# Determination of $3-RPR$ Planar Parallel Robot Assembly Modes by Jacobian Matrix Factorization

GEORGES FRIED, KARIM DJOUANI, DIANE BOROJENI, SOHAIL IQBAL
University of PARIS XII
LISSI-SCTIC Laboratory
120-122 rue Paul Aramangot, 94400 Vitry sur Seine
FRANCE
fried@univ-paris12.fr, djouani@univ-paris12.fr

*Abstract:* This paper presents a new approach for the determination of $3-RPR$ planar parallel robot assembly modes. In this approach, the parallel robot is considered as a multi robot system. The segments are then regarded as serial robots moving a common load. The proposed approach is based on a parallel robot Jacobian matrix factorization. This factorization is an extension of global formalism developed by Fijany. This approach allows to determine several parallel robot Jacobian matrices, which are used for the assembly mode determination of a $3-RPR$ planar parallel robot. Effectiveness of the proposed method is demonstrated by the simulation.

*Key–Words:* Parallel Robot, Kinematic Model, Jacobian Matrix, Assembly Modes.

## 1 Introduction

During past years parallel robot concept [1], [2], [3], [4], [5], proved to be an efficient solution to the accuracy problem in the end effector positioning, met on the serial manipulators. Parallel robots are primarily used in the applications where high accuracy, rigidity, and heavy load carrying capabilities are of fundamental importance. The teleoperation [6], machining tools [7], [8], and various other medical applications [9], [10] constitute some of the many possible applications of parallel robots. Kinematic modeling is an important problem of the robotics and particulary for the manipulator study. In the parallel robot case, the inverse kinematic model is usually straightforward for any parallel manipulator [2], [11]. On the other hand, the forward kinematics computation for a parallel robot is a complex problem. This forward kinematic model consists in finding the possible pose of the mobile paltform parallel robot for given active joint coordinates. Merlet proposed an algorithm based on interval analysis, which allows to solve the forward kinematic problem [12]. For control or simulation under the real time constraint, many authors have proposed the use of the Newton-Raphson algorithm [2], based on the Jacobian matrix computation. In parallel robot research area, the problem of Jacobian matrix determination is an open and interesting problem. Indeed, the computation of inverse Jacobian matrix is currently known and mastered [2], [13], [14], but its analytical expression still remains relatively complex. Thus analytical formulation of the Jacobian matrix, by symbolic inversion or even by using some formal computing tools, is difficult [2]. Its expression is generally obtained by a numerical method using any classical algorithm of matrix inversion or by a method based on an iterative scheme.

In the present paper, a Jacobian matrix factorization of a $3-RPR$ planar robot [13], [14], [15], [16] is presented. This approach is a generalization of the approach proposed by Fijany et al. [17][18][19] for serial robot. Thus we consider the parallel robot as a multi robot system with $k$ serial robots (the segments) moving a common load (the mobile platform) [20]. The basic idea is to compute the Jacobian matrix associated with each parallel robot segment considered as a serial robot and then to compute the Jacobian matrix of the parallel robot by considering the kinematic chain closing constraint. The proposed approach allows the computation of several Jacobian matrices due to the multiple solutions of the forward kinematic model. These matrices are used to determine the assembly modes of the $RPR$ planar parallel robot.

This paper is organized as follows. In the following section we describe the nomenclature and the used notation. In section $3$, the $RPR$ planar parallel robot architecure is described. The computation of the inverse kinematic model and the inverse Jacobian matrix are given in sections $4$ and $5$. The Jacobian matrix factorization is presented in section $6$. The determination of assembly modes is finally proposed in section $7$.

## 2 Preliminaries

We consider a parallel robot as a multi-robot system with $k$ serial robots (segments) moving a common load (mobile platform). Figure (1) shows the links, the frames and position vectors for the segment $i$ ($i = 1, \ldots, k$).

