall i \in \bar{p}$, $\forall j \in \bar{m}$, where $deg_{s_k}a_{ij}$ denotes the degree of the polynomial a_{ij} w.r.t. the variable s_k.

If all these inequalities are strict ones, $T(s;z)$ is said to be *strictly proper*. If $T(s;z) = \dfrac{1}{\pi(s)\theta(z)}M(s,z)$ where $\pi(s)$ and $\theta(z)$ are polynomials of the form $\pi(s) = \pi_1(s_1)\ldots\pi_q(s_q)$, $\theta(z) = \theta_1(z_1)\ldots\theta_r(z_r)$ and $M(s;z)$ is a polynomial matrix, then $T(s;z)$ is said to be with *separable denominator*.

The following characterization of $T_\Sigma(s;z)$ is a direct consequence of (3.16):

Proposition 3.11. *The transfer matrix of a* (q,r)*-D hybrid system is a rational proper* $p\times m$ *matrix with separable denominator. If* $D = 0$ *(the* $p\times m$ *null matrix) then* $T_\Sigma(s;z)$ *is strictly proper*.

We shall denote by $\mathcal{T}_S(s;z)$ the set of the proper rational matrices $T(s;z)$ with separable denominator which can be decomposed as a sum between a strictly proper matrix and a constant one.

Example 3.12. The system $\Sigma = (A_c, A_d, B, C, D)$ considered in Example 2.6 has the strictly proper separable transfer matrix

$$T_\Sigma(s,z) = C(sI - A_c)^{-1}(zI - A_d)^{-1}B + D$$

hence

$$T_\Sigma(s,z) = [1 \quad -1]\frac{1}{s^2+2s+1}\begin{bmatrix} s+2 & 1\\ -1 & s\end{bmatrix}\frac{1}{z-1}$$

$$\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}\cdot\cdot\begin{bmatrix} 1\\ 1\end{bmatrix} + 0 = \frac{4}{(s+1)^2(z-1)}.$$

4 Minimal realizations

In this section we shall give an algorithm which provides minimal realizations for proper matrices with separable denominator.

Definition 4.1. Given a rational matrix $T(s;z) \in \mathcal{T}_S(s;z)$, a system $\Sigma = (\{A_{ci}|i \in \bar{q}\}, \{A_{dj}|j \in \bar{r}\}, B, C, D)$ is said to be a *realization* of $T(s;z)$ is $T(s;z) = T_\Sigma(s;z)$, i.e. if

$$\begin{aligned} T(s;z) = C&\left(\prod_{i=1}^{q}(s_i - A_{ci})^{-1}\right)\cdot\\ &\cdot\left(\prod_{j=1}^{r}(z_j - A_{dj})^{-1}\right)B + D. \end{aligned} \qquad (4.1)$$

The realization Σ is said to be *minimal* if dim$\Sigma \le$ dim$\hat{\Sigma}$ for any realization $\hat{\Sigma}$ of $T(s;z)$.

Since the matrix D can be determined by (3.17), the *realization problem* will be formulated as follows: given a strictly proper $p\times m$ matrix $T(s;z) \in \mathcal{T}_S(s;z)$, determine the system $\Sigma = (\{A_{ci}|i \in \bar{q}\}, \{A_{dj}|j \in \bar{r}\}, B, C)$ such that

$$\begin{aligned} T(s;z) = C&\left(\prod_{i=1}^{q}(s_iI - A_{ci})^{-1}\right)\cdot\\ &\cdot\left(\prod_{j=1}^{r}(z_jI - A_{dj})^{-1}\right)B. \end{aligned} \qquad (4.2)$$

Following the lines of the proof in [7, Theorem 5.4], we obtain (see [8] and [9]):

Theorem 4.2. *A system* Σ *is a minimal realization of a strictly proper matrix* $T(s;z) \in \mathcal{T}_S(s;z)$ *if and only if* Σ *is completely reachable and completely observable*.

Now, let us expand $T(s;z)$ in Laurent series about infinity:

$$\begin{aligned} &T(s_1,\ldots,s_q;z_1,\ldots,z_r) =\\ &= \sum_{i_1=0}^{\infty}\cdots\sum_{i_q=0}^{\infty}\sum_{j_1=0}^{\infty}\cdots\sum_{j_r=0}^{\infty} M_{i_1,\ldots,i_q;j_1,\ldots,j_r}\cdot\\ &\cdot\left(\prod_{k=1}^{q} s_k^{-i_k-1}\right)\left(\prod_{l=1}^{r} z_l^{-j_k-1}\right). \end{aligned} \qquad (4.3)$$

The constant $p\times m$ matrices $M_{i_1,\ldots,i_q;j_1,\ldots,j_r}$ are called the *Markov parameters* of the matrix $T(s;z)$.

Theorem 4.3. *A system* $\Sigma = (\{A_{ci}|i \in \bar{q}\}, \{A_{dj}|j \in \bar{r}\}, B, C)$ *is a realization of the strictly proper matrix* $T(s;z)$ (4.3) *if and only if*

$$\begin{aligned} M_{i_1,\ldots,i_q;j_1,\ldots,j_r} = C\left(\prod_{k=1}^{q} A_{ck}^{i_k}\right)\left(\prod_{l=1}^{r} A_{dl}^{j_l}\right)B\\ \forall i_k \ge 0,\ k \in \bar{q},\ \forall j_l \ge 0,\ l \in \bar{r}. \end{aligned} \qquad (4.4)$$

Proof. For any square matrix A and $|s| > \max\limits_{\lambda\in\sigma(A)}|\lambda|$, the following Laurent series expansion holds: $(sI-A)^{-1}=\sum_{i=0}^{\infty}A^i s^{-i-1}$. Then (4.2) gives for any realization Σ of $T(s;z)$:

$$T(s;z)=\sum_{i_1=0}^{\infty}\cdots\sum_{i_q=0}^{\infty}\sum_{j_1=0}^{\infty}\cdots\sum_{j_r=0}^{\infty}C\left(\prod_{k=1}^{q}A_{ck}^{i_k}\right)\cdot \cdot\left(\prod_{l=1}^{r}A_{dl}^{j_l}\right)B\left(\prod_{k=1}^{q}s_k^{-i_k-1}\right)\left(\prod_{l=1}^{r}z_l^{-j_l-1}\right). \quad (4.5)$$

Therefore, Σ is a realization of $T(s;z)$ if and only if the Laurent series in (4.3) and (4.5) are equal, and this condition is equivalent to the equality of all their corresponding coefficients, i.e. with (4.4). □

Now we shall use the following notations: k and j denotes respectively $k_1,\ldots,k_q$ and $j_1,\ldots,j_r$ while $\hat{i}_a$ denotes $i_a,i_{a+1}\ldots,i_q$ and $\hat{j}_b$ $j_b,j_{b+1},\ldots,j_r$.

