An Inverse Dynamic Model of the Gough-Stewart Platform

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Abstract: In this paper a new formulation of the inverse dynamic model of the Gough-Stewart platform is proposed. This approach is based on the methodology developped by Khalil. The platform is considered as a multi robot system moving a common load. Using a global formalism, the Jacobian and inertia matrices of each segment are computed in a factorized form. This paper provides a basis for parallel algorithm development for a dynamic control under the real time consraint. The proposed sheme is validated by the simulation results.

Key–Words: Parallel Robot, Gough-Stewart Platform, Kinematic Model, Jacobian Matrix, Inverse Dynamic Model

1 Introduction

The parallel architecture manipulators [1], [2], [3], [4], [5] have some significant advantages in comparison with serial robots, in particular, a greater compactness and accuracy in the end effector positioning. These parallel robots are primarly used in the fields where the considered processes require a high degree of accuracy, high speeds or accelerations. Aircraft simulator [1], teleoperation [6], machining tools [7][8], and various other medical applications [9], [10] constitute some of the many possible applications of parallel robots.

The inverse dynamic model is essential for an effective robot control. In the field of parallel robots, many approaches have been developped. The formalism of d'Alembert has been used to obtain an analytical expression of the dynamics model [11] [12]. The principle of virtual works has been applied in [13] for solving the inverse dynamics of the Gough-Stewart platform. Lagrangian formalism is applied in [10] for the dynamics modeling of a parallel robot used like a haptic interface for a surgical simulator. These various approaches do not seem effective for a robot dynamic control under the real time constraint. The computation time reduction can be acquired by the development of approaches using recursive schemes, in particular, based on the Newton-Euler formulation. Thus Gosselin [14] proposed the inverse dynamic model of planar and spatial parallel robot, in which all the masses and inertias are taken into account. This proposed method is difficult to generalize for all the parallel architectures. Dasgupda et al [15] applied this method to several parallel manipulators. Khan [16] has developped a recursive algorithm for the inverse dynamics. This method is applied to a $3R$ planar parallel robot. Bi et al [17] use the Newton-Euler iterative scheme for the articular force computation of a tripod system. Khalil et al [18] proposed a general method for the inverse and direct dynamic model computation of parallel robots, wich is applied to several parallel manipulators [19].

In the present paper, the inverse dynamic modeling of the Gough-Stewart platform is presented. The parallel robot is considered as a multi robot system with k serial robots (the k parallel robot segments) moving a common load (the mobile platform). The proposed approach use the methodology developped by Khalil et al [18]. The purpose consists, using a global formalism, in highlighting explicitly inertia matrices expressed in articular and operational spaces. These matrices are obtained in factorized form, with the aim of parallel algorithm developement. The objective being the implementaton of a dynamic control under the real time constraint.

This paper is organized as follows. In the following section we describe the nomenclature and the used notation. In section 3, the Gough-Stewart platform architecure is described. Development of the inverse kinematic model and the inverse Jacobian matrix are briefly described in sections 4 and 5. The kinematic model of the segment is given in section 6. In section 7, the inverse dynamic model of the Gough-Stewart platform is developed. A simulation of this inverse dynamic model is provided in section 8 validating the proposed approach.

2 Preliminaries

In this section we define the required notation and preliminaries are presented. The parallel robot is considered as a multi-robot system with k serial robots (segments) moving a common load (mobile platform). Fig. 1 shows the links, the frames and position vectors for the segment i ($i = 1, \ldots, k$).

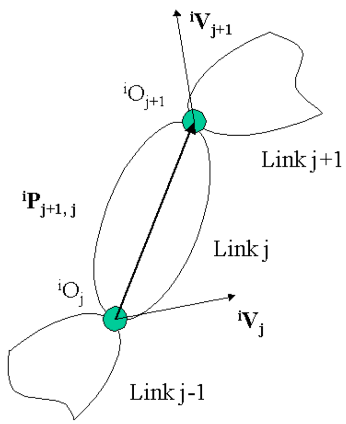

Figure 1: Links, frames and position vectors for the segment i

2.1 Nomenclature

2.1.1 Joint and link parameters

- ${}^i\boldsymbol{P_{j+1,j}}$: position vector from ${}^iO_j$ to ${}^iO_{j+1}$
- k: number of segments
- ${}^iM$: degrees of freedom (DOF)-number of segment i
- ${}^iN$: joint number of segment i
- S: active joint number by link
- θ_i^a, $\dot{\theta}_i^a$: position and velocity of active joint of the segment i
- ${}^i\theta_j^p$, ${}^i\dot{\theta}_j^p$: position and velocity of passive joint j of the segment i
- ${}^i\omega_j, {}^iv_j \in \Re^3$: angular and linear velocity of link j for the segment i

2.1.2 Spatial quantities

- ${}^iH_j$: spatial-axis (map matrix) of joint j for the segment i. For instance, for a joint with 2-DOF (rotation about z-axis and translation about x-axis), the matrix ${}^iH_j \in \Re^{6\times 2}$ is given by:

$$\begin{array}{cc} & \begin{array}{ccl} 1^{st} & 2^{nd} & -DOF \end{array} \\ {}^iH_j = & \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{array}\right] \begin{array}{l} \text{x-axis rotation} \\ \text{y-axis rotation} \\ \text{z-axis rotation} \\ \text{x-axis translation} \\ \text{y-axis translation} \\ \text{z-axis translation} \end{array} \end{array}$$

- ${}^i\boldsymbol{V_j} = \left[\begin{array}{c} {}^i\boldsymbol{\omega_j} \\ {}^i\boldsymbol{v_j} \end{array}\right] \in \Re^6$: spatial velocity of the link j for the segment i
- $\boldsymbol{V_{N+1}} = \left[\begin{array}{c} \boldsymbol{\omega_{N+1}} \\ \boldsymbol{v_{N+1}} \end{array}\right] \in \Re^6$: spatial velocity of the end effector

2.1.3 Global quantities

The following global quantities are defined for $j = {}^iN$ to 1 or $j = {}^iM$ to 1 and $i = k$ to 1

- $\boldsymbol{\dot{Q}_i} = Col\left(\ {}^i\dot{\theta}_j\ \right) \in \Re^{{}^iM}$: global vector of articular coordinate velocity of the segment i, taking into account passive and active joints
- $\boldsymbol{\dot{Q}} = Col\left(\ \dot{\theta}_i^a\ \right) \in \Re^k$: vector of generalized coordinate velocity of the system
- $\boldsymbol{\mathcal{V}_i} = Col\left(\ {}^iV_j\ \right) \in \Re^{6\,{}^iN}$: global vector of spatial velocities for the segment i
- $\mathcal{H}_i = Diag\left(\ {}^iH_j\ \right) \in \Re^{6\,{}^iN\times {}^iM}$: global matrix of spatial axis for the leg i

2.2 General notation

With any vector $\boldsymbol{V} = \left[\begin{array}{ccc} V_x & V_y & V_z \end{array}\right]^t$, a tensor $\tilde{V}$ can be associated whose representation in any frame is a skew symmetrical matrix:

$$\tilde{V} = \left[\begin{array}{ccc} 0 & -V_z & V_y \\ V_z & 0 & -V_x \\ -V_y & V_x & 0 \end{array}\right]$$

