Generalized Modus Ponens Using Fodor's Implication and a Parametric T-norm

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Abstract: - Using Generalized Modus Ponens reasoning, we examine the values of the inferred conclusion by using Fodor's implication in order to interpret a fuzzy if-then rule with a single input single output and the t-norm $t(x,y)=max((1+\lambda)(x+y-1)-\lambda xy)$, $\lambda \geq -1$, for composition operation. This t-norm is important to use because for $\lambda=-1$ and $\lambda=0$ it gives the commonly used t-norms $t_1(x,y)=xy$ and $t_2(x,y)=max(0,x+y-1)$, respectively.

Key-Words: - Fuzzy set, Fuzzy implication, Generalized modus ponens, t-norm, t-conorm, Strong negation

1 Introduction

In our daily life we often make inferences based on rules that contain imprecision. This makes it difficult to describe by means of natural language statements [1] the conditions and conclusions of the deduction rules. Another issue is the employment of these rules when the observed facts do not match the condition expressed in the premise of the rule, but are not too different.

These problems led Zadeh to outline the theory of approximate reasoning [2], which exemplifies the deduction of imprecise conclusions from a set of imprecise premises and is based on fuzzy logic. The development of fuzzy logic was motivated to a large degree by the need for a conceptual framework which can address the issue of uncertainty and lexical imprecision. Some essential characteristics of fuzzy logic are [3]:

• exact reasoning is viewed as a limiting case of approximate reasoning

• everything is a matter of degree

• knowledge is interpreted as a collection of elastic or fuzzy constraints on a collection of variables

• inference is viewed as a process of propagation of elastic constraints

• any logical system can be fuzzified.

There are two main characteristics of fuzzy systems that give them better performance for specific applications:

• fuzzy systems are suitable for uncertain or approximate reasoning, especially for the systems with a mathematical model that is difficult to derive

• fuzzy logic allows decision making with estimated values under incomplete or uncertain information.

In 1979 Zadeh introduced the theory of approximate reasoning [4]. This theory provides a powerful framework for reasoning in the face of imprecise and uncertain information. Central to this theory is the representation of propositions as statements assigning fuzzy sets as values to variables.

Zadeh extends the traditional modus ponens rule in order to deduce an imprecise conclusion from imprecise premises, obtaining the Generalized Modus Ponens rule. An investigation of inference processes in the fuzzy if-then rules is a subject of many papers in literature: [1-2, 5-22]. Fuzzy sets theory and its applications in different domains (control, classification, etc.) is, also, a common subject for several papers from WSEAS journals and conferences [23 - 27].

An elementary piece of information can be represented by a triple (attribute, object, value) which can be reduced to the canonical form

$$X \ is \ A,$$

where X is a variable representing the attribute of the entity and A is its value. The proposition

$$X \text{ is } A$$

can be understood as

the quantity X *satisfies the predicate* A

or

the variable X *takes its values in the set* A.

As pointed out Zadeh [1-4, 22] the semantic content of the proposition

$$X \text{ is } A$$

can be represented by

$$\pi_X = \mu_A,$$

where π_X is the possibility distribution restricting the possible value of X and μ_A is the membership function of the set A. The membership function μ_A can be expressed using parametric representation, which is achieved by the 5-tuple $(L_A, l_A, R_A, r_A, \theta)$ [28, 29]:

$$\mu_A(x) = \begin{cases} \theta & \text{for} \quad x \le L_A - l_A \text{ or } x \ge R_A + r_A \\ 1 & \text{for} \quad L_A \le x \le R_A \\ \psi_1(x) & \text{for} \quad L_A - l_A \le x \le L_A \\ \psi_2(x) & \text{for} \quad R_A \le x \le R_A + r_A \end{cases}$$

where $\theta \in [0,1]$ describes the uncertainty that accompanies the piece of information, ψ_1 is a non-decreasing function and ψ_2 is a non-increasing function (Fig. 1). Besides, the continuity conditions are necessary:

$$\psi_1(L_A) = \psi_2(R_A) = 1$$

and

$$\psi_1(L_A - l_A) = \psi_2(R_A + r_A) = \theta.$$

For $\theta = 0$ the piece of information is certain.

Because the majority of practical applications work with trapezoidal distributions ($\theta = 0$, ψ_1 and ψ_2 are linear functions) or triangular distributions (in addition, $L_A = R_A$) and these representations are still a subject of various recent papers ([30], for instance) we will analyze the trapezoidal representation of fuzzy sets (Fig. 2).