We associate to the strictly proper matrix $T(s,z)$ the following sequence of block Hankel matrices, for $i_\alpha\geq 0$, $k_\alpha\geq 1$, $\alpha\in\bar{q}$, $d_\beta\geq 0$, $l_\beta\geq 1$, $\beta\in\bar{r}$

$$H_{k_1}^{i_2,\ldots,i_q;j_1,\ldots,j_r}=$$

$$=\begin{bmatrix} M_{0,\hat{i}_2,;j} & M_{1,\hat{i}_2,;j} & \ldots & M_{k_1-1,\hat{i}_2,;j}\\ M_{1,\hat{i}_2,;j} & M_{2,\hat{i}_2,;j} & \ldots & M_{k_1,\hat{i}_2,;j}\\ . & . & \ldots & .\\ M_{k_1-1,\hat{i}_2,;j} & M_{k_1,\hat{i}_2,;j} & \ldots & M_{2k_1-2,\hat{i}_2,;j}\end{bmatrix}$$

(a $pk_1\times mk_1$ matrix).

$$H_{k_1,k_2}^{i_3,\ldots,i_q;j_1,\ldots,j_r}=$$

$$=\begin{bmatrix} H_{k_1}^{0,\hat{i}_3;j} & H_{k_1}^{1,\hat{i}_3;j} & \ldots & H_{k_1}^{k_2-1,\hat{i}_3;j}\\ H_{k_1}^{1,\hat{i}_3;j} & H_{k_1}^{2,\hat{i}_3;j} & \ldots & H_{k_1}^{k_2,\hat{i}_3;j}\\ . & . & \ldots & .\\ H_{k_1}^{k_2-1,\hat{i}_3;j} & H_{k_1}^{k_2,\hat{i}_3;j} & \ldots & H_{k_1}^{2k_2-2,\hat{i}_3;j}\end{bmatrix}$$

(a $pk_1k_2\times mk_1k_2$ matrix).

Generally, being determined the block Hankel matrices $H_{k_1,\ldots,k_{a-1}}^{i_a,i_{a+1},\ldots,i_q;j_1,\ldots,j_r}$, we define the $pk_1k_2\ldots k_a\times mk_1k_2\ldots k_a$ block Hankel matrix

$$H_{k_1,\ldots,k_{a-1},k_a}^{i_{a+1},\ldots,i_q;j_1,\ldots,j_r}=$$

$$=\begin{bmatrix} H_{k_1,\ldots,k_{a-1}}^{0,\hat{i}_{a+1},j} & H_{k_1,\ldots,k_{a-1}}^{1,\hat{i}_{a+1},j} & \ldots & H_{k_1,\ldots,k_{a-1}}^{k_a-1,\hat{i}_{a+1},j}\\ H_{k_1,\ldots,k_{a-1}}^{1,\hat{i}_{a+1},j} & H_{k_1,\ldots,k_{a-1}}^{2,\hat{i}_{a+1},j} & \ldots & H_{k_1,\ldots,k_{a-1}}^{k_a,\hat{i}_{a+1},j}\\ . & . & \ldots & .\\ H_{k_1,\ldots,k_{a-1}}^{k_a-1,\hat{i}_{a+1},j} & H_{k_1,\ldots,k_{a-1}}^{k_a,\hat{i}_{a+1},j} & \ldots & H_{k_1,\ldots,k_{a-1}}^{2k_a-2,\hat{i}_{a+1},j}\end{bmatrix}$$

Then

$$H_{k_1,\ldots,k_q;l_1}^{j_2,\ldots,j_r}=$$

$$=\begin{bmatrix} H_{k_1,\ldots,k_q}^{0,j_2,\ldots,j_r} & H_{k_1,\ldots,k_q}^{1,j_2,\ldots,j_r} & \ldots & H_{k_1,\ldots,k_q}^{l_1-1,j_2,\ldots,j_r}\\ H_{k_1,\ldots,k_q}^{1,j_2,\ldots,j_r} & H_{k_1,\ldots,k_q}^{2,j_2,\ldots,j_r} & \ldots & H_{k_1,\ldots,k_q}^{l_1,j_2,\ldots,j_r}\\ . & . & \ldots & .\\ H_{k_1,\ldots,k_q}^{l_1-1,j_2,\ldots,j_r} & H_{k_1,\ldots,k_q}^{l_1,j_2,\ldots,j_r} & \ldots & H_{k_1,\ldots,k_q}^{2l_1-2,j_2,\ldots,j_r}\end{bmatrix}$$

($pk_1\ldots k_ql_1\times mk_1\ldots k_ql_1$ matrix).

Finally, we obtain the $pk_1\ldots k_ql_1\ldots l_2\times mk_1\ldots k_ql_1\ldots l_r$ matrix

$$H_{k_1,\ldots,k_q;l_1,\ldots,l_{r-1},l_2}= \quad (4.6)$$

$$=\begin{bmatrix} H^0_{k;l_1,\ldots,l_{r-1}} & H^1_{k;l_1,\ldots,l_{r-1}} & \ldots & H^{l_r-1}_{k;l_1,\ldots,l_{r-1}}\\ H^1_{k;l_1,\ldots,l_{r-1}} & H^2_{k;l_1,\ldots,l_{r-1}} & \ldots & H^{l_r}_{k;l_1,\ldots,l_{r-1}}\\ . & . & \ldots & .\\ H^{l_1-1}_{k;l_1,\ldots,l_{r-1}} & H^{l_1}_{k;l_1,\ldots,l_{r-1}} & \ldots & H^{2l_r-1}_{k;l_1,\ldots,l_{r-1}}\end{bmatrix}.$$

Sometimes the matrix (4.6) will be denoted $H_{k;l}$.