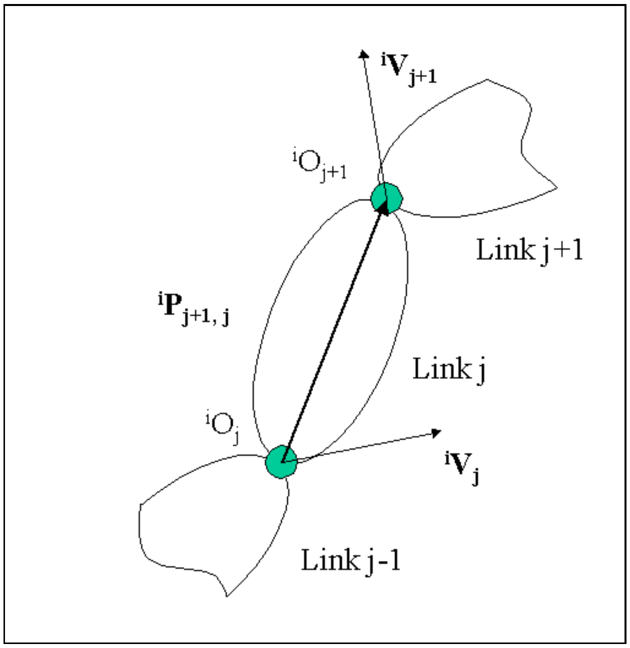


Figure 1: Links, frames and position vectors for the segment $i$

### 2.1 Nomenclature

#### 2.1.1 Joint and link parameters

- $^i\boldsymbol{P_{j+1,j}}$: position vector from $^iO_j$ to $^iO_{j+1}$
- $k$: number of segments
- $^iM$: DOF-number of segment $i$
- $^iN$: joint number of segment $i$
- $\theta_i^a$, $\dot{\theta}_i^a$: position and velocity of active joint of the segment $i$
- $^i\theta_j^p$, $^i\dot{\theta}_j^p$: position and velocity of passive joint $j$ of the segment $i$
- $^i\omega_j$, $^iv_j \in \Re^3$: angular and linear velocity of link $j$ for the segment $i$

#### 2.1.2 Spatial quantities

- $^iH_j$: spatial-axis (map matrix) of joint $j$ for the segment $i$. For instance, for a joint with 2-DOF (rotation about $z$-axis and translation about $x$-axis), the matrix $^iH_j \in \Re^{6\times 2}$ is given by:

$$
\begin{array}{cccl}
 & 1^{st} & 2^{nd} & -DOF \\
 & & & \\
\end{array}
$$

$$
^iH_j = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}
\begin{array}{l} \text{x-axis rotation} \\ \text{y-axis rotation} \\ \text{z-axis rotation} \\ \text{x-axis translation} \\ \text{y-axis translation} \\ \text{z-axis translation} \end{array}
$$

- $^i\boldsymbol{V_j} = \begin{bmatrix} ^i\boldsymbol{\omega_j} \\ ^i\boldsymbol{v_j} \end{bmatrix} \in \Re^6$: spatial velocity of the link $j$ for the segment $i$
- $\boldsymbol{V_{N+1}} = \begin{bmatrix} \boldsymbol{\omega_{N+1}} \\ \boldsymbol{v_{N+1}} \end{bmatrix} \in \Re^6$: spatial velocity of the end effector

#### 2.1.3 Global quantities

The following global quantities are defined for $j = {}^iN$ to 1 or $j = {}^iM$ to 1 and $i = k$ to 1

- $\boldsymbol{\dot{Q}_i} = Col\left( {}^i\dot{\theta}_j \right) \in \Re^{^iM}$: global vector of articular coordinate velocity of the segment $i$, taking into account passive and active joints
- $\boldsymbol{\dot{Q}} = Col\left( \dot{\theta}_i^a \right) \in \Re^k$: vector of generalized coordinate velocity of the system
- $\boldsymbol{\mathcal{V}_i} = Col\left( {}^iV_j \right) \in \Re^{6\,^iN}$: global vector of spatial velocities for the segment $i$
- $\mathcal{H}_i = Diag\left( {}^iH_j \right) \in \Re^{6\,^iN \times {}^iM}$: global matrix of spatial-axis for the leg $i$