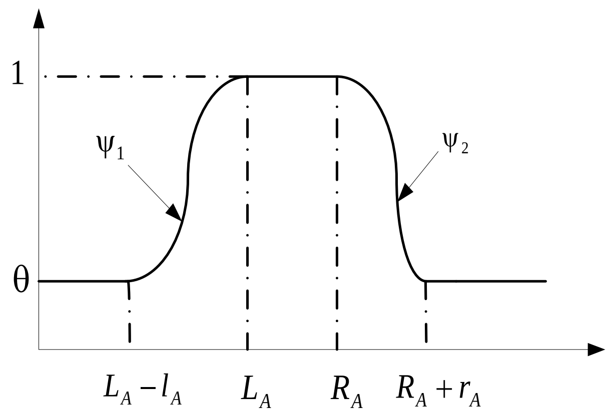


Fig. 1 Parametric representation of a fuzzy set

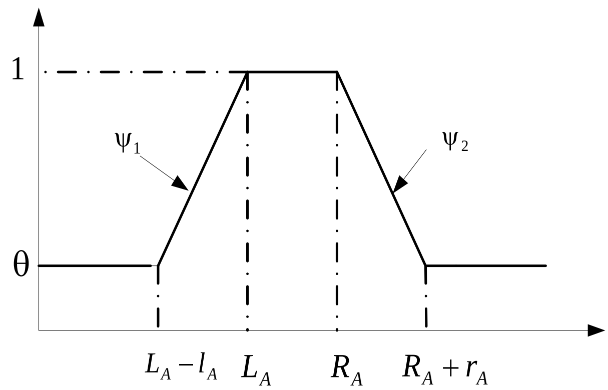


Fig. 2 Trapezoidal representation of a fuzzy set

Let X and Y be two variables whose domains are U and V, respectively. A causal link from X to Y is represented as a conditional possibility distribution [2, 22] which restricts the possible values of Y for a given value of X. For the rule

$$\text{if } X \text{ is } A \text{ then } Y \text{ is } B$$

we have

$$\forall u \in U,\ \forall v \in V,\ \pi_{Y/X}(v,u) = \mu_A(u) \to \mu_B(v)$$

where $\to$ is an implication operator and μ_A and μ_B are the possibility distributions of the propositions

$$X \text{ is } A$$

and

$$Y \text{ is } B,$$

respectively. For simplicity we note

$$\mu_A(u) \to \mu_B(v) = I(u,v).$$

If $\mu_{A'}$ is the possibility distribution of the proposition

$$X \text{ is } A'$$

then from the rule

$$\text{if } X \text{ is } A \text{ then } Y \text{ is } B$$

and the fact

$$X \text{ is } A'$$

the Generalized Modus Ponens rule computes the possibility distribution $\mu_{B'}$ of the conclusion

$$Y \text{ is } B'$$

as

$$\mu_{B'}(v) = sup_{u \in U}\ t(\mu_{A'}(u), \pi_{Y/X}(v,u))$$

where t is a t-norm.

Taking into account the following reasons, we shall work with rules having a single input single output:

a) a rule with multiple consequent can be treated as a set of rules with a single conclusion; for instance, the rule

if antecedent then C_1 *and* C_2*....and* C_n

is equivalent to the rules

if antecedent then C_1
if antecedent then C_2
....
if antecedent then C_n

b) a rule with multiple premise can be broken up into simple rules [31] when the rules are represented with any S-implication or any R-implication and the observations are normalized fuzzy sets.

2 Basic Concepts

We recall the definitions of basic concepts used in Generalized Modus Ponens reasoning.

Definition 1 *A function*

$$T : [0,1]^2 \rightarrow [0,1]$$

is a triangular norm (t-norm for short) iff it is commutative, associative, non-decreasing in each argument and

$$T(x,1) = x, \forall x \in [0,1].$$

In other words, any t-norm T *satisfies the properties:*

symmetry:

$$T(x, y) = T(y, x) \quad \forall x, y \in [0,1]$$

associativity:

$$T(x, T(y, z)) = T(T(x, y), z) \ \forall x, y, z \in [0,1]$$

monotonicity:

$$T(x, y) \le T(x', y') \text{ if } x \le x' \text{ and } y \le y' \quad \forall x, y, x', y' \in [0,1]$$

one identity:

$$T(x,1) = x, \forall x \in [0,1].$$

Definition 2 *A function*

$$S : [0,1]^2 \rightarrow [0,1]$$

is a t-conorm iff it satisfies the properties:

symmetry:

$$S(x, y) = S(y, x) \quad \forall x, y \in [0,1]$$

associativity:

$$S(x, S(y, z)) = S(S(x, y), z) \ \forall x, y, z \in [0,1]$$

monotonicity:

$$S(x, y) \le S(x', y') \text{ if } x \le x' \text{ and } y \le y' \quad \forall x, y, x', y' \in [0,1]$$

zero identity:

$$S(x, 0) = x, \forall x \in [0,1].$$

Definition 3 *A function*

$$N : [0,1] \rightarrow [0,1]$$

is a strong negation iff it satisfies the properties:

N1) $N(0) = 1,\ N(1) = 0$

N2) is an involutive function

$$N(N(x)) = x \quad \forall x \in [0,1]$$

N3) is a decreasing function

$$N(x) \le N(y) \text{ if } x \ge y \quad \forall x, y \in [0,1]$$

N4) is a continuous function.

Definition 4 *A fuzzy implication is a function*

$$I : [0,1]^2 \rightarrow [0,1]$$

satisfying the following conditions:

***I1**: If* $x \le z$ *then* $I(x, y) \ge I(z, y)$ *for all* $x, y, z \in [0,1]$

I2: *If* $y \le z$ *then* $I(x,y) \le I(x,z)$ *for all* $x, y, z \in [0,1]$

I3: $I(0,y) = 1$ *(falsity implies anything) for all* $y \in [0,1]$

I4: $I(x,1) = 1$ *(anything implies tautology) for all* $x \in [0,1]$

I5: $I(1,0) = 0$ *(Booleanity).*

The following properties could be important in some applications:

I6: $I(1,x) = 1$ (tautology cannot justify anything) for all $x \in [0,1]$

I7: $I(x, I(y,z)) = I(y, I(x,z))$(exchange principle) for all $x, y, z \in [0,1]$

I8: $x \le y$ if and only if $I(x,y) = 1$ (implication defines ordering) for all $x, y \in [0,1]$

I9: $I(x,0) = N(x)$ for all $x \in [0,1]$ is a strong negation

I10: $I(x,y) \ge y$ for all $x, y \in [0,1]$

I11: $I(x,x) = 1$ (identity principle) for all $x \in [0,1]$

I12: $I(x,y) = I(N(y), N(x))$ for all $x, y \in [0,1]$ and a strong negation N

I13: I is a continuous function

The most important families of implications are given [32] by

Definition 5 *A* S*-implication associated with a t-conorm* S *and a strong negation* N *is defined by*

$$I_{S,N}(x,y) = S(N(x), y) \quad \forall x, y \in [0,1].$$

A R*-implication associated with a t-norm* T *is defined by*

$$I_T(x,y) = sup\{z \in [0,1] / T(x,z) \le y\} \quad \forall x, y \in [0,1]$$

A QL-implication is defined by

$$I_{T,S,N}(x,y) = S(N(x), T(x,y)) \quad \forall x, y \in [0,1]$$

We shall work with Fodor's implication

$$I_F(x,y) = \begin{cases} 1 & if \quad x \le y \\ max(1-x, y) & otherwise \end{cases}$$

which is [32] a R-implication for $T = min_0$, a S-implication for $S = max_0$ and a QL-implication for $T = min$ and $S = max_0$, where

$$min_0(x,y) = \begin{cases} 0 & if \quad x+y \le 1 \\ min(x,y) & if \quad x+y > 1 \end{cases}$$

$$max_0(x,y) = \begin{cases} 1 & if \quad x+y \ge 1 \\ max(x,y) & if \quad x+y < 1 \end{cases}$$

and $N(x) = 1 - x$. Besides, the Fodor's implication verifies the properties I1-I12.

3 Previous Results

In our paper [13] we analyzed the Generalized Modus Ponens reasoning with t-norm

$$t(x,y) = max((1+\lambda)(x+y-1) - \lambda xy), \; \lambda \ge -1,$$

and the following implication operators:

Reichenbach implication

$$I_R(u,v) = 1 - \mu_A(u) + \mu_A(u)\mu_B(v)$$

Willmott implication

$$I_W(u,v) = max(1 - \mu_A(u), min(\mu_A(u), \mu_B(v)))$$

Mamdani implication

$$I_M(u,v) = min(\mu_A(u), \mu_B(v))$$

Rescher-Gaines implication

$$I_{RG}(u,v) = \begin{cases} 1 & if \quad \mu_A(u) \le \mu_B(v) \\ 0 & otherwise \end{cases}$$

Kleene-Dienes implication

$$I_{KD}(u,v) = max(1 - \mu_A(u), \mu_B(v))$$

Brouwer-Gödel implication

$$I_{BG}(u,v) = \begin{cases} 1 & if \quad \mu_A(u) \le \mu_B(v) \\ \mu_B(v) & otherwise \end{cases}$$

Goguen implication

$$I_G(u,v) = \begin{cases} min\left(\dfrac{\mu_B(v)}{\mu_A(u)}, 1\right) & if \quad \mu_A(u) \ne 0 \\ 1 & otherwise \end{cases}$$

Lukasiewicz implication

$$I_L(u,v) = min(1 - \mu_A(u) + \mu_B(v), 1)$$

We obtained the results given by the following three theorems.