Proposition 4.4. For any realization Σ of $T(s;z)\in\mathcal{T}_S(s;z)$ and any $k_a\geq 1$, $l_b\geq 1$, $a\in\bar{q}$, $b\in\bar{r}$,

$$\text{rank}H_{k_1,\ldots,k_q;l_1,\ldots,l_b}\leq \dim\Sigma. \quad (4.7)$$

Proof. Let us consider a realization as in Definition 2.1, hence $n=\dim\Sigma$. We shall define by recurrence the following controllability-type block matrices:

$$\mathcal{C}(A_{c1};B;k_1)=[B\ A_{c1}B\ A_{c1}^2B\ldots A_{c1}^{k_1-1}B]$$
$$\mathcal{C}(A_{c1},A_{c2};B;k_1,k_2)=[\mathcal{C}(A_{c1};B;k_1)$$
$$\mathcal{C}(A_{c1};A_{c2}B;k_1)\ldots\mathcal{C}(A_{c1};A_{c2}^{k_2-1}B;k_1)]$$

$$\mathcal{C}(A_{c1},A_{c2},\ldots,A_{c,i-1},A_{ci};B;k_1,k_2,\ldots,k_{i-1},k_i)=$$
$$=[\mathcal{C}(A_{c1},A_{c2},\ldots,A_{c,i-1};B;k_1,k_2,\ldots,k_{i-1})$$
$$\mathcal{C}(A_{c1},A_{c2},\ldots,A_{c,i-1};A_{ci}B;k_1,k_2,\ldots,k_{i-1})\ldots$$
$$\mathcal{C}(A_{c1},A_{c2},\ldots,A_{c,i-1};A_{ci}^{k_i-1}B;k_1,k_2,\ldots,k_{i-1})],$$

$\forall i$, $2\leq i\leq q$. For $A_c=(A_{c1},A_{c2},\ldots,A_{c,q-1},A_{cq})$ and $k=(k_1,k_2,\ldots,k_{q-1},k_q)$ we denote

$$\mathcal{C}(A_c;B;k)=$$
$$\mathcal{C}(A_{c1},A_{c2},\ldots,A_{c,q-1},A_{cq};B;k_1,k_2,\ldots,k_{q-1},k_q).$$

Then

$$\mathcal{C}(A_c; A_{d1}; B; k; l_1) =$$
$$[\mathcal{C}(A_c; B; k)\ \mathcal{C}(A_c; A_{d1}B; k) \ldots \mathcal{C}(A_c; A_{d1}^{l_1-1}B; k)],$$
$$\mathcal{C}(A_c; A_{d1}, A_{d2}; B; k; l_1, l_2) = [\mathcal{C}(A_c; A_{d1}, B; k; l_1)$$
$$\mathcal{C}(A_c; A_{d1}, A_{d2}B; k; l_1) \ldots \mathcal{C}(A_c; A_{d1}; A_{d2}^{l_2-1}B; k; l_1)],$$
$$\ldots\ldots\ldots\ldots$$

$$\mathcal{C}(A_c; A_{d1}, A_{d2}, \ldots, A_{d,j-1}, A_{dj}; B; k; l_1, l_2, \ldots, l_{j-1}, l_j)$$
$$= [\mathcal{C}(A_c; A_{d1}, A_{d2}, \ldots, A_{d,j-1}; B; k; l_1, l_2, \ldots, l_{j-1})$$
$$\mathcal{C}(A_c; A_{d1}, A_{d2}, \ldots, A_{d,j-1}; A_{dj}B; k; l_1, l_2, \ldots, l_{j-1}) \ldots$$
$$\mathcal{C}(A_c; A_{d1}, A_{d2}, \ldots, A_{d,j-1}; A_{dj}^{l_j-1}B; k_j; l_1, l_2, \ldots, l_{j-1})],$$

$\forall j,\ 2 \le j \le r$. Finally, we denote, for $A_d = (A_{d1}, A_{d2}, \ldots, A_{d,r-1}, A_{dr})$ and $l = (l_1, l_2, \ldots, l_{r-1}, l_r)$

$$\mathcal{C}(A_c; A_d; B; k; l) := \mathcal{C}(A_c; A_{d1}, A_{d2}, \ldots, A_{d,r-1}, A_{dr}; B; k; l_1, l_2, \ldots, l_{r-1}, l_r). \tag{4.8}$$

Similarly, we define by reccurence the observability type matrices:

$$\mathcal{O}(A_{c1}; C; k_1) = \begin{bmatrix} C \\ CA_{c1} \\ \ldots \\ CA_{c1}^{k_1-1} \end{bmatrix},$$

$$\mathcal{O}(A_{c1}, A_{c2}; C; k_1, k_2) = \begin{bmatrix} \mathcal{O}(A_{c1}; C; k_1) \\ \mathcal{O}(A_{c1}; CA_{c2}; k_1) \\ \ldots \\ \mathcal{O}(A_{c1}; CA_{c2}^{k_2-1}; k_1) \end{bmatrix}$$

and so on. Finally, we obtain the matrix (as in (4.7)):

$$\mathcal{O}(A_c; A_d; C; k; l) := \mathcal{O}(A_c; A_{d1}, A_{d2}, \ldots, A_{d,r-1}; C; k; l_1, l_2, \ldots, l_{r-1}, l_r). \tag{4.9}$$

Using (4.6), (4.8) and (4.9), we can prove that

$$H_{k_1,\ldots,k_q;l_1,\ldots,l_r} = = \mathcal{O}(A_c; A_d; C; k; l)\mathcal{C}(A_c; A_d; B; k; l) \tag{4.10}$$

Now we shall employ Sylvester's inequalities. If P is a $p \times n$ matrix and M is an $n \times m$ matrix, then

$$\text{rank}P + \text{rank}M - n \le \text{rank}PM \le \le \min(\text{rank}P, \text{rank}M). \tag{4.11}$$

Obviously, $\mathcal{O}(A_c; A_d; B; k; l)$ is a $p\tilde{k}\tilde{l} \times n$ matrix and $\mathcal{O}(A_c; A_d; B; k; l)$ is an $n \times m\tilde{k}\tilde{l}$ matrix, where $\tilde{k} = k_1 k_2 \ldots k_q$ and $\tilde{l} = l_1 l_2 \ldots l_r$.