This tensor $\tilde{V}$ has the properties that $\tilde{V} = -\tilde{V}^t$ and $\tilde{V}_1\boldsymbol{V_2} = \boldsymbol{V_1} \wedge \boldsymbol{V}_2$ i.e., is the vector cross-product of $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$.
A matrix $\hat{V}$ associated to the vector $\boldsymbol{V}$ is defined as:

$$\hat{V} = \begin{bmatrix} U & \tilde{V} \\ 0 & U \end{bmatrix}$$

and

$$\hat{V}^t = \begin{bmatrix} U & 0 \\ -\tilde{V} & U \end{bmatrix}$$

where U and 0 stand for unit and zero matrices of appropriate size.
In our derivation, we also make use of global matrices and vectors which lead to a compact representation of various factorizations. A bidiagonal block matrix $\mathcal{P}_i \in \Re^{6\,^iN \times 6\,^iN}$ is defined as:

$$\mathcal{P}_i = \begin{bmatrix} U & & & & \\ -^i\hat{P}_{N-1} & U & 0 & & \\ 0 & -^i\hat{P}_{N-2} & U & & \\ 0 & 0 & & & \\ \vdots & \vdots & & & \\ 0 & 0 & 0 & -^i\hat{P}_1 & U \end{bmatrix}$$

Note that, according to our notation, $\boldsymbol{^iP_{j+1,j} = {^iP_j}}$.
The inverse of $\mathcal{P}_i$ is a lower triangular block matrix given by:

$$\mathcal{P}_i^{-1} = \begin{bmatrix} U & & & & \\ ^i\hat{P}_{N,N-1} & U & & 0 & \\ ^i\hat{P}_{N,N-2} & ^i\hat{P}_{N-1,N-2} & U & & \\ \vdots & \vdots & & & \\ ^i\hat{P}_{N,1} & ^i\hat{P}_{N-1,1} & \dots & ^i\hat{P}_{2,1} & U \end{bmatrix}$$

3 Parallel robot description

The robot considered in this paper is of fully parallel type. This robot consists of 6 segments linking a fixed base to a mobile platform. The extremities of each leg are fitted with a 2-DOF universal joint at the base and a 3-DOF spherical joint at the platform (Fig. 2).
The universal joint center and the spherical joint center are denoted by A_i and B_i ($i = 1, \cdots, 6$), respectively. The lenght of each leg i is actuated using an active prismatic joint.

The used notations to describe the parallel robot are defined in the following.

- R_b is the absolute frame, tied to the fixed base (see Fig. 2). $R_b = (0, x, y, z)$.
- R_p is the mobile frame, tied to the mobile part. $R_p = (C, x_p, y_p, z_p)$.

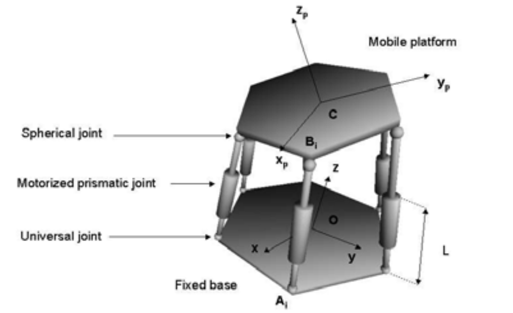

Figure 2: Parallel robot representation.

- Let O be the origin of the absolute coordinate system
- Let C (or O_{N+1}) be the origin of the mobile frame R_p, whose coordinates are in the absolute frame R_b:

$$\boldsymbol{OC}_{/R_b} = \begin{bmatrix} x_c & y_c & z_c \end{bmatrix}^t$$

- A_i (or $^iO_1$)is the center of the joint between the segment i and the fixed base:

$$\boldsymbol{OA_{i}}_{/R_b} = \begin{bmatrix} a_i^x & a_i^y & a_i^z \end{bmatrix}^t$$

- B_i (or ^iO_N) is the center of the joint between the segment i and the mobile part:

$$\boldsymbol{CB_{i}}_{/R_p} = \begin{bmatrix} b_i^x & b_i^y & b_i^z \end{bmatrix}^t$$

- $[R]$ is the rotation matrix of r_{ij} elements (in the RPY formalism), expressing the orientation of the R_p mobile frame withrespect to the R_b absolute frame. The expression for this matrix is given by:

$$[R] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \quad (1)$$

where:
$r_{11} = \cos\beta\cos\gamma$
$r_{12} = -\cos\beta\sin\gamma$
$r_{13} = \sin\beta$
$r_{21} = \sin\gamma\cos\alpha + \cos\gamma\sin\beta\sin\alpha$
$r_{22} = \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma$
$r_{23} = -\cos\beta\sin\alpha$
$r_{31} = \sin\gamma\sin\alpha - \cos\gamma\sin\beta\cos\alpha$
$r_{32} = \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma$
$r_{33} = \cos\beta\cos\alpha$

- α, β and γ are the Bryan angles [24], describing the rotation of the mobile platform with respect to the fixed base.
- $\boldsymbol{X}$ is the task coordinate vector.

$$\boldsymbol{X} = \begin{bmatrix} \alpha & \beta & \gamma & x_c & y_c & z_c \end{bmatrix}^t$$

- R_{b_i} is the frame tied to the segment i. $R_{b_i} = (A_i, x_{b_i}, y_{b_i}, z_{b_i})$.
- α_{b_i}, β_{b_i}, γ_{b_i} are the angles, in the RPY formalism, describing frame R_{b_i} rotation with respect to the absolute frame R_b.
- $\left[{}^bR_{b_i}\right]$ is the rotation matrix of ${}^br_{b_{i_{jk}}}$ elements (in the RPY formalism), expressing the orientation of the R_{b_i} frame withrespect to the R_b absolute frame. The expression for this matrix is given by:

$$\left[{}^bR_{b_i}\right] = \begin{bmatrix} c\beta_{b_i}c\gamma_{b_i} & -c\beta_{b_i}s\gamma_{b_i} & s\beta_{b_i} \\ s\gamma_{b_i} & c\gamma_{b_i} & 0 \\ -c\gamma_{b_i}s\beta_{b_i} & s\beta_{b_i}s\gamma_{b_i} & c\beta_{b_i} \end{bmatrix} \tag{2}$$

where c represents the function $\cos$ and s the function $\sin$

4 Inverse kinematic model

The inverse geometric model relates the active joint variables (Q) to the operational variables which define the position and the orientation of the end effector (X). This relation is given by the following equation:

$$\theta_i^a = \|\boldsymbol{A_iB_i}\| = \|\boldsymbol{A_iO}_{/R_b} + \boldsymbol{OC}_{/R_b} + [R]\,\boldsymbol{CB_{i/R_p}}\| \tag{3}$$

Thus:

$$\theta_i^a = \sqrt{X_i^2 + Y_i^2 + Z_i^2} \tag{4}$$

where:

$$\begin{aligned} X_i &= x_c - a_i^x + r_{11}b_i^x + r_{12}b_i^y + r_{13}b_i^z \\ Y_i &= y_c - a_i^y + r_{21}b_i^x + r_{22}b_i^y + r_{23}b_i^z \\ Z_i &= z_c - a_i^z + r_{31}b_i^x + r_{32}b_i^y + r_{33}b_i^z \end{aligned} \tag{5}$$

5 Determination of the inverse Jacobian matrix

For parallel robots, the inverse Jacobian matrix computation ($\mathcal{J}^{-1}$) stays, in principle, relatively easy. $\mathcal{J}^{-1}$ matrix is obtained by the determination of point B_i velocity [2][23]:

$$\dot{\boldsymbol{OB_i}} = \boldsymbol{v_{N+1}} + \boldsymbol{B_iC} \wedge \boldsymbol{\omega_{N+1}} \tag{6}$$

The following relationship is verified:

$$\dot{\theta}_i^a = \dot{\boldsymbol{OB_i}}\,\boldsymbol{n_i} \tag{7}$$

Inserting (6) into (7), we obtain:

$$\dot{\theta}_i^a = \boldsymbol{n_i v_{N+1}} + \boldsymbol{\omega_{N+1}}\left(\boldsymbol{n_i} \wedge \boldsymbol{B_iC}\right) \tag{8}$$

The inverse Jacobian matrix is written as:

$$\mathcal{J}^{-1} = \begin{bmatrix} (\boldsymbol{n_6} \wedge \boldsymbol{B_6C})^t & \boldsymbol{n_6^t} \\ & \vdots \\ (\boldsymbol{n_1} \wedge \boldsymbol{B_1C})^t & \boldsymbol{n_1^t} \end{bmatrix} \tag{9}$$