Theorem 1 *If the premise contains the observation* $(i.e.\ \mu_{A'}(u) \le \mu_A(u) \quad \forall u \in U)$ *then*

1) $\mu_{B'}(v) = \mu_B(v)$ *for every of the cases*

1.1 $I = I_R$ *and* $\lambda \ge 0$

1.2 $I = I_R$, $\lambda < 0$ *and* $\mu_B(v) \ge \dfrac{\lambda}{\lambda - 1}$

1.3 $I = I_W$ and $\lambda \ge 0$

1.4 $I = I_W$, $\lambda < 0$ *and* $\mu_B(v) \ge -\dfrac{\lambda}{4}$

1.5 $I = I_M$

1.6 $I \in \{I_{KD}, I_L\}$ *and* $\lambda \ge 0$

1.7 $I = I_{KD}$, $\lambda < 0$ *and* $\mu_B(v) \ge \dfrac{-\lambda}{4}$

1.8 $I \in \{I_{BG}, I_G\}$

2) $\mu_{B'}(v) \le \mu_B(v)$ *for* $I = I_{RG}$

3) $\mu_{B'}(v) < -\dfrac{\lambda}{4}$ *for every of the cases*

3.1 $I = I_W, \lambda < 0$ *and* $\mu_B(v) < -\dfrac{\lambda}{4}$

3.2 $I = I_{KD}, \lambda < 0$ *and* $\mu_B(v) < -\dfrac{\lambda}{4}$

4) $\mu_{B'}(v) \le -\dfrac{(\mu_B(v)(1+\lambda) - \lambda)^2}{4\lambda(1 - \mu_B(v))}$

for $I = I_R$, $\lambda < 0$ *and* $\mu_B(v) \le \dfrac{\lambda}{\lambda - 1}$

5) $\mu_{B'}(v) < \dfrac{-\lambda(1 + \mu_B(v))^2 + 4(1+\lambda)\mu_B(v)}{4}$

for $I = I_L$ *and* $\lambda < 0$.

Theorem 2 *If the premise and the observation are identical then:*

1) *for* $\lambda \ge 0$ *we have*

$\mu_{B'}(v) = \mu_B(v)$

$\forall I \in \{I_R, I_W, I_M, I_{RG}, I_{KD}, I_{BG}, I_G, I_L\}$

2) *for* $\lambda < 0$ *we have*

a) $\mu_{B'}(v) = \mu_B(v)$ *for*

a_1 *)* $I \in \{I_M, I_{RG}, I_{BG}, I_G\}$

a_2 *)* $I = I_R$ *and* $\dfrac{\lambda}{\lambda - 1} \le \mu_B(v) \le 1$

b) $\mu_{B'}(v) = max\left(-\dfrac{\lambda}{4}, \mu_B(v)\right)$

for $I \in \{I_W, I_{KD}\}$

c) $\mu_{B'}(v) = -\dfrac{[(1+\lambda)\mu_B(v) - \lambda]^2}{4\lambda(1 - \mu_B(v))}$

for $I = I_R$ *and* $\mu_B(v) \le \dfrac{\lambda}{\lambda - 1}$

d) $\mu_{B'}(v) = \dfrac{-\lambda\mu_B^2(v) + 2\mu_B(v)(\lambda + 2) - \lambda}{4}$

for $I = I_L$

Theorem 3 *If the observation contains the premise (i. e.* $\mu_{A'}(u) \le \mu_A(u) \ \forall u \in U$ *) then*

a) $\mu_{B'}(v) = \mu_B(v)$ *if* $I = I_M$

b) $\mu_{B'}(v) \ge \mu_B(v)$ *if*

$I \in \{I_R, I_W, I_{RG}, I_{KD}, I_{BG}, I_G, I_L\}$

4 Main Result

Taking into account the properties verified by Fodor's implication, it results that it is one of the most important implication operators. Consequently, we shall continue the research in [13] by using the same t-norm, but Fodor's implication. We shall analyze five cases, depending on the relation between μ_A and $\mu_{A'}$.