### 2.2 General notation

With any vector $\boldsymbol{V} = \begin{bmatrix} V_x & V_y & V_z \end{bmatrix}^t$, a tensor $\tilde{V}$ can be associated whose representation in any frame is a skew symmetrical matrix:

$$
\tilde{V} = \begin{bmatrix} 0 & -V_z & V_y \\ V_z & 0 & -V_x \\ -V_y & V_x & 0 \end{bmatrix}
$$

The tensor $\tilde{V}$ has the properties that $\tilde{V} = -\tilde{V}^t$ and $\tilde{V}_1\boldsymbol{V_2} = \boldsymbol{V_1} \wedge \boldsymbol{V}_2$ i.e., it is the vector cross-product. A matrix $\hat{V}$ associated to the vector $\boldsymbol{V}$ is defined as:

$$\hat{V} = \begin{bmatrix} U & V \\ 0 & U \end{bmatrix}$$

and

$$\hat{V}^t = \begin{bmatrix} U & 0 \\ -\tilde{V} & U \end{bmatrix}$$

where $U$ and $0$ stand for unit and zero matrices of appropriate size.
In our derivation, we also make use of global matrices and vectors which lead to a compact representation of various factorizations. A bidiagonal block matrix $\mathcal{P}_i \in \Re^{6\,^iN \times 6\,^iN}$ is defined as:

$$\mathcal{P}_i = \begin{bmatrix} U & & & & \\ -^i\hat{P}_{N-1} & U & 0 & & \\ 0 & -^i\hat{P}_{N-2} & U & & \\ 0 & 0 & & & \\ \vdots & \vdots & & & \\ 0 & 0 & 0 & -^i\hat{P}_1 & U \end{bmatrix}$$

Note that according to our notation, $\boldsymbol{^iP_{j+1,j}} = \boldsymbol{^iP_j}$.
The inverse of $\mathcal{P}_i$ is a lower triangular block matrix given by:

$$\mathcal{P}_i^{-1} = \begin{bmatrix} U & & & \\ ^i\hat{P}_{N,N-1} & U & 0 & \\ ^i\hat{P}_{N,N-2} & ^i\hat{P}_{N-1,N-2} & U & \\ \vdots & \vdots & & \\ ^i\hat{P}_{N,1} & ^i\hat{P}_{N-1,1} & \dots\ ^i\hat{P}_{2,1} & U \end{bmatrix}$$

# 3 Planar parallel robot description

The robot considered in this study is symmetric and composed of three identical legs connecting the fixed base to the end effector triangle as shown in Fig. 2 $3-RPR$ planar robot [13], [14], [15], [16]. Each leg is of $RPR$ design, with two passive swivel joints and an active primatic joint. These three linear links are used in order to move the mobile triangle defined by the triplet $B_1$, $B_2$, $B_3$.

The used notation to describe the planar parallel robot is defined as following.

- $R_b$ is the absolute frame, tied to the fixed base. $R_b = (0, x, y)$.
- $R_p$ is the mobile frame, tied to the mobile part. $R_p = (C, x_p, y_p)$.