6 Kinematic Model of the segment i

In this section, the segment i is considered as a serial robot. The point B_i is the robot terminal tool. This serial robot have 2 joints:

- A passive joint ${}^i\theta_1$ with 2 degrees of freedom (along y_{b_i} et z_{b_i} axis)
- An active joint ${}^i\theta_2$ with 1 degree of freedom (along x_{b_i} axis)

We define the following vectors:

- $\boldsymbol{Q_i}$, the articular coordinate vector of the segment i:

$$\boldsymbol{Q_i} = \begin{bmatrix} \theta_i^a & \beta_{b_i} & \gamma_{b_i} \end{bmatrix}^t \tag{10}$$

- $\dot{\boldsymbol{Q}}_{\boldsymbol{i}}$, the articular velocity vector of the segment i:

$$\dot{\boldsymbol{Q}}_{\boldsymbol{i}} = \begin{bmatrix} \dot{\theta}_i^a & \dot{\beta}_{b_i} & \dot{\gamma}_{b_i} \end{bmatrix}^t \tag{11}$$

- $\ddot{\boldsymbol{Q}}_{\boldsymbol{i}}$, the articular acceleration vector of the segment i:

$$\ddot{\boldsymbol{Q}}_{\boldsymbol{i}} = \begin{bmatrix} \ddot{\theta}_i^a & \ddot{\beta}_{b_i} & \ddot{\gamma}_{b_i} \end{bmatrix}^t \tag{12}$$

The velocity propagation for a serial chain of interconnected bodies is given by the following intrinsic equation [20][21][22]:

$${}^i\boldsymbol{V_j} - {}^i\hat{P}_{j-1}^t\,{}^i\boldsymbol{V_{j-1}} = {}^iH_j\,{}^i\dot{\theta}_j \tag{13}$$

By using the matrix $\mathcal{P}_i$, (13) can be expressed in a global form by:

$$\mathcal{P}_i^t \boldsymbol{\mathcal{V}_i} = \mathcal{H}_i \dot{\boldsymbol{Q}}_{\boldsymbol{i}} \quad (14)$$

thus:

$$\boldsymbol{\mathcal{V}_i} = \left(\mathcal{P}_i^t\right)^{-1} \mathcal{H}_i \dot{\boldsymbol{Q}}_{\boldsymbol{i}} \quad (15)$$

The end effector spatial velocity $\boldsymbol{V_{B_i}}$ is obtained by the following relation:

$$\boldsymbol{V_{B_i}} - {}^i\hat{P}_N^t \, {}^{\boldsymbol{i}}\boldsymbol{V_N} = \boldsymbol{0} \quad (16)$$

thus:

$$\boldsymbol{V_{B_i}} = {}^i\hat{P}_N^t \, {}^{\boldsymbol{i}}\boldsymbol{V_N} \quad (17)$$

Let $\beta_i \in \Re^{6\times {}^iN}$ be the matrix defined by

$$\beta_i = \left[\begin{array}{cccc} {}^i\hat{P}_N^t & 0 & \cdots & 0 \end{array} \right]$$

The equation (17) becomes:

$$\boldsymbol{V_{B_i}} = \beta_i \boldsymbol{\mathcal{V}_i} \quad (18)$$

Thus, inserting the expression of $\boldsymbol{\mathcal{V}_i}$ from (15), we obtain:

$$\boldsymbol{V_{B_i}} = \beta_i \left(\mathcal{P}_i^t\right)^{-1} \mathcal{H}_i \dot{\boldsymbol{Q}}_{\boldsymbol{i}} \quad (19)$$

The spatial velocity of point B_i is defined by the following relation:

$$\boldsymbol{V_{B_i}} = \mathcal{J}_{B_i} \dot{\boldsymbol{Q}}_{\boldsymbol{i}} \quad (20)$$

Where $\mathcal{J}_{B_i} \in \Re^{6\times 6}$ is the Jacobian matrix of the segment i expressed in the base frameR_b.
Thus we deduce, considering (19), a factorized expression of the Jacobian matrix $\mathcal{J}_{B_i}$:

$$\mathcal{J}_{B_i} = \beta_i \, \mathcal{P}_i^{-t} \, \mathcal{H}_i \quad (21)$$

with:

$$\begin{cases} \beta_i = \left[\begin{array}{cc} {}^i\hat{P}_2 & 0 \end{array} \right] \in \Re^{6\times 12} \\ \mathcal{P}_i^{-t} = \begin{bmatrix} U & {}^i\hat{P}_1^t \\ 0 & U \end{bmatrix} \in \Re^{12\times 12} \\ \mathcal{H}_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \Re^{12\times 3} \end{cases} \quad (22)$$

By using (22) in (21), a new Jacobian matrix formulation is obtained:

$$\mathcal{J}_{B_i} = \begin{bmatrix} {}^bR_{b_i} & 0 \\ 0 & {}^bR_{b_i} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & \theta_i^a \\ 0 & -\theta_i^a & 0 \end{bmatrix} \quad (23)$$

This expression can be rewritten as:

$$\mathcal{J}_{B_i} = \begin{bmatrix} \mathcal{J}_{B_i}^\omega \\ \mathcal{J}_{B_i}^v \end{bmatrix} = \begin{bmatrix} {}^bR_{b_i} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ {}^bR_{b_i} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \theta_i^a \\ 0 & -\theta_i^a & 0 \end{bmatrix} \end{bmatrix} \quad (24)$$

The forward kinematic model of the segment i is defined as the relation linking the linear velocity of the segment i terminal tool (point B_i) to articular velocity vector $\dot{\boldsymbol{Q}}_{\boldsymbol{i}}$:

$$\boldsymbol{v_{B_i}} = \mathcal{J}_{B_i}^v \dot{\boldsymbol{Q}}_{\boldsymbol{i}} = {}^bR_{b_i} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \theta_i^a \\ 0 & -\theta_i^a & 0 \end{bmatrix} \dot{\boldsymbol{Q}}_{\boldsymbol{i}} \quad (25)$$

The inverse Jacobian matrix $\left(\mathcal{J}_{B_i}^v\right)^{-1}$ is directly obtained by:

$$\left(\mathcal{J}_{B_i}^v\right)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{\theta_i^a} \\ 0 & \frac{1}{\theta_i^a} & 0 \end{bmatrix} {}^bR_{b_i}^t \quad (26)$$

Thus:

$$\left(\mathcal{J}_{B_i}^v\right)^{-1} = \begin{bmatrix} \cos\beta_{b_i}\cos\gamma_{b_i} & \sin\gamma_{b_i} & -\cos\gamma_{b_i}\sin\beta_{b_i} \\ -\frac{\sin\beta_{b_i}}{\theta_i^a} & 0 & -\frac{\cos\beta_{b_i}}{\theta_i^a} \\ -\frac{\cos\beta_{b_i}\sin\gamma_{b_i}}{\theta_i^a} & \frac{\cos\gamma_{b_i}}{\theta_i^a} & \frac{\sin\gamma_{b_i}\sin\beta_{b_i}}{\theta_i^a} \end{bmatrix} \quad (27)$$

The second-order inverse kinematic model of the segment i is given by:

$$\ddot{\boldsymbol{Q}}_{\boldsymbol{i}} = \left(\mathcal{J}_{B_i}^v\right)^{-1} \dot{\boldsymbol{v}}_{\boldsymbol{B_i}} + \frac{d}{dt}\left(\left(\mathcal{J}_{B_i}^v\right)^{-1}\right) \boldsymbol{v_{B_i}} \quad (28)$$