4.1 The Premise Contains the Observation: $\mu_{A'}(u) \le \mu_A(u)$

Theorem 4 *If the premise contains the observation then*

(i) $\mu_{B'}(v) < -\dfrac{\lambda}{4}$ *if* $\lambda < 0$ *and* $\mu_B(v) < -\dfrac{\lambda}{4}$

(ii) $\mu_{B'}(v) = \mu_B(v)$ *otherwise*.

Proof. This case is illustrated in the following figure

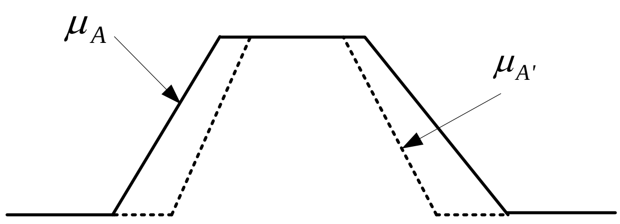


Fig. 3 The premise contains the observation

i1) value on the set $U_1 = \{u \in U \,/\, \mu_A(u) \le \mu_B(v)\}$

Because

$$I_F(\mu_A(u), \mu_B(v)) = 1,$$

we have

$$\mu_{B'}(v) = sup_{u \in U_1}\; max(\mu_{A'}(u), 0)$$
$$\le sup_{u \in U_1}\; max(\mu_A(u),\, 0) \le \mu_B(v).$$

i2) value on the set

$$U_2 = \{u \in U \,/\, \mu_A(u) > \mu_B(v) \ge 0.5\}$$

We have

$$1 - \mu_A(u) < 1 - \mu_B(v) \le \mu_B(v)$$

and therefore

$$I_F(\mu_A(u), \mu_B(v)) = \mu_B(v).$$

Further on we obtain

$$\mu_{B'}(v) = sup_{u \in U_2}\; max\,\big((1+\lambda)(\mu_{A'}(u) + \mu_B(v) - 1)$$
$$- \lambda \mu_{A'}(u)\mu_B(v), 0\big)$$
$$= sup_{u \in U_2}\; max\,\big(\mu_{A'}(u)(1 + \lambda - \lambda\mu_B(v))$$
$$-(1+\lambda)(1 - \mu_B(v)), 0\big)$$
$$= \mu_B(v)$$

and this value is obtained for

$$u_0 \in U \text{ with } \mu_{A'}(u_0) = 1.$$

i3) value on the set

$$U_3 = \{u \in U \,/\, \mu_A(u) > 1 - \mu_B(v) > 0.5\}$$

We have

$$I_F(\mu_A(u), \mu_B(v)) = \mu_B(v)$$

and, as in the previous case, we obtain

$$\mu_{B'}(v) = \mu_B(v).$$

i4) value on the set

$$U_4 = \{u \in U \,/\, \mu_B(v) < \mu_A(u) \le 1 - \mu_B(v)\}$$

In this case

$$I_F(\mu_A(u), \mu_B(v)) = 1 - \mu_A(u)$$

and therefore

$$\mu_{B'}(v) = sup_{u \in U_4}\; max\,(\mu_{A'}(u)(1 + \lambda\mu_A(u))$$
$$-(1+\lambda)\mu_A(u), 0)$$
$$\le sup_{u \in U_4}\; max(\lambda\mu_A^2(u) - \lambda\mu_A(u), 0).$$

It results

$$\mu_{B'}(v) = 0 \text{ for } \lambda \ge 0$$

and

$$\mu_{B'}(v) < -\frac{\lambda}{4} \text{ for } \lambda < 0.$$

Remark 1 If the observation is more precise than the premise of the rule then it gives more information than the premise. However, it does not seem reasonable to think that the generalized modus ponens allows obtaining a conclusion more precise than that of the rule. The result of the inference is valid if

$$\mu_{B'}(v) = \mu_B(v) \quad \forall v \in V.$$

Sometimes, the deduction operation allows the reinforcement of the conclusion, as in the following example [17].

Rule: *If the tomato is red then the tomato is ripe.*
Observation: *This tomato is very red.*

If we know that the maturity degree increases with respect to color, we can infer

this tomato is very ripe.

On the other hand, in the example

Rule: *If the melon is ripe then it is sweet*
Observation: *The melon is very ripe*

we do not infer that

the melon is very sweet

because it can be so ripe that it can be rotten.

Remark 2 This examples show that the expert must choose the deduction operation depending on the knowledge base. If he/she has no supplementary information about the connection between the variation of the premise and the conclusion, he/she must be satisfied with the conclusion

$$\mu_{B'}(v) = \mu_B(v), \quad \forall v \in V.$$

Theorem 4 says that for this we can choose $\lambda \ge 0$.