By the second inequality (4.11) and by (4.10) we get

$$\text{rank}H_{k_1,\ldots,k_q;l_1,\ldots,l_r} \le \min(\text{rank}\mathcal{O}(A_c; A_d; C; k; l),$$
$$\text{rank}\mathcal{O}(A_c; A_d; C; k; l)) \le n,$$

i.e. (4.7), since $n = \text{rank}\Sigma$. □

Now let us assume that

$$T(s; z) = \left(\prod_{a=1}^{q} \pi_a(s_a)\right)^{-1} \left(\prod_{b=1}^{r} \theta_b(z_b)\right)^{-1} M(s; z)$$

where $M(s; z) = M(s_1, \ldots, s_q; z_1, \ldots, z_r)$ is a polynomial matrix and $\pi_a(s_i)$, $\theta_b(z_j)$ are polynomials of degree k_a and l_b respectively, $a \in \bar{q}$, $b \in \bar{r}$.

We define the *first level shift operators* $\tilde{\sigma}_a^\alpha$, $a \in \bar{q}$, $\alpha \ge 1$ by

$$\tilde{\sigma}_a^\alpha H_{k_1,\ldots,k_{a-1},k_a}^{i_{a+1},\ldots,i_q;j_1,\ldots,j_r} = \tag{4.12}$$

$$= \begin{bmatrix} H_{k_1,\ldots,k_{a-1}}^{\alpha,\hat{i}_{a+1},j} & H_{k_1,\ldots,k_{a-1}}^{\alpha+1,\hat{i}_{a+1},j} \ldots H_{k_1,\ldots,k_{a-1}}^{\alpha+k_a-1,\hat{i}_{a+1},j} \\ H_{k_1,\ldots,k_{a-1}}^{\alpha+1,\hat{i}_{a+1},j} & H_{k_1,\ldots,k_{a-1}}^{\alpha+2,\hat{i}_{a+1},j} \ldots H_{k_1,\ldots,k_{a-1}}^{\alpha+k_a,\hat{i}_{a+1},j} \\ . & . \quad \ldots \quad . \\ H_{k_1,\ldots,k_{a-1}}^{\alpha+k_a-1,\hat{i}_{a+1},j} & H_{k_1,\ldots,k_{a-1}}^{\alpha+k_a,\hat{i}_{a+1},j} \ldots H_{k_1,\ldots,k_{a-1}}^{\alpha+2k_a-2,\hat{i}_{a+1},j} \end{bmatrix}$$

Similarly, the first level operators $\tilde{\delta}_b^\beta$, $b \in \bar{r}$, $\beta \ge 1$ are defined by

$$\tilde{\delta}_b^\beta H_{k;l_1,\ldots,l_{b-1},l_b}^{j_{b+1},\ldots,j_r} = \tag{4.13}$$

$$= \begin{bmatrix} H_{k;l_1,\ldots,l_{b-1}}^{\beta,\hat{j}_{b+1}} & H_{k;l_1,\ldots,l_{b-1}}^{\beta+1,\hat{j}_{b+1}} \ldots H_{k;l_1,\ldots,l_{b-1}}^{\beta+l_b-1,\hat{j}_{b+1}} \\ H_{k;l_1,\ldots,l_{b-1}}^{\beta+1,\hat{j}_{b+1}} & H_{k;l_1,\ldots,l_{b-1}}^{\beta+2,\hat{j}_{b+1}} \ldots H_{k;l_1,\ldots,l_{b-1}}^{\beta+l_b,\hat{j}_{b+1}} \\ . & . \quad \ldots \quad . \\ H_{k;l_1,\ldots,l_{b-1}}^{\beta+l_b-1,\hat{j}_{b+1}} & H_{k;l_1,\ldots,l_{b-1}}^{\beta+l_b,\hat{j}_{b+1}} \ldots H_{k;l_1,\ldots,l_{b-1}}^{\beta+2l_b-2,\hat{j}_{b+1}} \end{bmatrix}$$

The *second level shift operators* σ_a^α and δ_b^β, $a \in \bar{q}$, $b \in \bar{r}$, $\alpha, \beta \ge 1$ acting on the block Hankel matrix $H_{k;l} = H_{k_1,\ldots,k_q;l_1,\ldots,l_r}$ (4.6) are defined as follows: $\sigma_a^\alpha H_{k;l}$ is the matrix obtained by recurrence as $H_{k;l}$ (4.6) by replacing $H_{k_1,\ldots,k_{a-1},k_a}^{i_{a+1},\ldots,i_k;j_1,\ldots,j_r}$ by $\tilde{\sigma}_a^\alpha H_{k_1,\ldots,k_{a-1},k_a}^{i_{a+1},\ldots,i_k;j_1,\ldots,j_r}$; $\delta_b^\beta H_{k;l}$ is the matrix obtained by recurrence as $H_{k;l}$ (4.6) by replacing $H_{k;l_1,\ldots,l_{b-1},l_b}^{j_{b+1},\ldots,j_r}$ by $\tilde{\delta}_b^\beta H_{k,l_1,\ldots,l_{b-1},l_b}^{j_{b+1},\ldots,j_r}$.

We shall denote σ_a^1 and δ_b^1 by σ_a and δ_b respectively.

Assume that the polynomials in the denominator of $T(s;z)$ are

$$\pi_a(s_a) = s_a^{k_a} + \alpha_{a,k_a-1}s_a^{k_a-1} + \ldots + \alpha_{a,1}s_a + \\ +\alpha_{a,0}, \quad a \in \bar{q}, \tag{4.14}$$

and

$$\theta_b(z_b) = z_b^{l_b} + \beta_{b,l_b-1}z_b^{l_b-1} + \ldots + \beta_{b,1}z_b + \\ +\beta_{b,0}, \quad b \in \bar{r}, \tag{4.15}$$

We associate to the polynomials π_a and θ_b the companion cells $K_a =$

$$\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ . & . & . & \ldots & . & . \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -\alpha_{a,0} & -\alpha_{a,1} & -\alpha_{a,2} & \ldots & -\alpha_{a,k_a-2} & -\alpha_{a,k_a-1} \end{bmatrix}$$

and $L_b =$

$$\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ . & . & . & \ldots & . & . \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -\beta_{b,0} & -\beta_{b,1} & -\beta_{b,2} & \ldots & -\beta_{b,l_b-2} & -\beta_{b,l_b-1} \end{bmatrix}.$$