Thus:

$$\ddot{\boldsymbol{Q}}_{\boldsymbol{i}} = \left(\mathcal{J}_{B_i}^v\right)^{-1} \left(\dot{\boldsymbol{v}}_{\boldsymbol{B_i}} - \dot{\mathcal{J}}_{B_i}^v \dot{\boldsymbol{Q}}_{\boldsymbol{i}}\right) \quad (29)$$

The linear velocity of the point B_i can be expressed as a function of linear and angular velocities of the mobile platform as:

$$\boldsymbol{v_{B_i}} = \boldsymbol{v_{N+1}} + \boldsymbol{B_i C} \wedge \boldsymbol{\omega_{N+1}} \quad (30)$$

Thus, in a matrical form:

$$\boldsymbol{v_{B_i}} = \begin{bmatrix} \widetilde{B_iC} & U \end{bmatrix} \boldsymbol{V_{N+1}} \tag{31}$$

The linear acceleration of the point B_i is then given by the following relation:

$$\dot{\boldsymbol{v}}_{\boldsymbol{B_i}} = \begin{bmatrix} \widetilde{B_iC} & U \end{bmatrix} \dot{\boldsymbol{V}}_{\boldsymbol{N+1}} + \begin{bmatrix} \dot{\widetilde{B_iC}} & 0 \end{bmatrix} \boldsymbol{V_{N+1}} \tag{32}$$

Inserting (32) in (29), the second-order inverse kinematic model is finally obtained as:

$$\ddot{\boldsymbol{Q}}_{\boldsymbol{i}} = \left(\mathcal{J}^v_{B_i}\right)^{-1} \left(\boldsymbol{\mathcal{B}_i}\, \dot{\boldsymbol{V}}_{\boldsymbol{N+1}} + \dot{\boldsymbol{\mathcal{B}}}_{\boldsymbol{i}}\, \boldsymbol{V_{N+1}} - \dot{\mathcal{J}}^v_{B_i}\, \dot{\boldsymbol{Q}}_{\boldsymbol{i}}\right) \tag{33}$$

Where $\boldsymbol{\mathcal{B}_i} = \begin{bmatrix} \widetilde{B_iC} & U \end{bmatrix}$

6.1 Computation of $\dot{\mathcal{J}}^v_{B_i}\, \dot{\boldsymbol{Q}}_{\boldsymbol{i}}$

Considering the relation given in (25), we obtain:

$$\dot{\mathcal{J}}^v_{B_i} = \frac{d}{dt}\left({}^b R_{b_i} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \theta^a_i \\ 0 & -\theta^a_i & 0 \end{bmatrix} \right) \tag{34}$$

Thus:

$$\dot{\mathcal{J}}^v_{B_i} = {}^b \dot{R}_{b_i} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \theta^a_i \\ 0 & -\theta^a_i & 0 \end{bmatrix} + {}^b R_{b_i} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \dot{\theta}^a_i \\ 0 & -\dot{\theta}^a_i & 0 \end{bmatrix} \tag{35}$$

We define the following vectors:

- Translation velocity vector along x_{b_i} axis:

$$\dot{\boldsymbol{\theta}}_{\boldsymbol{i}} = \begin{bmatrix} \dot{\theta}^a_i & 0 & 0 \end{bmatrix}^t$$

- Rotation velocity vector along y_{b_i} axis:

$$\dot{\boldsymbol{\beta}}_{\boldsymbol{b_i}} = \begin{bmatrix} 0 & \dot{\beta}_{b_i} & 0 \end{bmatrix}^t$$

- Rotation velocity vector along z_{b_i} axis:

$$\dot{\boldsymbol{\gamma}}_{\boldsymbol{b_i}} = \begin{bmatrix} 0 & 0 & \dot{\gamma}_{b_i} \end{bmatrix}^t$$

and their associated skew symmetrical matrices:

$$\left\{ \begin{array}{l} \tilde{\dot{\theta}}_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\theta}^a_i \\ 0 & \dot{\theta}^a_i & 0 \end{bmatrix} \\ \tilde{\dot{\beta}}_{b_i} = \begin{bmatrix} 0 & 0 & \dot{\beta}_{b_i} \\ 0 & 0 & 0 \\ -\dot{\beta}_{b_i} & 0 & 0 \end{bmatrix} \\ \tilde{\dot{\gamma}}_{b_i} = \begin{bmatrix} 0 & -\dot{\gamma}_{b_i} & 0 \\ \dot{\gamma}_{b_i} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \right. \tag{36}$$

Taking into account theses vectors and matrices, (35) becomes:

$$\dot{\mathcal{J}}^v_{B_i} = {}^b \dot{R}_{b_i} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \theta^a_i \\ 0 & -\theta^a_i & 0 \end{bmatrix} - {}^b R_{b_i}\, \tilde{\dot{\theta}}_i \tag{37}$$

${}^b R_{b_i}$ have been defined as the rotation matrix expressing the orientation of the i R_{bi} frame withrespect to the R_b absolute frame. This rotation matrix can be decomposed into product of two matrices as:

$${}^b R_{b_i} = R_{\beta_{b_i}}\, R_{\gamma_{b_i}} \tag{38}$$

Thus, we deduce:

$${}^b \dot{R}_{b_i} = \dot{R}_{\beta_{b_i}} R_{\gamma_{b_i}} + R_{\beta_{b_i}} \dot{R}_{\gamma b_i} = \tilde{\dot{\beta}}_{b_i} \underbrace{R_{\beta_{b_i}}\, R_{\gamma_{b_i}}}_{{}^b R_{b_i}} + R_{\beta_{b_i}} \tilde{\dot{\gamma}}_{b_i} R_{\gamma_{b_i}} \tag{39}$$

Inserting the relation given by (39) in (37), we deduce a new expression of the matrix $\dot{\mathcal{J}}^v_{B_i}$ as:

$$\dot{\mathcal{J}}^v_{B_i} = \left(\tilde{\dot{\beta}}_{b_i}\, {}^b R_{b_i} + R_{\beta_{b_i}}\, \tilde{\dot{\gamma}}_{b_i}\, R_{\gamma_{b_i}}\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \theta^a_i \\ 0 & -\theta^a_i & 0 \end{bmatrix} - {}^b R_{b_i} \tilde{\dot{\theta}}_i \tag{40}$$

Considering (25) and (31) following relation can be deduced:

$$\dot{\boldsymbol{Q}}_{\boldsymbol{i}} = \left(\mathcal{J}^v_{B_i}\right)^{-1} \begin{bmatrix} \widetilde{B_iC} & U \end{bmatrix} \boldsymbol{V_{N+1}} \tag{41}$$

Thus by inserting (26) :

$$\dot{\boldsymbol{Q}}_{\boldsymbol{i}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{\theta^a_i} \\ 0 & \frac{1}{\theta^a_i} & 0 \end{bmatrix} {}^b R^t_{b_i} \begin{bmatrix} \widetilde{B_iC} & U \end{bmatrix} \boldsymbol{V_{N+1}} \tag{42}$$

The expression of $\dot{\mathcal{J}}^v_{B_i}\, \dot{\boldsymbol{Q}}_{\boldsymbol{i}}$ is determined from (40) and (42) as:

$$\begin{aligned} \dot{\mathcal{J}}^v_{B_i}\, \dot{\boldsymbol{Q}}_{\boldsymbol{i}} = {} & \left(\tilde{\dot{\beta}}_{b_i}\, {}^b R_{b_i} + R_{\beta_{b_i}}\, \tilde{\dot{\gamma}}_{b_i}\, R_{\gamma_{b_i}}\right) {}^b R^t_{b_i}\, \boldsymbol{\mathcal{B}_i}\, \boldsymbol{V_{N+1}} \\ & + {}^b R_{b_i}\, \frac{\tilde{\dot{\theta}}_i}{\left(\theta^a_i\right)^2}\, {}^b R^t_{b_i}\, \boldsymbol{\mathcal{B}_i}\, \boldsymbol{V_{N+1}} \end{aligned} \tag{43}$$