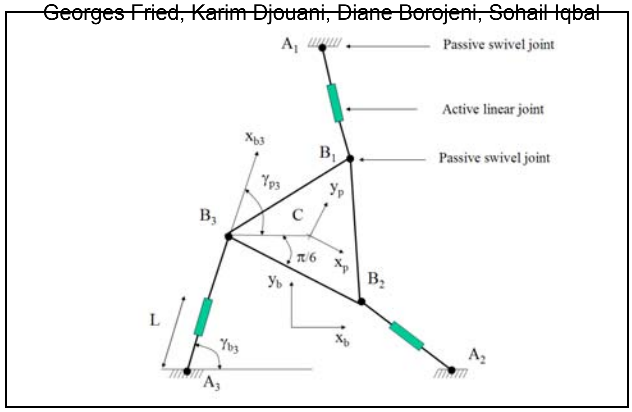


Figure 2: Planar parallel manipulator

- Let $O$ be the origin of the absolute coordinate system
- Let $C$ (or $O_{N+1}$) be the origin of the mobile coordinate system, whose coordinates are in the absolute frame:

$$\boldsymbol{OC}_{/R_b} = \begin{bmatrix} x_c & y_c \end{bmatrix}^t$$

- $A_i$ (or $^iO_1$)is the center of the joint between the segment $i$ and the fixed base:

$$\boldsymbol{OA_i}_{/R_b} = \begin{bmatrix} a_i^x & a_i^y \end{bmatrix}^t$$

- $B_i$ (or $^iO_N$) is the center of the joint between the segment $i$ and the mobile part:

$$\boldsymbol{CB_i}_{/R_p} = \begin{bmatrix} b_i^x & b_i^y \end{bmatrix}^t$$

- $[R]$ is the rotation matrix of $r_{ij}$ elements (in the $RPY$ formalism), expressing the orientation of the $R_p$ coordinate system withrespect to the $R_b$ coordinate system. The expression for this matrix is given by:

$$[R] = \begin{bmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{bmatrix} \quad (1)$$

- $\boldsymbol{X}$ is the task coordinate vector.

$$\boldsymbol{X} = \begin{bmatrix} \gamma & x_c & y_c \end{bmatrix}^t$$

- $R_{b_i}$ is the frame tied to the segment $i$. $R_{b_i} = (A_i, x_{b_i}, y_{b_i})$.
- $\gamma_{b_i}$ is the angle, in the $RPY$ formalism, describing frame $R_{b_i}$ rotation with respect to the absolute frame $R_b$.

- $\left[{}^{b}R_{b_i}\right]$ is the rotation matrix of ${}^{b}r_{b_{i_{jk}}}$ elements (in the $RPY$ formalism), expressing the orientation of the $R_{b_i}$ coordinate system withrespect to the $R_b$ coordinate system. The expression for this matrix is given by:

$$\left[{}^{b}R_{b_i}\right] = \begin{bmatrix} \cos\gamma_{b_i} & -\sin\gamma_{b_i} \\ \sin\gamma_{b_i} & \cos\gamma_{b_i} \end{bmatrix} \quad (2)$$

- $\gamma_{p_i}$ is the angle, in the $RPY$ formalism, describing frame $R_{p_i}$ rotation with respect to the frame $R_{b_i}$.

- $\left[{}^{b_i}R_{p_i}\right]$ is the rotation matrix of ${}^{b_i}r_{p_{i_{jk}}}$ elements (in the $RPY$ formalism), expressing the orientation of the $R_{p_i}$ coordinate system with respect to the $R_{b_i}$ coordinate system. The expression for this matrix is given by:

$$\left[{}^{b_i}R_{p_i}\right] = \begin{bmatrix} \cos\gamma_{p_i} & -\sin\gamma_{p_i} \\ \sin\gamma_{p_i} & \cos\gamma_{p_i} \end{bmatrix} \quad (3)$$

- ${}^{\boldsymbol{i}}\boldsymbol{P_N} = \begin{bmatrix} x_i & y_i & z_i \end{bmatrix}^t$ is the propagation vector from $B_i$ to $C$ in $R_{b_i}$ the frame tied to the segment $i$:

$$ {}^{\boldsymbol{i}}\boldsymbol{P_N} = \boldsymbol{B_iC}_{/R_{b_i}} = \left[{}^{b}R_{b_i}\right]^t \ [R] \ \boldsymbol{B_iC}_{/R_p} \quad (4)$$