We consider (for $a \in \bar{q}$ and $b \in \bar{r}$) the matrices

$$\tilde{F}_a = \left(\bigotimes_{i=a+1}^{q} I_{k_i}\right) \otimes \left(\bigotimes_{j=1}^{r} I_{l_j}\right) \otimes K_a \otimes \left(\bigotimes_{i=1}^{a-1} I_{k_i}\right) \otimes I_p \tag{4.16}$$

$$\hat{F}_a = \left(\bigotimes_{i=a+1}^{q} I_{k_i}\right) \otimes \left(\bigotimes_{j=1}^{r} I_{l_j}\right) \otimes K_a^T \otimes \left(\bigotimes_{i=1}^{a-1} I_{k_i}\right) \otimes I_m \tag{4.17}$$

$$\tilde{G}_b = \left(\bigotimes_{j=b+1}^{r} I_{l_j}\right) \otimes L_b \otimes \left(\bigotimes_{j=1}^{b-1} I_{l_j}\right) \otimes \left(\bigotimes_{i=1}^{q} I_{k_i}\right) \otimes I_p \tag{4.18}$$

$$\hat{G}_b = \left(\bigotimes_{j=b+1}^{r} I_{l_j}\right) \otimes L_b^T \otimes \left(\bigotimes_{j=1}^{b-1} I_{l_j}\right) \otimes \left(\bigotimes_{i=1}^{q} I_{k_i}\right) \otimes I_m \tag{4.19}$$

where $\otimes$ denotes the Kroneker product of matrices,

$$A \otimes B = \begin{bmatrix} a_{11}B & \ldots & a_{1n}B \\ . & \ldots & . \\ a_{m1}B & \ldots & a_{mn}B \end{bmatrix} \text{ if } A = \begin{bmatrix} a_{11} \ldots a_{1n} \\ . \quad \ldots \quad . \\ a_{m1} \ldots a_{mn} \end{bmatrix}$$

(hence $\tilde{F}_a = I_{k_{a+1}} \ldots k_q l_1 \ldots l_2 \otimes K_a \otimes I_{k_1 \ldots k_a} p$ and $\hat{F}_a, \tilde{G}_b, \hat{G}_b$ have similar representations).

Proposition 4.5. The matrices $\tilde{F}_a$, $a \in \bar{q}$ and $\tilde{G}_b$, $b \in \bar{r}$ are commutative; $\hat{F}_a$, $a \in \bar{q}$ and $\hat{G}_b$, $b \in \bar{r}$ are commutative matrices.

Proof. For $a, c \in \bar{q}$, $a < c$, we have

$$\tilde{F}_a\tilde{F}_c = \left[\left(\bigotimes_{i=a+1}^{q} I_{k_i}\right) \otimes \left(\bigotimes_{j=1}^{r} I_{l_j}\right) \otimes K_a \otimes \left(\bigotimes_{i=1}^{a-1} I_{k_i}\right) \otimes I_p\right]$$

$$\cdot \left[\left(\bigotimes_{i=c+1}^{q} I_{k_i}\right) \otimes \left(\bigotimes_{j=1}^{r} I_{l_j}\right) \otimes K_a \otimes \left(\bigotimes_{i=1}^{c-1} I_{k_i}\right) \otimes I_p\right] =$$

$$= \left[\left(\bigotimes_{j=1}^{r} I_{l_j}\right) \otimes \left(\bigotimes_{i=a+1}^{q} I_{k_i}\right) \otimes I_{k_c} \otimes \left(\bigotimes_{i=a+1}^{c-1} I_{k_i}\right) \otimes \right.$$

$$\left. \otimes K_a \otimes \left(\bigotimes_{i=1}^{a-1} I_{k_i}\right) \otimes I_p\right] \cdot$$

$$\cdot \left[\left(\bigotimes_{j=1}^{r} I_{l_j}\right) \otimes \left(\bigotimes_{i=c+1}^{q} I_{k_i}\right) \otimes K_c \otimes \left(\bigotimes_{i=a+1}^{c-1} I_{k_i}\right) \otimes \right.$$

$$\left. \otimes I_{k_a} \otimes \left(\bigotimes_{i=1}^{a-1} I_{k_i}\right) \otimes I_p\right] =$$

$$= \left(\bigotimes_{j=1}^{r} I_{l_j}\right) \otimes \left(\bigotimes_{i=c+1}^{q} I_{k_i}\right) \otimes K_c \otimes \left(\bigotimes_{i=a+1}^{c-1} I_{k_i}\right) \otimes$$

$$\otimes K_a \otimes \left(\bigotimes_{i=1}^{a-1} I_{k_i}\right) \otimes I_p = \tilde{F}_c\tilde{F}_a$$

We used the properties of the Kronecker product: $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ for matrices A, B, C, D with suitable dimensions; $I_n \otimes I_m = I_{nm} = I_m \otimes I_n$. Similarly, one obtains $\tilde{F}_a\tilde{G}_b = \tilde{G}_b\tilde{F}_a$, $\forall a \in \bar{q}$, $b \in \bar{r}$, $\tilde{G}_b\tilde{G}_d = \tilde{G}_d\tilde{G}_b$, $\forall b, d \in \bar{r}$, $\hat{F}_a\hat{F}_c = \hat{F}_c\hat{F}_a$, $\hat{F}_a\hat{G}_b = \hat{G}_b\hat{F}_a$, $\hat{G}_b\hat{G}_d = \hat{G}_d\hat{G}_b$. □

Proposition 4.6. The second level shift operators σ_a and δ_b verify the equalities

$$\sigma_a H_{k;l} = \tilde{F}_a H_{k;l} = H_{k;l}\hat{F}_a, \quad a \in \bar{q}, \tag{4.20}$$

$$\delta_b H_{k;l} = \tilde{G}_b H_{k;l} = H_{k;l}\hat{G}_b, \quad b \in \bar{r}. \tag{4.21}$$

Proof. The main idea of the proof is the fact that the product

$$\pi_a(s_a)T(s;z) =$$

$$= \pi_a(s_a)T(s_1, \ldots, s_a, \ldots, s_q; z_1, \ldots, z_r) =$$

$$= \left(\prod_{\substack{i=1 \\ i \neq a}}^{q} \pi_i(s_i)\right)^{-1} \left(\prod_{j=1}^{r} \theta_j(z_j)\right)^{-1} M(s;z)$$

is a polynomial matrix with respect to s_a, $a \in \bar{q}$, hence the coefficient of the negative powers of s_a vanish. This gives recurrence formulas for the Markov parameters of $T(s;z)$, taking into account (4.14), for instance

$$M_{i_1,\ldots,i_{a-1},i_a+k_a,i_{a+1},\ldots,i_q;j_1,\ldots,j_r} = $$
$$= -\sum_{c=0}^{k_a-1} \alpha_{a,c} M_{i_1,\ldots,i_{a-1},i_a+c,i_{a+1},\ldots,i_q;j_1,\ldots,j_r}$$
$$\forall i_1,\ldots,i_a,\ldots,i_q,j_1,\ldots,j_r \geq 0,\ \forall a \in \bar{q}.$$