4.2 Identity Between Premise and Observation: $\mu_{A'}(u) = \mu_A(u) \quad \forall u \in U$

Theorem 5 *If the premise and the observation coincide then*

$(i)\quad \mu_{B'}(v) = max\left(-\frac{\lambda}{4}, \mu_B(v)\right)$

$\qquad if\ \ \lambda < 0\ \ and\ \ \mu_B(v) < 0.5$

$(ii)\quad \mu_{B'}(v) = \mu_B(v)\quad otherwise.$

Proof: In this case all inequalities (generated by inequality $\mu_{A'}(u) \le \mu_A(u)$ from the proof of the Theorem 4 become equalities. Thus we obtain the relations proposed by Theorem 5.

Remark 3 When the observation and the premise of the rule coincide then the natural behavior of the fuzzy deduction is to obtain an identical conclusion. However, Theorem 5, in the case of $\lambda < 0$, can give a different conclusion. This indicates the appearance of an uncertainty in the conclusion, which is completely unreasonable. In order to avoid this possibility we suggest using a value $\lambda \ge 0$.

4.3 The Observation Contains the Premise: $\mu_A(u) \le \mu_{A'}(u)\quad \forall u \in U$

Theorem 6 *If the observation contains the premise then* $\mu_{B'}(v) \ge \mu_B(v)\quad \forall v \in V$.

Proof:
This case is presented in the next figure

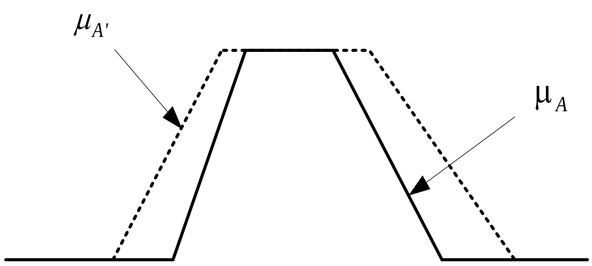


Fig. 4 The observation contains the premise

i) value on the set

$$U_1 = \{u \in U \,/\, \mu_A(u) \le \mu_B(v)\}$$

We have

$$I_F(\mu_A(u), \mu_B(v)) = 1$$

and therefore

$$\mu_{B'}(v) = sup_{u \in U_1}\ max(\mu_{A'}(u), 0).$$

On the set

$$U_1^1 = \{u \in U_1 \,/\, \mu_{A'}(u) \le \mu_B(v)\}$$

we obtain

$$\mu_{B'}(v) = \mu_B(v)$$

and on the set

$$U_1^2 = \{u \in U_1 \,/\, \mu_{A'}(u) > \mu_B(v)\}$$

we obtain

$$\mu_{B'}(v) > \mu_B(v).$$

i2)-i3) value on the sets

$$U_2 = \{u \in U \,/\, \mu_A(u) > \mu_B(v) \ge 0.5\}$$

and

$$U_3 = \{u \in U \,/\, \mu_A(u) > 1 - \mu_B(v) > 0.5\}.$$

As in the cases i2) and i3) from the Theorem 4 it results

$$\mu_{B'}(v) = \mu_B(v).$$

i4) value on the set

$$U_4 = \{u \in U \,/\, \mu_B(v) < \mu_A(u) \le 1 - \mu_B(v)\}$$

We have

$$I_F(\mu_A(u), \mu_B(v)) = 1 - \mu_A(u)$$

and therefore

$$\begin{aligned}\mu_{B'}(v) &= sup_{u \in U_4}\ max\,(\ \mu_{A'}(u)(1 + \lambda\mu_A(u)) \\ &\qquad - (1+\lambda)\mu_A(u), 0\ ) \\ &\ge sup_{u \in U_4}\ max\left(\lambda\mu_A^2(u) - \lambda\mu_A(u), 0\right).\end{aligned}$$

But,

a) for $\lambda \ge 0$,

$$max\left(\lambda\mu_A^2(u) - \lambda\mu_A(u), 0\right) = 0$$

and

b) for $\lambda < 0$,

$$max\left(\lambda\mu_A^2(u) - \lambda\mu_A(u), 0\right) \ge \lambda\mu_B^2(v) - \lambda\mu_B(v);$$

in addition,

$$\lambda\mu_B^2(v) - \lambda\mu_B(v) < \mu_B(v).$$

The final conclusion is

$$\mu_{B'}(v) \ge \mu_B(v).$$

Remark 4 The result obtained by Theorem 6 is very general and it does not offer enough information about the inferred conclusion. The inference result

depends on compatibility between observation and the premise of the rule.
To express this compatibility, the following quantities [7, 14, 19] are frequently used:

(a) $D.I = sup_{\{u \in U \,/\, \mu_A(u)=0\}}\ \mu_{A'}(u)$,

named uniform degree of non-determination; it appears when the support of the premise does not contain the support of the observation;

(b) $I = sup_{\{u \in U \,/\, \mu_{A'}(u)\ \geq \mu_A(u)\}} \left(\mu_{A'}(u) - \mu_A(u)\right)$.