## 4 Inverse kinematics

The inverse geometric model relates the active joint variables ($\boldsymbol{Q}$) to the operational variables which define the position and the orientation of the end effector ($\boldsymbol{X}$). This relation is given by the following equation [2], [13]:

$$\theta_i^a = \|\boldsymbol{A_iB_i}\| = \|\boldsymbol{A_iO}_{/R_b} + \boldsymbol{OC}_{/R_b} + [R]\,\boldsymbol{CB_{i/R_p}}\| \quad (5)$$

Thus:

$$\theta_i^a = \sqrt{X_i^2 + Y_i^2} \quad (6)$$

where:

$$\begin{aligned} X_i &= x_c - a_i^x + r_{11}b_i^x + r_{12}b_i^y \\ Y_i &= y_c - a_i^y + r_{21}b_i^x + r_{22}b_i^y \end{aligned} \quad (7)$$

The intermediate passive joint angles $\gamma_{b_i}$ and $\gamma_{p_i}$ are:

$$\gamma_{b_i} = \arctan\left(\frac{Y_i}{X_i}\right) \quad (8)$$

$$\gamma_{p_i} = \gamma - \gamma_{b_i} + \frac{\pi}{6} \quad (9)$$

## 5 Determination of the inverse Jacobian matrix

For parallel robots, the inverse Jacobian matrix computation ($\mathcal{J}^{-1}$) stays in principle relatively easy. $\mathcal{J}^{-1}$ matrix is obtained by the determination of point $B_i$ velocity [2][21]:

$$\dot{\boldsymbol{OB_i}} = \boldsymbol{v_{N+1}} + \boldsymbol{B_iC} \wedge \boldsymbol{\omega_{N+1}} \quad (10)$$

The following relationship is verified:

$$\dot{\theta}_i^a = \dot{\boldsymbol{OB_i}}\,\boldsymbol{n_i} \quad (11)$$

Inserting equation (10) into (11), we also obtain:

$$\dot{\theta}_i^a = \boldsymbol{n_i v_{N+1}} + \boldsymbol{\omega_{N+1}}\left(\boldsymbol{n_i} \wedge \boldsymbol{B_iC}\right) \quad (12)$$

The inverse Jacobian matrix is written as:

$$\mathcal{J}^{-1} = \begin{bmatrix} (\boldsymbol{n_3} \wedge \boldsymbol{B_3C})^t & \boldsymbol{n_3^t} \\ (\boldsymbol{n_2} \wedge \boldsymbol{B_2C})^t & \boldsymbol{n_2^t} \\ (\boldsymbol{n_1} \wedge \boldsymbol{B_1C})^t & \boldsymbol{n_1^t} \end{bmatrix} \quad (13)$$

Thus, the inverse Jacobian matrix for the mobile triangle case is given by:

$$\mathcal{J}^{-1} = \begin{bmatrix} \sin\gamma'_{b_3}\, b_3^x - \cos\gamma'_{b_3}\, b_3^y & \cos\gamma_{b_3} & \sin\gamma_{b_3} \\ \sin\gamma'_{b_2}\, b_2^x - \cos\gamma'_{b_2}\, b_2^y & \cos\gamma_{b_2} & \sin\gamma_{b_2} \\ \sin\gamma'_{b_1}\, b_1^x - \cos\gamma'_{b_1}\, b_1^y & \cos\gamma_{b_1} & \sin\gamma_{b_1} \end{bmatrix} \quad (14)$$