Then a long calculus which is omitted verifies (4.20) and (4.21). □

We obtain by induction

Corollary 4.7. *For any* $\alpha, \beta \geq 1$, $a \in \bar{q}$, $b \in \bar{r}$,

$$\sigma_a^\alpha H_{k;l} = \tilde{F}_a^\alpha H_{k;l} = H_{k;l}\hat{F}_a^\alpha, \tag{4.22}$$

$$\delta_b^\beta H_{k;l} = \tilde{G}_b^\beta H_{k;l} = H_{k;l}\hat{G}_b^\beta. \tag{4.23}$$

We shall use the following notations: 0_p^m is the null matrix with p rows and m columns; I_p is the unit matrix of order p; E_p^m is the $p \times m$ matrix defined by

$$E_p^m = \begin{cases} [I_p\ 0_p^{m-p}] & \text{if} \quad p < m \\ I_p & \text{if} \quad p = m \\ \begin{bmatrix} I_m \\ 0_{p-m}^m \end{bmatrix} & \text{if} \quad p > m. \end{cases}$$

Obviously, these matrices have the following properties:

i) If $p < m$ and $M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$ is a matrix with M_1 $p \times q$ and M_2 $(m-p) \times q$, then

$$E_p^m M = M_1 \tag{4.24}$$

ii) If $p > m$ and $M = [M_1\ M_2]$ is a matrix with M_1 $q \times m$ and M_2 $q \times (p-m)$ then

$$M E_p^m = M_1 \tag{4.25}$$

iii) If $n \leq p$, $n \leq m$ then

$$E_n^p E_p^n = I_n \text{ and } E_p^n E_n^m = \begin{bmatrix} I_n & 0_n^{m-n} \\ 0_{p-n}^n & 0_{p-n}^{m-n} \end{bmatrix}. \tag{4.26}$$

Algorithm 4.8. (of minimal realization). Let $T(s;z) = T(s_1,\ldots,s_q;z_1,\ldots,z_r)$ be a strictly proper matrix, $T(s;z) \in \mathcal{T}_S(s;z)$.

Stage I. Expand $T(s;z)$ in Laurent series (4.3) about infinity:

$$T(s;z) = \sum_{i_1=0}^{\infty}\ldots\sum_{i_q=0}^{\infty}\sum_{j_1=0}^{\infty}\ldots\sum_{j_r=0}^{\infty} M_{i_1,\ldots,i_q;j_1,\ldots,j_r} \cdot \left(\prod_{k=1}^{q} s_k^{-i_k-1}\right)\left(\prod_{l=1}^{r} z_l^{-j_k-1}\right).$$

Stage II. Determine the degrees k_i, $i \in \bar{q}$ and l_j, $j \in \bar{r}$ of the polynomials $\pi_i(s_i)$ and $\theta_j(z_j)$ respectively in the l.c.d of the entries of $T(s;z)$.

Stage III. Using the Markov parameters $M_{i_1,\ldots,i_q;j_1,\ldots,j_r}$ write the block Hankel matrices $H_{k;l}$, $\sigma_a H_{k;l}$, $a \in \bar{q}$, $\delta_b H_{k;l}$, $b \in \bar{r}$ for $k = (k_1,\ldots,k_q)$ and $l = (l_1,\ldots,l_r)$ and write the array

$$\mathcal{A} = \begin{bmatrix} I_{\tilde{p}} & H_{k;l} \\ & I_{\tilde{m}} \end{bmatrix}$$

where $\tilde{p} = k_1\ldots k_q l_1 \ldots l_r p$ and $\tilde{m} = k_1\ldots k_q l_1\ldots l_r m$.

Stage IV. By applying elementary rows operations (ERO) on the first block rows of $\mathcal{A}$ (i.e. on $[I_{\tilde{p}}\ H_{k;r}]$) and elementary column operations (ECO) on the second block column of $\mathcal{A}$ (i.e. on $\begin{bmatrix} H_{k;r} \\ I_{\tilde{m}} \end{bmatrix}$) transform $\mathcal{A}$ into the array

$$\tilde{\mathcal{A}} = \begin{bmatrix} P & \tilde{H} \\ & M \end{bmatrix} \text{ where } \tilde{H} = \begin{bmatrix} I_n & 0_n^{\tilde{m}-n} \\ 0_{\tilde{p}-n} & 0_{\tilde{p}-n}^{\tilde{m}-n} \end{bmatrix} \tag{4.27}$$

Stage V. Determine the minimal realization $\Sigma = (\{A_{ci}, i \in \bar{q}\}, \{A_{dj}, j \in \bar{r}\}, B, C$ by the following formulas:

$$A_{ca} = E_n^{\tilde{p}} P[\sigma_a H_{k;l}] M E_{\tilde{m}}^n, \quad a \in \bar{q} \tag{4.28}$$

$$A_{db} = E_n^{\tilde{p}} P[\delta_b H_{k;l}] M E_{\tilde{m}}^n, \quad b \in \bar{r} \tag{4.29}$$

$$B = E_n^{\tilde{p}} P\ H_{k;l} E_{\tilde{m}}^m, \tag{4.30}$$

$$C = E_p^{\tilde{p}} H_{k;l} M E_{\tilde{m}}^n, \tag{4.31}$$

Proof. The matrices P and M being the results of ERO and respectively of ECO on the unit matrix, they are products of the corresponding elementary matrices, which are nonsingular, hence P and M are nonsingular too. Moreover, by (4.26) and (4.27) we get

$$P H_{k;l} M = \tilde{H} = E_{\tilde{p}}^n E_n^{\tilde{m}} \tag{4.32}$$

The matrix $Q = M E_{\tilde{m}}^n E_n^{\tilde{p}} P$ is the pseudoinverse of $H_{k;l}$, i.e.

$$H_{k;l} Q H_{k;l} = H_{k;l}. \tag{4.33}$$

Indeed

$$PH_{k;l}QH_{k;l}M = PH_{k;l}ME^n_{\tilde{m}}E^{\tilde{p}}_nPH_{k;l}M \overset{(4.32)}{=}$$

$$E^n_{\tilde{p}}E^{\tilde{m}}_nE^n_{\tilde{m}}E^{\tilde{p}}_nE^n_{\tilde{p}}E^{\tilde{m}}_n \overset{(4.26)}{=} E^n_{\tilde{p}}E^{\tilde{m}}_n \overset{(4.26)}{=} PH_{k;l}M.$$

By premultiplying and postmultiplying this equality by P^{-1} and M^{-1} respectively, we get (4.33).