The uncertainty propagated is expressed with the help of $D.I$ and I and it corresponds to the value $\mu_{B'}$ on the set $\{v \in V \,/\, \mu_B(v) = 0\}$.

Theorem 7 *If* $\mu_{A'}(u) \geq \mu_A(u) \quad \forall u \in U$ *then the uncertainty propagated during the inference is*

$$\mu_{B'}(v) < I \quad if \quad \lambda > 0$$
$$\mu_{B'}(v) = I \quad if \quad \lambda = 0$$
$$\mu_{B'}(v) > I \quad if \quad \lambda < 0 .$$

Proof: The result is obtained from the expression of $\mu_{B'}(v)$ for $\mu_B(v) = 0$:

$$\mu_{B'}(v) = sup_{u \in U}\ max\,(\ \mu_{A'}(u) - \mu_A(u) - \lambda\mu_A(u)(1 - \mu_{A'}(u)), 0\ ).$$

Remark 5 This theorem says that the value $\lambda > 0$ is indicated to be used.

4.4 There is a Partial Overlapping Between Premise and Observation

Theorem 8 *If there is a partial overlapping between the sets* A *and* A' *then*

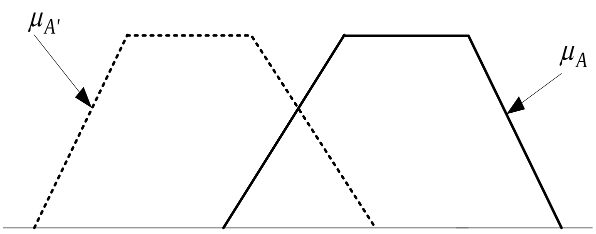
$$\mu_{B'}(v) = 1 \ \ if \ \ core(A') \not\subset A_{\mu_B(v)}$$

and

$$\mu_{B'}(v) \geq \mu_B(v) \ \ otherwise,$$

where A_α *denotes the* α*-cut of* A.

Proof: Some of these possibilities are illustrated in the following figures:



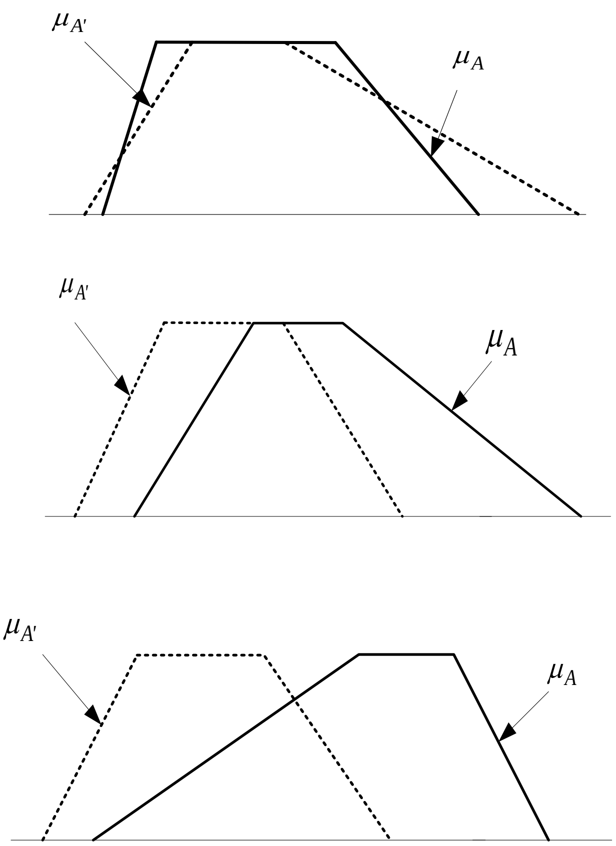


Fig. 5 Partial overlapping between premise and observation

i) the case $core(A') \not\subset A_{\mu_B(v)}$

For $\mu_A(u) \leq \mu_B(v)$ we have

$$I_F\left(\mu_A(u), \mu_B(v)\right) = 1$$

and therefore

$$\mu_{B'}(v) = sup_{\{u \in U \,/\, \mu_A(u)\, \leq \mu_B(v)\}}\ max\left(\mu_{A'}(u), 0\right) = 1$$