Where $\gamma'_{b_i} = \gamma_{b_i} - \gamma$ with $i = 1,\ 2$ or $3$

## 6 Factorized expression of the Jacobian matrix

The differential kinematic model of a manipulator can be defined by the relationship between the spatial velocity of the end effector and the vector of generalized coordinate velocities of the robot: $\boldsymbol{V_{N+1}} = \mathcal{J}\dot{\boldsymbol{Q}}$, where $\mathcal{J}$ is the Jacobian matrix.
In the proposed approach, the parallel robot is considered as a multi robot system, composed of serial robots (the segments) moving a common load (the mobile platform). A relationship linking the Jacobian matrix of the parallel robot ($\mathcal{J}$) to the Jacobian matrix of each segment ($\mathcal{J}_i$) is presented.
The principle of this approach consists of first computing the Jacobian matrix for each leg considered as an open serial chain. Secondly, the closing constraint is determined, allowing the computation of the parallel robot Jacobian matrix.

### 6.1 $\mathcal{J}_i$ matrix determination

Velocity propagation for a serial chain of interconnected bodies is given by the following intrinsic equation[17][18][19]:

$$^{i}\boldsymbol{V_j} - {}^{i}\hat{P}^t_{j-1}\,{}^{i}\boldsymbol{V_{j-1}} = {}^{i}H_j\,{}^{i}\dot{\theta}_j \qquad (15)$$

By using the matrix $\mathcal{P}$, equation (15) can be expressed in a global form by:

$$\mathcal{P}^t_i\,\boldsymbol{\mathcal{V}_i} = \mathcal{H}_i\dot{\boldsymbol{Q}}_{\boldsymbol{i}} \qquad (16)$$

thus:

$$\boldsymbol{\mathcal{V}_i} = \left(\mathcal{P}^t_i\right)^{-1}\mathcal{H}_i\dot{\boldsymbol{Q}}_{\boldsymbol{i}} \qquad (17)$$

The end effector spatial velocity $\boldsymbol{V_{N+1}}$ is obtained by the following relation:

$$\boldsymbol{V_{N+1}} - {}^{i}\hat{P}^t_N\,{}^{i}\boldsymbol{V_N} = \boldsymbol{0} \qquad (18)$$

thus:

$$\boldsymbol{V_{N+1}} = {}^{i}\hat{P}^t_N\,{}^{i}\boldsymbol{V_N} \qquad (19)$$

Let $\beta_i \in \Re^{6\times\,{}^iN}$ be the matrix defined by:

$$\beta_i = \left[\begin{array}{cccc} {}^{i}\hat{P}^t_N & 0 & \cdots & 0 \end{array}\right] \qquad (20)$$

Equation (19) becomes:

$$\boldsymbol{V_{N+1}} = \beta_i\boldsymbol{\mathcal{V}_i} \qquad (21)$$

Thus, inserting the expression of $\mathcal{V}_i$ from equation (17), we obtain:

$$\boldsymbol{V_{N+1}} = \beta_i\left(\mathcal{P}^t_i\right)^{-1}\mathcal{H}_i\dot{\boldsymbol{Q}}_{\boldsymbol{i}} \qquad (22)$$

Thus:

$$\mathcal{J}_i = \beta_i\left(\mathcal{P}^t_i\right)^{-1}\mathcal{H}_i \qquad (23)$$

For the $3-RPR$ planar robot case this matrix is given by:

$$\mathcal{J}_i = \left[\begin{array}{ccc} 1 & 0 & 1 \\ -\sin\gamma_i\,x_i & \cos\gamma_{p_i} & -\sin\gamma_i\,x_i - \sin\gamma_{b_i}\theta^a_i \\ \cos\gamma_i\,x_i & \sin\gamma_{p_i} & \cos\gamma_i\,x_i + \cos\gamma_{b_i}\theta^a_i \end{array}\right] \qquad (24)$$