Now let as show that A_{ca} and A_{db} are commutative matrices. By (4.20), (4.28) and (4.33) we have, for $a_1, a_2 \in \bar{q}$: $A_{ca_1}A_{ca_2} =$

$$= (E^{\tilde{p}}_nP[\sigma_{a_1}H_{k;l}]ME^n_{\tilde{m}})(E^{\tilde{p}}_np[\sigma_{a_2}H_{k;l}]ME^n_{\tilde{m}}) =$$

$$= E^{\tilde{p}}_nP\tilde{F}_{a_1}H_{k;l}QH_{k;l}\hat{F}_{a_2}ME^n_{\tilde{m}} =$$

$$= E^{\tilde{p}}_nP\tilde{F}_{a_1}\tilde{F}_{a_2}H_{k;l}ME^n_{\tilde{m}} = A_{ca_2}A_{ca_1}$$

since $\tilde{F}_{a_1}\tilde{F}_{a_2} = \tilde{F}_{a_2}\tilde{F}_{a_1}$ by Proposition 4.5. Similarly, we can prove that $A_{ca}A_{db} = A_{db}A_{ca}$ and $A_{db_1}A_{db_2} = A_{db_2}A_{db_1}$ and by induction we get

$$\left(\prod_{a=1}^{q} A^{i_a}_{ca}\right)\left(\prod_{b=1}^{r} A^{j_b}_{db}\right) =$$

$$= E^{\tilde{p}}_nP\left(\prod_{a=1}^{q} \tilde{F}^{i_a}_a\right)\left(\prod_{b=1}^{r} \tilde{G}^{j_b}_b\right)H_{k;l}ME^n_{\tilde{m}} \qquad (4.34)$$

$$\forall i_a \geq 0,\ a \in \bar{q},\ \forall j_b \geq 0,\ b \in \bar{r}.$$

Now, let us prove that (4.4) holds. Firstly, by (4.30), (4.31), (4.33), (4.24), (4.25) and by the definition of $H(k;l)$ we have

$$CB = (E^{\tilde{p}}_pH_{k;l}ME^n_{\tilde{m}})(E^{\tilde{p}}_nPH_{k;l}E^m_{\tilde{m}}) =$$

$$= E^{\tilde{p}}_pH_{k;l}QE^m_{\tilde{m}} = E^{\tilde{p}}_pH_{k;l}E^m_{\tilde{m}} = M_{0,\dots,0;0,\dots,0}.$$

For $i_1, \dots i_q; j_1, \dots, j_r \geq 0$, by (4.34), (4.30), (4.31), (4.33), (4.22) and (4.23) we obtain

$$C\left(\prod_{a=1}^{q} A^{i_a}_{ca}\right)\left(\prod_{b=1}^{r} A^{j_b}_{db}\right)B =$$

$$= (E^{\tilde{p}}_pH_{k;l}ME^n_{\tilde{m}})\left(E^{\tilde{p}}_nP\left(\prod_{a=1}^{q} \tilde{F}^{i_a}_a\right)\cdot\right.$$

$$\left.\cdot\left(\prod_{b=1}^{r} \tilde{G}^{j_b}_b\right)H_{k;l}ME^n_{\tilde{m}}\right)\cdot E^{\tilde{p}}_nPH_{k;l}E^m_{\tilde{m}}) =$$

$$= E^{\tilde{p}}_pH_{k;l}Q\left(\prod_{a=1}^{q} \tilde{F}^{i_a}_a\right)\left(\prod_{b=1}^{r} \tilde{G}^{j_b}_b\right)H_{k;l}QH_{k;l}E^m_{\tilde{m}} =$$

$$= E^{\tilde{p}}_pH_{k;l}QH_{k;l}\left(\prod_{a=1}^{q} \hat{F}^{i_a}_a\right)\left(\prod_{b=1}^{r} \hat{G}^{j_b}_b\right)E^m_{\tilde{m}} =$$

$$= E^{\tilde{p}}_p\left[\left(\prod_{a=1}^{q} \sigma^{i_a}_a\right)\left(\prod_{b=1}^{r} \delta^{j_b}_b\right)H_{k;l}\right]E^m_{\tilde{m}} =$$

$$= M_{i_1,\dots,i_q;j_1,\dots,j_r}$$

i.e. the first $p\times m$ block of $\left(\prod_{a=1}^{q} \sigma^{i_a}_a\right)\left(\prod_{b=1}^{r} \sigma^{j_b}_b\right)H_{k;l}$, by the definitions of $H_{k;l}$ and of the shift operators σ_a and δ_b. Therefore (4.4) holds and by Theorem 3.2 Σ (4.28)-(4.31) is a realization of $T(s;z)$.

Obviously, the dimension of this realization is dim$\Sigma = n$ where n is determined in (4.27). Since P and M are nonsingular matrices, we get by (4.32) and (4.27) rank$H_{k;l}$ = rank$\tilde{H}_{k;l} = n =$ dimΣ. It follows from Proposition 4.4 that dimΣ = rank$H_{k;l} \leq$ dim$\hat{\Sigma}$ for any realization $\hat{\Sigma}$ of $T(s;z)$, hence Σ is a minimal realization.