Thus:

$$\dot{\mathcal{J}}^v_{B_i}\, \dot{\boldsymbol{Q}}_{\boldsymbol{i}} = \Psi_i\, \boldsymbol{\mathcal{B}_i}\, \boldsymbol{V_{N+1}} \tag{44}$$

Where:

$$\Psi_i = \tilde{\dot{\beta}}_{b_i} + R_{\beta_{b_i}}\, \tilde{\dot{\gamma}}_{b_i}\, R^t_{\beta_{b_i}} + {}^b R_{b_i}\, \frac{\tilde{\dot{\theta}}_i}{\left(\theta^a_i\right)^2}\, {}^b R^t_{b_i} \tag{45}$$

6.2 Computation of $\dot{\widetilde{B_iC}}$

The rotation matrix R expressing the orientation of the R_p mobile frame withrespect to the R_b absolute framee can be defined by the following matrical product:

$$R = R_\alpha\, R_\beta\, R_\gamma \tag{46}$$

We define the following vectors:

- Rotation velocity vector along x axis:

$$\dot{\boldsymbol{\alpha}} = \begin{bmatrix} \dot\alpha & 0 & 0 \end{bmatrix}^t$$

- Rotation velocity vector along y axis:

$$\dot{\boldsymbol{\beta}} = \begin{bmatrix} 0 & \dot\beta & 0 \end{bmatrix}^t$$

- Rotation velocity vector along z axis:

$$\dot{\boldsymbol{\gamma}} = \begin{bmatrix} 0 & 0 & \dot\gamma \end{bmatrix}^t$$

And their associated skew symmetrical matrices:

$$\begin{cases} \tilde{\dot\alpha} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot\alpha \\ 0 & \dot\alpha & 0 \end{bmatrix} \\ \tilde{\dot\beta} = \begin{bmatrix} 0 & 0 & \dot\beta \\ 0 & 0 & 0 \\ -\dot\beta & 0 & 0 \end{bmatrix} \\ \tilde{\dot\gamma} = \begin{bmatrix} 0 & -\dot\gamma & 0 \\ \dot\gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{cases} \tag{47}$$

$\widetilde{B_iC}$ can be also defined by:

$$\widetilde{B_iC} = \left(R\, \widetilde{B_iC_{/R_p}} \right) = R\, \widetilde{B_iC}_{/R_p}\, R^t \tag{48}$$

Or:

$$\widetilde{B_iC} = (R_\alpha\, R_\beta\, R_\gamma)\, \widetilde{B_iC}_{/R_p}\, (R_\alpha\, R_\beta\, R_\gamma)^t \tag{49}$$

Thus:

$$\dot{\widetilde{B_iC}} = \mathcal{R}_{\alpha\beta\gamma}\, \underbrace{R\, \widetilde{B_iC}_{/R_p}\, R^t}_{\widetilde{B_iC}} + \underbrace{R\, \widetilde{B_iC}_{/R_p}\, R^t}_{\widetilde{B_iC}}\, \mathcal{R}^t_{\alpha\beta\gamma} \tag{50}$$

With:

$$\mathcal{R}_{\alpha\beta\gamma} = \tilde{\dot\alpha} + R_\alpha\, \tilde{\dot\beta}\, R^t_\alpha + R_\alpha\, R_\beta\, \tilde{\dot\gamma}\, R^t_\beta\, R^t_\alpha \tag{51}$$

We finally obtain:

$$\dot{\widetilde{B_iC}} = \mathcal{R}_{\alpha\beta\gamma}\, \widetilde{B_iC} + \widetilde{B_iC}\, \mathcal{R}^t_{\alpha\beta\gamma} \tag{52}$$

The second-order inverse kinematic model can be expressed by inserting (44) and (52) in (33):

$$\ddot{\boldsymbol{Q}}_{\boldsymbol{i}} = \left(\mathcal{J}^v_{B_i}\right)^{-1} \left(\boldsymbol{\mathcal{B}_i}\, \dot{\boldsymbol{V}}_{\boldsymbol{N+1}} + \boldsymbol{\mathcal{A}_i}\, \boldsymbol{V_{N+1}} \right) \tag{53}$$

Where:

$$\boldsymbol{\mathcal{A}_i} = \begin{bmatrix} (\mathcal{R}_{\alpha\beta\gamma} - \Psi_i)\, \widetilde{B_iC} + \widetilde{B_iC}\, \mathcal{R}^t_{\alpha\beta\gamma} & -\Psi_i \end{bmatrix} \tag{54}$$

7 Inverse dynamic model

The equation describing the dynamic behaviour of a closed loop system, in the articular space, is given by the following equation:

$$\mathcal{M}_i\, \ddot{\boldsymbol{Q}}_{\boldsymbol{i}} + \boldsymbol{C_i} + \boldsymbol{G_i} + \left(\mathcal{J}^v_{B_i}\right)^t\, \boldsymbol{\phi_i} = \boldsymbol{\Gamma_i} \tag{55}$$

Or:

$$\mathcal{M}_i\, \ddot{\boldsymbol{Q}}_{\boldsymbol{i}} = {}^{\boldsymbol{i}}\boldsymbol{\mathcal{F}_T} \tag{56}$$

Where

$${}^{\boldsymbol{i}}\boldsymbol{\mathcal{F}_T} = Col\left\{ {}^{\boldsymbol{i}}\boldsymbol{F_{T_j}} \right\} = \boldsymbol{\Gamma_i} - \left(\boldsymbol{C_i} + \boldsymbol{G_i} + \left(\mathcal{J}^v_{B_i}\right)^t\, \boldsymbol{\phi_i} \right)$$

${}^{\boldsymbol{i}}\boldsymbol{F_{T_j}}$ represents the acceleration-dependent component of the control force at the level of joint j for the segment i

7.1 Computation of the matrix $\mathcal{M}_i$

The propagation of accelerations and forces among the links of serial chain are given by:

$${}^{\boldsymbol{i}}\dot{\boldsymbol{V}}_{\boldsymbol{j}} = {}^{i}\hat{P}^t_{j-1}\, {}^{\boldsymbol{i}}\dot{\boldsymbol{V}}_{\boldsymbol{j-1}} + {}^{i}H_j\, {}^{\boldsymbol{i}}\ddot{\boldsymbol{\theta}}_{\boldsymbol{j}} \tag{57}$$

$${}^{\boldsymbol{i}}\boldsymbol{F_j} = {}^{i}I_j\, {}^{\boldsymbol{i}}\dot{\boldsymbol{V}}_{\boldsymbol{j}} + {}^{i}\hat{P}_j\, {}^{\boldsymbol{i}}\boldsymbol{F_{j+1}} \tag{58}$$

Equations (57)-(58) represent the simplified N-E algorithm (with nonlinear terms being excluded) for the serial chain.
Using the global notation (57) and (58) can be written by the following equation system:

$$\mathcal{P}^t_i\, \dot{\boldsymbol{\mathcal{V}}}_{\boldsymbol{i}} = \mathcal{H}_i\, \ddot{\boldsymbol{Q}}_{\boldsymbol{i}} \tag{59}$$

$$\mathcal{P}_i\, \boldsymbol{\mathcal{F}_i} = \mathcal{I}_i\, \dot{\boldsymbol{\mathcal{V}}}_{\boldsymbol{i}} \tag{60}$$

The determination of the matrix $\mathcal{M}_i$ expression is based on a rather unconventional decomposition of inter body force of the form:

$${}^{\boldsymbol{i}}\boldsymbol{F_j} = {}^{i}H_j\, {}^{\boldsymbol{i}}\boldsymbol{F_{T_j}} + {}^{i}W_j\, {}^{\boldsymbol{i}}\boldsymbol{F_{S_j}} \tag{61}$$