Where $\gamma_i = \gamma_{p_i} + \gamma_{b_i}$

### 6.2 Jacobian matrix $\mathcal{J}$ of the parallel robot determination

#### 6.2.1 Forward kinematic problem

The Jacobian matrix $\mathcal{J}$ of the parallel robot is obtained by the closing constraint determination of the kinematic chain. This determination can be obtained by expressing vectors $\dot{\boldsymbol{Q}}_{\boldsymbol{i}}$ associated to each segment $i$ in function of the actuated joint velocity $\dot{\boldsymbol{Q}}$ of the parallel robot. Let the matrix $\Pi_i$ be characterized by:

$$\dot{\boldsymbol{Q}}_{\boldsymbol{i}} = \Pi_i\,\dot{\boldsymbol{Q}} \qquad (25)$$

Inserting equation (25) into (22), we obtain:

$$\boldsymbol{V_{N+1}} = \beta_i\left(\mathcal{P}^t_i\right)^{-1}\mathcal{H}_i\Pi_i\dot{\boldsymbol{Q}} \qquad (26)$$

Therefore, a factorized expression of the parallel robot Jacobian matrix is given by:

$$\mathcal{J} = \beta_i\left(\mathcal{P}^t_i\right)^{-1}\mathcal{H}_i\Pi_i \qquad (27)$$

The matrices $\mathcal{J}$ and $\mathcal{J}_i$ are linked by the following relationship:

$$\mathcal{J} = \mathcal{J}_i\,\Pi_i \qquad (28)$$

#### 6.2.2 $\Pi_i$ matrix determination

The matrix $\Pi_i$ is obtained by expressing vectors $\dot{\boldsymbol{Q}}_{\boldsymbol{i}}$ associated to each segment $i$ in function of the actuated joint velocity $\dot{\boldsymbol{Q}}$ of the parallel robot:

$$\left[\begin{array}{c} \dot{\gamma}_{p_i} \\ \dot{\theta}^a_i \\ \dot{\gamma}_{b_i} \end{array}\right] = \left[\begin{array}{ccc} \pi_{i_{11}} & \pi_{i_{12}} & \pi_{i_{13}} \\ \pi_{i_{21}} & \pi_{i_{22}} & \pi_{i_{23}} \\ \pi_{i_{31}} & \pi_{i_{32}} & \pi_{i_{33}} \end{array}\right]\left[\begin{array}{c} \dot{\theta}^a_3 \\ \dot{\theta}^a_2 \\ \dot{\theta}^a_1 \end{array}\right] \qquad (29)$$

where:

- $\dot{\gamma}_{p_i}$, $\dot{\theta}^a_i$, $\dot{\gamma}_{b_i}$ are the elements of the joint velocity vector of the leg $i$:

$$\dot{\boldsymbol{Q}}_{\boldsymbol{i}} = \left[\begin{array}{ccc} \dot{\gamma}_{p_i} & \dot{\theta}^a_i & \dot{\gamma}_{b_i} \end{array}\right]^t \qquad (30)$$

- $\dot{\theta}^a_3$, $\dot{\theta}^a_2$, $\dot{\theta}^a_1$ are the elements of the generalized coordinate velocity vector of the system:

$$\dot{\boldsymbol{Q}} = \left[\begin{array}{ccc} \dot{\theta}^a_3 & \dot{\theta}^a_2 & \dot{\theta}^a_1 \end{array}\right]^t \qquad (31)$$

- $\pi_{i_{jk}}$ are the elements of the matrix $\Pi_i$

From the inverse kinematic model given in Eq. (14), we obtain for $j = 1$ to 3:

$$\dot{\theta}^a_j = \left(\sin\left(\gamma_{b_j} - \gamma\right)b^x_j - \cos\left(\gamma_{b_j} - \gamma\right)b^y_j\right)\dot{\gamma} + \cos\gamma_{b_j}V^x_{N+1} + \sin\gamma_{b_j}V^y_{N+1} \qquad (32)$$

In inserting the relation given in Eq. (24) in Eq. (32), we obtain:

$$\dot{\theta}^a_j = \mathcal{A}_{j,i}\,\dot{\gamma}_{p_i} + \mathcal{B}_{j,i}\,\dot{\gamma}_{b_i} + \mathcal{C}_{j,i}\,\dot{\theta}^a_i \qquad (33)$$