Example 4.9. Let us consider the strictly proper separable function obtained in Example 3.12 $T(s;z) = \dfrac{4}{(s+1)^2(z-1)}$. Therefore $\pi(s) = s^2 + 2s + 1$, $\theta(z) = z - 1$, $q = r = 1$, $p = m = 1$, $l = 2$, $l = 1$. Using the geometric series $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x| < 1$ and its derivative $\dfrac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$ one obtains for $|s| > 1$ and $|z| > 1$:

$$T(s;z) = \frac{4}{s^2z}\cdot\frac{1}{\left(1+\dfrac{1}{s}\right)}\cdot\frac{1}{1-\dfrac{1}{z}} =$$

$$= \frac{4}{s^2z}\left(1 - \frac{2}{s} + \frac{3}{s^2} - \frac{4}{s^3} + \dots\right)\cdot$$

$$\cdot\left(1 + \frac{1}{z} + \frac{1}{z^2} + \dots\right) =$$

$$= \frac{4}{s^2z} - \frac{8}{s^3z} + \frac{12}{s^4z} - \frac{16}{s^5z} + \dots +$$

$$+ \frac{4}{s^2z^2} - \frac{8}{s^3z^2} + \frac{12}{s^4z^2} - \frac{16}{s^5z^2} + \dots$$

Since $T(s;z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} M_{ij}s^{-i-1}z^{-j-1}$ we get $M_{0j} = 0$, $M_{1j} = 4$, $M_{2j} = -8$, $M_{3j} = 12$, $M_{4j} = -16$, $\forall j \geq 0$. Since $k = 2$ and $l = 1$, we determine the Hankel matrices $H^0_2 = \begin{bmatrix} M_{00} & M_{10} \\ M_{10} & M_{20} \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 4 & -8 \end{bmatrix}$, $H_{2,1} = [H^0_2]$ and the action of the shift operators is $\tilde{\sigma}H^0_2 = \begin{bmatrix} M_{10} & M_{20} \\ M_{20} & M_{30} \end{bmatrix} = \begin{bmatrix} 4 & -8 \\ -8 & 12 \end{bmatrix}$, $\sigma H_{2,1} = \tilde{\sigma}H^0_2 = \begin{bmatrix} 4 & -8 \\ -8 & 12 \end{bmatrix}$ and $\sigma_2H_{2,1} = [H^1_2] = \begin{bmatrix} M_{01} & M_{11} \\ M_{11} & M_{21} \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 4 & -8 \end{bmatrix}$.

We have the array

$$\mathcal{A} = \begin{bmatrix} I_2 & H_{2,1} \\ & I_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 4 & -8 \\ & & 1 & 0 \\ & & 0 & 1 \end{bmatrix}.$$

By the permutation of the first two rows and then by the addition of the second row multiplied by 2 to the first row we transform $\mathcal{A}$ into $\tilde{\mathcal{A}} = \begin{bmatrix} 1/2 & 1/4 & 1 & 0 \\ 1/4 & 0 & 0 & 1 \\ & & 1 & 0 \\ & & 0 & 1 \end{bmatrix}$, hence $\tilde{H} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1/2 & 1/4 \\ 1/4 & 0 \end{bmatrix}$, $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. By (4.28) - (4.31) we get the minimal realization

$$A_c = E_2^2 P[\sigma_1 H_{2,1}] M E_2^2 =$$

$$= \begin{bmatrix} 1/2 & 1/4 \\ 1/4 & 0 \end{bmatrix} \begin{bmatrix} 4 & -8 \\ -8 & 12 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix},$$

$$A_d = E_2^2 P[\sigma_2 H_{2,1}] M E_2^2 =$$

$$= \begin{bmatrix} 1/2 & 1/4 \\ 1/4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 4 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$B = E_2^2 P H_{2,1} E_2^1 =$$

$$= \begin{bmatrix} 1/2 & 1/4 \\ 1/4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 4 & -8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$C = E_1^2 H_{2,1} M E_2^2 = [1\ 0] \begin{bmatrix} 0 & 4 \\ 4 & -8 \end{bmatrix} = [0\ 4].$$

Aknowledgement. This paper is partially supported by the Grant CNCSIS 86/ 2007 and by the 15-th Italian-Romanian Executive Programme of S&T Co-operation for 2006-2008, University Politehnica of Bucharest.

References:

[1] S. Attasi, Introduction d'une classe de systèmes linéaires reccurents à deux indices, *Comptes Rendus Acad. Sc. Paris* 277, série A (1973), 1135.

[2] E. Fornasini, A 2D systems approach to river pollution modelling, *Multi-dimensional Systems and Signal Processing*, 2 (1991), pp. 233-265.

[3] K. Gałkovski, E. Rogers and D.H. Owens, New 2D models and a transition matrix for discrete linear repetitive processes, *Int. J. Control*, 72, 15 (1999), pp. 1365-1380.

[4] B. L. Ho and R. E. Kalman, *Effective construction of linear, state-variable models from input/output functions*, Regelungstechnik, **14**, *12* (1966), 545-548.

[5] T. Kaczorek, *Controllability and minimum energy control of* 2D *continuous-discrete linear systems*. Appl.Math. and Comp.Sci.,**5**, *1* (1995), 5-21.

[6] J. Kurek and M.B. Zaremba, Iterative learning control synthesis on 2D system theory, *IEEE Trans. Aut. Control*, AC-38, 1 (1993), pp. 121-125.

[7] V. Prepelita, Reachability, Observability and Minimality for a Class of 2D Continuous-Discrete Systems, Proceedings of the 7th WSEAS International Conference on SYSTEMS THEORY and SCIENTIFIC COMPUTATION (ISTASC'07), Vouliagmeni, Athens, Greece, August 24-26, 2007, ISSN 1790-5117, ISBN 978-960-8457-98-0.

[8] V. Prepelita, Necessary and sufficient conditions of reachability for a model of multidimensional hybrid systems, Proceedings of the 8th WSEAS International Conference on SYSTEMS THEORY AND SCIENTIFIC COMPUTATION (ISTASC'08), Rhodes Island, Grece, August 20-22, 2008, pp. 247-254.

[9] V. Prepelita, Observability of a model of (q,r)-D continuous-discrete systems, Proceedings of the 4th WSEAS International Conference on DYNAMICAL SYSTEMS AND CONTROL (CONTROL'08), Corfu Island, Greece, October 26-28, 2008.

[10] V. Prepelita, Multiple Hybrid Laplace Transformation and Applications to Multidimensional Hybrid Systems, (to appear).

[11] E. Rogers, D.H. Owens, Stability Analysis for Linear Repetitive Processes, *Lecture Notes in Control and Information Sciences*, 175, Ed. Thoma H, Wyner W., Springer Verlag Berlin, 1999.

[12] K. Smyth, Computer aided analysis for linear repetitive processes, PhD Thesis, *University of Strathclyde, Glasgow*, UK, 1992.