Where ${}^{\boldsymbol{i}}\boldsymbol{F_{S_j}}$ represents the constraint force.
Contrary to degrees of freedom (dof) we introduce

degrees of constraint (doc) notion ($doc = 6 - dof$). For a joint with n_i dof, ${}^iW_j \in \Re^{6\times(6-n_i)}$
In the Gough-Stewart platform case, these prjection matrices in the constraint space are defined by:

$$ \begin{array}{c} \quad 1^{st} \quad 2^{nd} \quad 3^{rd} \quad 4^{th}\ doc \\ {}^iW_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \quad (62) $$

$$ \begin{array}{c} \quad 1^{st} \quad 2^{nd} \quad 3^{rd} \quad 4^{th} \quad 5^{th}\ doc \\ {}^iW_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \quad (63) $$

The projection matrices ${}^iH_j$ and ${}^iW_j$ are taken to satisfy the following orthogonality conditions:

$$ {}^iW_j^t\, {}^iH_j = 0 \quad (64) $$

$$ {}^iH_j\, {}^iH_j^t + {}^iW_j\, {}^iW_j^t = U \quad (65) $$

$$ {}^iH_j^t\, {}^iH_j = {}^iW_j^t\, {}^iW_j = U \quad (66) $$

Or in a global form:

$$ \mathcal{W}_i^t\, \mathcal{H}_i = 0 \quad (67) $$

$$ \mathcal{H}_i\, \mathcal{H}_i^t + \mathcal{W}_i\, \mathcal{W}_i^t = U \quad (68) $$

$$ \mathcal{H}_i^t\, \mathcal{H}_i = \mathcal{W}_i^t\, \mathcal{W}_i = U \quad (69) $$

With $\mathcal{W}_i = Diag\left\{{}^iW_j\right\} = \begin{bmatrix} {}^iW_2 & 0 \\ 0 & {}^iW_1 \end{bmatrix} \in \Re^{9\times 12}$

By multiplying (61) by ${}^iH_j^t$, and considering the relations given by (64) and (66), we deduce the following relationship:

$$ {}^i\boldsymbol{F_{T_j}} = {}^iH_j^t\, {}^i\boldsymbol{F_j} \quad (70) $$

Or in a global form:

$$ {}^i\boldsymbol{\mathcal{F}_T} = \mathcal{H}_i^t\, \boldsymbol{\mathcal{F}} \quad (71) $$

Considering (71), (59) and (60), we also obtain:

$$ {}^i\boldsymbol{\mathcal{F}_T} = \mathcal{H}_i^t\, \mathcal{P}_i^{-1}\, \mathcal{I}_i\, \dot{\mathcal{V}}_i = \mathcal{H}_i^t\, \mathcal{P}_i^{-1}\, \mathcal{I}_i\, \mathcal{P}_i^{-t}\, \mathcal{H}_i\, \boldsymbol{\ddot{Q}_i} \quad (72) $$

A factorized expression of the matrix $\mathcal{M}_i$ can be also deduced from (56) and (72) as:

$$ \mathcal{M}_i = \mathcal{H}_i^t\, \mathcal{P}_i^{-1}\, \mathcal{I}_i\, \mathcal{P}_i^{-t}\, \mathcal{H}_i \quad (73) $$

7.2 Computation of vectors $\boldsymbol{C_i}$ et $\boldsymbol{G_i}$

$\boldsymbol{C_i} + \boldsymbol{G_i}$ is computed by using the Newton-Euler recursive algorithm considering the angular acceleration to be zero. This algorithm is given by each segment i:

1. We determine the linear and angular velocities and accelerations of segment i links, starting link $j = 1$ to link $j = N$

   - Initialization : ${}^i\boldsymbol{\omega_0} = {}^i\boldsymbol{\dot{\omega}_0} = \boldsymbol{0}$ and ${}^i\boldsymbol{\dot{v}_0} = -\boldsymbol{g}$ ($\boldsymbol{g}$ represents the gravity vector)
   - Angular velocities :

   $$ {}^i\boldsymbol{\omega_j} = {}^jR_{j-1}\left({}^i\boldsymbol{\omega_{j-1}} + {}^i\sigma_j\, {}^ih_j^r\, {}^i\dot{\theta}_j\right) \quad (74) $$

   Where:
     - ${}^i\sigma_j = 1$ if the joint j of the segment i is a rotation, else ${}^i\sigma_j = 0$.
     - ${}^ih_j^r$ represents rotation part of the projection matrix ${}^iH_j$: ${}^iH_j = \begin{bmatrix} {}^ih_j^r \\ {}^ih_j^p \end{bmatrix}$
     - ${}^jR_{j-1}$ is the matrix expressing the orientation of the $j-1$ frame withrespect to the j frame.
   - Angular accelerations:

   $$ {}^i\boldsymbol{\dot{\omega}_j} = {}^jR_{j-1}\left({}^i\boldsymbol{\dot{\omega}_{j-1}} + {}^i\sigma_j\, {}^i\tilde{\omega}_{j-1}\, {}^ih_j^r\, {}^i\dot{\theta}_j\right) \quad (75) $$

   - Linear accelerations:

   $$ {}^i\boldsymbol{\dot{v}_j} = {}^jR_{j-1}\left[{}^i\boldsymbol{\dot{v}_{j-1}} + (1 - {}^i\sigma_j)\, 2\, {}^i\tilde{\omega}_{j-1}\, {}^ih_j^p\, {}^i\dot{\theta}_j\right] + \left({}^i\dot{\tilde{\omega}}_j + {}^i\tilde{\omega}_j\, {}^i\tilde{\omega}_j\right)\, {}^i\boldsymbol{P_j} \quad (76) $$

2. We determine torques and forces of interaction with the following recursive scheme, starting of link $j = N$ to link $j = 1$

   - Initialization : ${}^i\boldsymbol{n_{N+1}} = {}^i\boldsymbol{f_{N+1}} = \boldsymbol{0}$
   - Computation of the link j center of mass linear acceleration belonging to the segment i: ${}^i\boldsymbol{\dot{v}_{C_j}}$:

   $$ {}^i\boldsymbol{\dot{v}_{C_j}} = {}^i\boldsymbol{\dot{v}_j} + \left({}^i\dot{\tilde{\omega}}_j + {}^i\tilde{\omega}_j\, {}^i\tilde{\omega}_j\right)\, {}^i\boldsymbol{S_j} \quad (77) $$

   - Computation of applied force at center of mass of the link j belonging to the segment i: ${}^i\boldsymbol{f_{C_j}}$

   $$ {}^i\boldsymbol{f_{C_j}} = {}^im_j\, {}^i\boldsymbol{\dot{v}_{C_j}} \quad (78) $$

- Computation of applied torque at center of mass of the link j belonging to the segment i: ${}^{i}\boldsymbol{n_{C_j}}$

$$ {}^{i}\boldsymbol{n_{C_j}} = {}^{i}I_{C_j}\,{}^{i}\boldsymbol{\dot{\omega}_j} + {}^{i}\tilde{\omega}_j\,{}^{i}I_{C_j}\,{}^{i}\boldsymbol{\omega_j} \quad (79) $$

- Computation of the vector giving the exercised force by link $j-1$ to link j belonging to the segment i: ${}^{i}\boldsymbol{f_j}$

$$ {}^{i}\boldsymbol{f_j} = {}^{j}R_{j+1}\,{}^{i}\boldsymbol{f_{j+1}} + {}^{i}\boldsymbol{f_{C_j}} \quad (80) $$

- Computation of the vector giving the exercised torque by link $j-1$ to link j belonging to the segment i: ${}^{i}\boldsymbol{n_j}$

$$ {}^{i}\boldsymbol{n_j} = {}^{j}R_{j+1}{}^{i}\boldsymbol{n_{j+1}} + {}^{i}\tilde{P}_j{}^{i}\boldsymbol{f_j} + {}^{i}\boldsymbol{n_{C_j}} + {}^{i}\tilde{S}_j{}^{i}\boldsymbol{f_{C_j}} \quad (81) $$