ii) the case $core(A') \subset A_{\mu_B(v)}$

a) For $\mu_B(v) \geq 0.5$, on the set

$U_1 = \{u \in U \,/\, \mu_A(u) > \mu_B(v) \geq 0.5\}$

we have

$$I_F\left(\mu_A(u), \mu_B(v)\right) = \mu_B(v)$$

and

$$\mu_{B'}(v) = sup_{u \in U_1}\ max\,(\ \mu_{A'}(u)(1 + \lambda - \lambda\mu_B(v)) + (1+\lambda)(\mu_B(v) - 1), 0\ )$$
$$= \mu_B(v)$$

because there is $u_0 \in U_1$ such that $\mu_{A'}(u_0) = 1$.

b) For $\mu_B(v) < 0.5$ we consider the sets

$$U_2 = \{u \in U \,/\, \mu_A(u) > 1 - \mu_B(v) > 0.5\},$$

$$U_3 = \{u \in U \,/\, \mu_B(v) < \mu_A(u) \le 1 - \mu_B(v)\}$$

and

$$U' = U_2 \cup U_3 .$$

Because there is $u_0 \in U'$ such that $\mu_{A'}(u_0) = 1$, we have

b1) on the set U_2, likewise on the set U_1 from the case a), we obtain $\mu_{B'}(v) = \mu_B(v)$.

b2) on the set U_3 we obtain

$$I_F(\mu_A(u), \mu_B(v)) = 1 - \mu_A(u)$$

and therefore

$$\mu_{B'}(v) = sup_{u \in U_2}\, max\,(\mu_{A'}(u)(1 + \lambda\mu_A(u)) - (1+\lambda)\mu_A(u), 0\,).$$

But, for u_0 mentioned above we obtain

$$max(\mu_{A'}(u_0)(1 + \lambda\mu_A(u_0)) - (1+\lambda)\mu_A(u_0), 0) = 1 - \mu_A(u_0) \ge \mu_B(v).$$

Finally, it results that the value of $\mu_{B'}(v)$ is at least $\mu_B(v)$.

4.5 The Premise and the Observation are Contradictory: $\mu_{A'}(u) = 1 - \mu_A(u)\ \forall u \in U$

Theorem 9 *If the premise and the observation are contradictory then* $\mu_{B'}(v) = 1 \quad \forall v \in V$.

Proof: The following figure presents this case

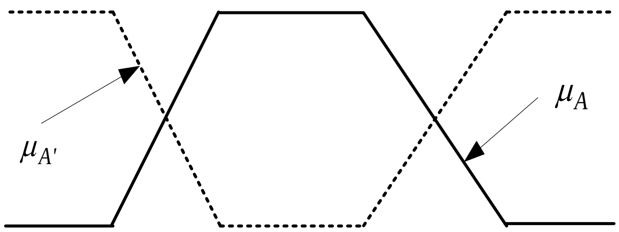


Fig. 6 The premise and the observation are contradictory

On the set

$$U_1 = \{u \in U \,/\, \mu_A(u) \le \mu_B(v)\}$$

we have

$$I_F(\mu_A(u), \mu_B(v)) = 1$$

and therefore

$$\mu_{B'}(v) = sup_{u \in U_1}\, max(\mu_{A'}(u), 0) = sup_{u \in U_1}\, max(1 - \mu_A(u), 0) = 1$$

because there is $u_0 \in U_1$ with $\mu_A(u_0) = 0$.

Remark 6 This result represents an indeterminate conclusion: all values of V are possible.

5 Conclusion

The results presented in this paper show how the generalized modus ponens rule works with the parametric t-norm

$$t(x, y) = max((1+\lambda)(x + y - 1) - \lambda xy),\ \lambda \ge -1,$$

and Fodor's implication. Five cases, depending on the relation between observation and the premise of the rule, are analyzed. For $\lambda = -1$ and $\lambda = 0$ we recover some results from [12].

The previous results are important because in the cases specified in Theorems 4, 5, 9 (and a part of Theorem 8) the inferred results are obtained directly, avoiding the typical calculus for Generalized Modus Ponens reasoning. Such a calculus is necessary only in the conditions of Theorems 6 and 8. Consequently, because in some cases the calculus required to obtain the conclusion is avoided, the usage of the previous results is recommended in practical applications based on fuzzy reasoning.

One of our future preoccupations is the improvement of these results by using a genetic algorithms technique, like in [33], in order to determine the parameters which define trapezoidal fuzzy sets.

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