- Computation of ${}^{i}\boldsymbol{C_j} + {}^{i}\boldsymbol{G_j}$

$$ {}^{i}\boldsymbol{C_j} + {}^{i}\boldsymbol{G_j} = {}^{i}\sigma_j\,{}^{i}\boldsymbol{n}_{\boldsymbol{j}}^{t}\,{}^{j}R_{j+1}\,{}^{i}h_j^r + (1 - {}^{i}\sigma_j)\,{}^{i}\boldsymbol{f}_{\boldsymbol{j}}^{t}\,{}^{j}R_{j+1}\,{}^{i}h_j^p \quad (82) $$

7.3 Computation of the contact forces $\boldsymbol{\phi_i}$

The contact forces are computed from (55):

$$ \boldsymbol{\phi_i} = -\left(\mathcal{J}_{B_i}^{v}\right)^{-t}\left(\mathcal{M}_i\,\boldsymbol{\ddot{Q}_i} + \boldsymbol{C_i} + \boldsymbol{G_i}\right) + \left(\mathcal{J}_{B_i}^{v}\right)^{-t}\boldsymbol{\Gamma_i} \quad (83) $$

Considering (27), the term $\left(\mathcal{J}_{B_i}^{v}\right)^{-t}\boldsymbol{\Gamma_i}$ of the previous equation can be expressed as:

$$ \left(\mathcal{J}_{B_i}^{v}\right)^{-t}\boldsymbol{\Gamma_i} = \boldsymbol{\mathcal{C}_i}\begin{bmatrix} {}^{i}\Gamma_2 \\ 0 \\ 0 \end{bmatrix} \quad (84) $$

Where:

$$ \boldsymbol{\mathcal{C}_i} = \begin{bmatrix} \cos\beta_{b_i}\cos\gamma_{b_i} & -\frac{\sin\beta_{b_i}}{\theta_i^a} & -\frac{\cos\beta_{b_i}\sin\gamma_{b_i}}{\theta_i^a} \\ \sin\gamma_{b_i} & 0 & \frac{\cos\gamma_{b_i}}{\theta_i^a} \\ -\cos\gamma_{b_i}\sin\beta_{b_i} & -\frac{\cos\beta_{b_i}}{\theta_i^a} & \frac{\sin\gamma_{b_i}\sin\beta_{b_i}}{\theta_i^a} \end{bmatrix} \quad (85) $$

Only the joint 2 is active (linear joint). Thus:

$$ \left(\mathcal{J}_{B_i}^{v}\right)^{-t}\boldsymbol{\Gamma_i} = \boldsymbol{n_i}\,{}^{i}\Gamma_2 \quad (86) $$

Equation (83) can be rewritten as:

$$ \boldsymbol{\phi_i} = -\left(\mathcal{J}_{B_i}^{v}\right)^{-t}\underbrace{\left(\mathcal{M}_i\,\boldsymbol{\ddot{Q}_i} + \boldsymbol{C_i} + \boldsymbol{G_i}\right)}_{\boldsymbol{\mathcal{D}_i}} + \boldsymbol{n_i}\,{}^{i}\Gamma_2 \quad (87) $$

7.4 Dynamic behaviour of the mobile platform

The dynamic behavior of the mobile platform is given by the following relation:

$$ \boldsymbol{F_{N+1}} = \Lambda_C\,\boldsymbol{\dot{V}_{N+1}} - (\boldsymbol{G_C} + \boldsymbol{C_C}) \quad (88) $$

Where:

- $\boldsymbol{F_{N+1}}$ is the spatial force applied to the point C, representing the contribution of the contact forces $\boldsymbol{\phi_i}$ propagated in the point C:

$$ \boldsymbol{F_{N+1}} = \begin{bmatrix} \boldsymbol{n_{N+1}} \\ \boldsymbol{f_{N+1}} \end{bmatrix} = \sum_{i=1}^{6}\begin{bmatrix} \widetilde{B_iC}^t \\ U \end{bmatrix}\boldsymbol{\phi_i} \quad (89) $$

- $\Lambda_C \in \Re^{6\times 6}$ is the spatial inertia matrix of the mobile platform:

$$ \Lambda_C = \begin{bmatrix} I_C & m_C\,\widetilde{GC} \\ -m_C\,\widetilde{GC} & m_C\,U \end{bmatrix} \quad (90) $$

  - m_C is the platform mass
  - $I_C \in \Re^{3\times 3}$ is the inertia tensor of the mobile platform expressed in the mobile platform center of mass, and projected ine the fixed frame R_b:

$$ I_C = R\,I_{C_{/R_m}}\,R^t \quad (91) $$

  - $\boldsymbol{C_C} \in \Re^6$ is the vector of Coriolis and centrigugal forces:

$$ \boldsymbol{C_C} = \begin{bmatrix} -\tilde{\omega}_{N+1}\,I_C\,\boldsymbol{\omega_{N+1}} \\ m_C\,\tilde{\omega}_{N+1}\,\widetilde{GC}\,\boldsymbol{\omega_{N+1}} \end{bmatrix} \quad (92) $$

  - $\boldsymbol{G_C} \in \Re^6$ is the vector of gravitational forces:

$$ \boldsymbol{G_C} = \begin{bmatrix} m_C\,\widetilde{GC} \\ m_C\,U \end{bmatrix}\boldsymbol{g} \quad (93) $$

  $\boldsymbol{g}$ being the acceleration vector of gravity

7.5 Computation of active articular force vector Γ

Substituting (87) in (89), we obtain:

$$ \boldsymbol{F_{N+1}} = \sum_{i=1}^{6}\left(\begin{bmatrix} \widetilde{B_iC}^t \\ U \end{bmatrix}\left(-\left(\mathcal{J}_{B_i}^{v}\right)^{-t}\boldsymbol{\mathcal{D}_i} + \boldsymbol{n_i}\,{}^{i}\Gamma_2\right)\right) \quad (94) $$

With:

$$\sum_{i=1}^{6}\left(\left[\begin{array}{c}\widetilde{B_iC}^t\\ U\end{array}\right]\boldsymbol{n_i}\,{}^i\Gamma_2\right)=\left[\begin{array}{c}\sum_{i=1}^{6}\widetilde{B_iC}^t\,\boldsymbol{n}_i\,{}^i\Gamma_2\\ \sum_{i=1}^{6}\boldsymbol{n_i}\,{}^i\Gamma_2\end{array}\right] \quad (95)$$

Or by highlighting the vector

$$\boldsymbol{\Gamma}=\left[\begin{array}{cccccc}{}^6\Gamma_2 & {}^5\Gamma_2 & {}^4\Gamma_2 & {}^3\Gamma_2 & {}^2\Gamma_2 & {}^1\Gamma_2\end{array}\right]^t$$

and considering the inverse Jacobian matrix expression of the parallel robot:

$$\sum_{i=1}^{6}\left(\left[\begin{array}{c}\widetilde{B_iC}^t\\ U\end{array}\right]\boldsymbol{n_i}\,{}^i\Gamma_2\right)=\mathcal{J}^{-t}\,\boldsymbol{\Gamma} \quad (96)$$

with:

$$\mathcal{J}^{-t}=\left[\begin{array}{ccc}\widetilde{B_6C}^t\,\boldsymbol{n}_6 & \cdots & \widetilde{B_1C}^t\,\boldsymbol{n}_1\\ \boldsymbol{n_6} & \cdots & \boldsymbol{n_1}\end{array}\right]$$

The inverse dynamic model of the parallel robot is determined by inserting (96) into (94):

$$\boldsymbol{\Gamma}=\mathcal{J}^t\left[\boldsymbol{F_{N+1}}+\sum_{i=1}^{6}\left(\left[\begin{array}{c}\widetilde{B_iC}^t\\ U\end{array}\right]\left\{-\left(\mathcal{J}^v_{B_i}\right)^{-t}\boldsymbol{\mathcal{D}_i}\right\}\right)\right] \quad (97)$$

8 Simulation of the inverse dynamic model

To validate our inverse dynamic model of the Gough-Stewart platform, a simulation under Matlab environment is presented.
The trajectory profile used for this study is the following one:

- Fig. 3 shows the terminal tool Cartesian trajectory for a constant orientation ($\alpha = 15°, \beta = 10°$ and $\gamma = 5°$)
- Fig. 4 and 5 show respectively the end effector cartesian velocity profile and the end effector cartesian acceleration profile
- The used dynamic parameter for this simulation are summarized in Table 1 (these parameters are identical for all segments).
- The active joint positions are computed using inverse kinematic model given by the equation (3). Fig. 6 shows the active joint position variation along the trajectory.

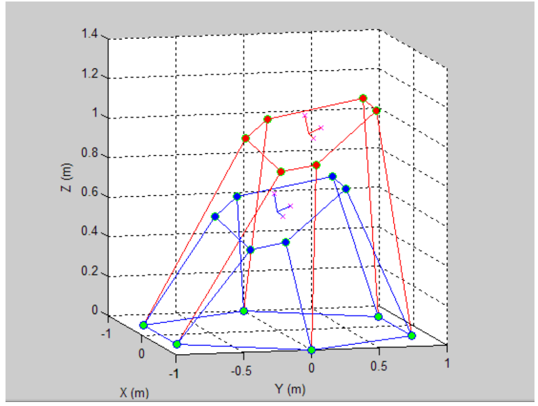

Figure 3: Cartesian trajectory of the end effector.

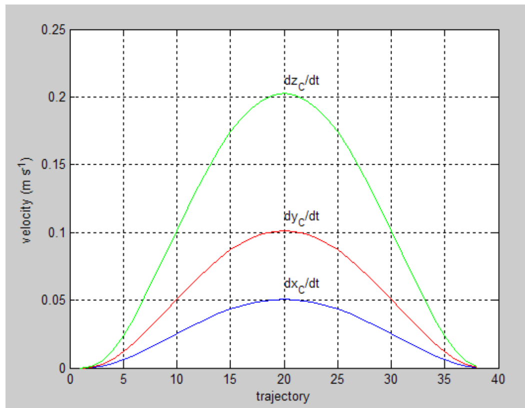

Figure 4: Cartesian velocity profile

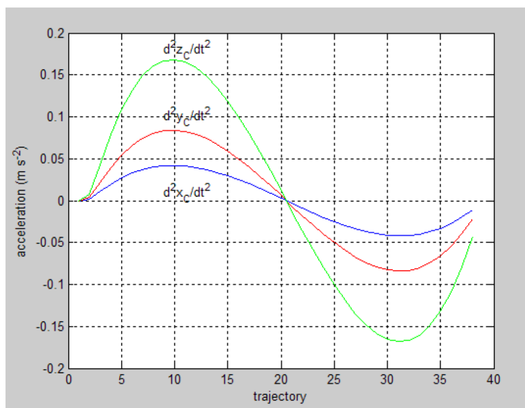

Figure 5: Cartesian acceleration profile.

- The active joint velocities are computed using (9). Fig. 7 shows the active joint velocity evolution along the trajectory.
- The active joint accelerations are computed using (53). Fig. 8 shows the active joint acceleration evolution along the trajectory.
- The active joint forces are computed using in-

Table 1: Gough-Stewart platform dynamic parameters.

Link	Mass	Inertia
1	$0.5\ kg$	$I_{C_1} = \begin{bmatrix} 0.0002 & 0 & 0 \\ 0 & 0.0038 & 0 \\ 0 & 0 & 0.0038 \end{bmatrix}$
2	$1\ kg$	$I_{C_2} = 10^{-4} \begin{bmatrix} 13 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$ with $a = 6.25 + 830\,(\theta_i^a - 0.83)^2$
Mobile platform	$3\ kg$	$I_C = \begin{bmatrix} 0.375 & 0 & 0 \\ 0 & 0.1875 & 0 \\ 0 & 0 & 0.1875 \end{bmatrix}$

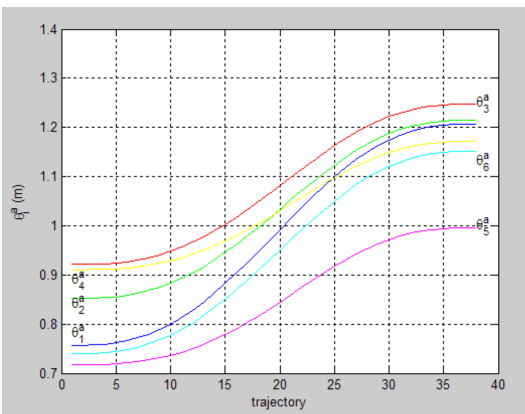

Figure 6: Active joint positions along the trajectory

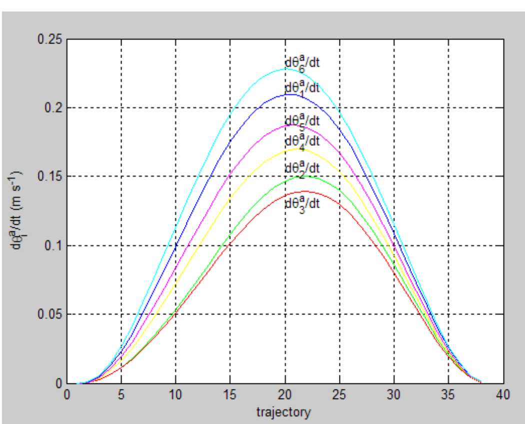

Figure 7: Active joint velocities along the trajectory

verse dynamic model given by (97). Fig. 9 shows the active joint force evolution along the trajectory.

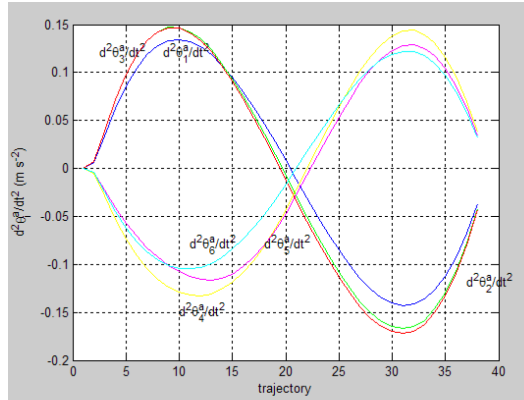

Figure 8: Active joint accelerations along the trajectory

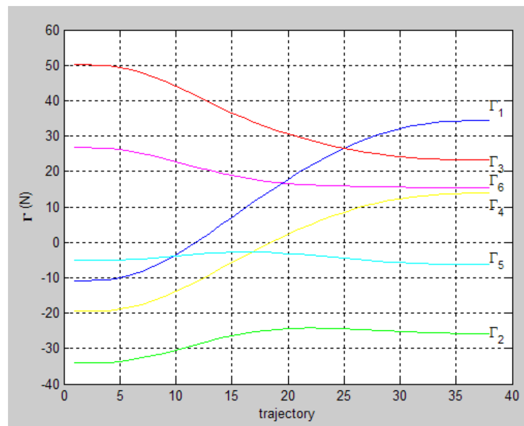

Figure 9: Active joint forces Γ_i along the trajectory.

9 Conclusion

In this paper an inverse dynamic model of the Gough Stewart platform has been presented. Parallel robot is considered as a multi robot system moving a common load. The proposed approach, based on a global formalism, highlights the inertia matrices $\mathcal{M}_i$ of each robot segment, expressed in the articular space, in a factorized form. These factorizations allow to provide a basis for the development of a parallel algorithm, to minimize the computation time, for a dynamic control under the real time constraint.

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