Thus:

$$\dot{\theta}_j^a - \mathcal{C}_{j,i}\,\dot{\theta}_i^a = \mathcal{A}_{j,i}\,\dot{\gamma}_{p_i} + \mathcal{B}_{j,i}\,\dot{\gamma}_{b_i} \tag{34}$$

where:

$$\mathcal{A}_{j,i} = \sin\left(\gamma_{b_j} - \gamma\right) b_j^x - \cos\left(\gamma_{b_j} - \gamma\right) b_j^y + x_i \sin\left(\gamma_{p_i} + \gamma_{b_i} - \gamma_{b_j}\right) \tag{35}$$

$$\mathcal{B}_{j,i} = \mathcal{A}_{j,i} + \theta_i^a \sin\left(\gamma_{b_i} - \gamma_{b_j}\right) \tag{36}$$

$$\mathcal{C}_{j,i} = \cos\left(\gamma_{b_j} - \gamma_{p_i}\right) \tag{37}$$

**$\Pi_1$ matrix computation** The matrix $\Pi_1$ is obtained by:

$$\Pi_1 = \begin{bmatrix} \pi_{1_{11}} & \pi_{1_{12}} & \pi_{1_{13}} \\ 0 & 0 & 1 \\ \pi_{1_{31}} & \pi_{1_{32}} & \pi_{1_{33}} \end{bmatrix} \tag{38}$$

The equation (34) is written for $i = 1$ and $j = 1$ to 3 in a matrix form as:

$$\underbrace{\begin{bmatrix} 0 & 0 & 1-\mathcal{C}_{1,1} \\ 0 & 1 & -\mathcal{C}_{2,1} \\ 1 & 0 & -\mathcal{C}_{3,1} \end{bmatrix}}_{\mathcal{E}_1} \dot{\boldsymbol{Q}} = \underbrace{\begin{bmatrix} \mathcal{A}_{1,1} & \mathcal{B}_{1,1} \\ \mathcal{A}_{2,1} & \mathcal{B}_{2,1} \\ \mathcal{A}_{3,1} & \mathcal{B}_{3,1} \end{bmatrix}}_{\mathcal{D}_1} \begin{bmatrix} \dot{\gamma}_{p_1} \\ \dot{\gamma}_{b_1} \end{bmatrix} \tag{39}$$

Two solutions with the calculation of this matrix $\Pi_1$ can then be considered:

1. $$\begin{bmatrix} \pi_{1_{11}} & \pi_{1_{12}} & \pi_{1_{13}} \\ \pi_{1_{31}} & \pi_{1_{32}} & \pi_{1_{33}} \end{bmatrix} = \mathcal{D}_1^+\,\mathcal{E}_1 \tag{40}$$

2. $$\begin{bmatrix} \pi_{1_{11}} & \pi_{1_{12}} & \pi_{1_{13}} \\ \pi_{1_{31}} & \pi_{1_{32}} & \pi_{1_{33}} \end{bmatrix} = \left(\mathcal{E}_1^{-1}\,\mathcal{D}_1\right)^+ \tag{41}$$

where $X^+$ is the Moore-Penrose inverse matrix of $X$. The compute of this pseudo inverse matrix is obtained by using the Greville algorithm [22].

**$\Pi_2$ matrix computation** The matrix $\Pi_2$ is obtained by:

$$\Pi_2 = \begin{bmatrix} \pi_{2_{11}} & \pi_{2_{12}} & \pi_{2_{13}} \\ 0 & 1 & 0 \\ \pi_{2_{31}} & \pi_{2_{32}} & \pi_{2_{33}} \end{bmatrix} \tag{42}$$

The equation (34) is written for $i = 2$ and $j = 1$ to 3 in a matrix form as:

$$\underbrace{\begin{bmatrix} 0 & -\mathcal{C}_{1,2} & 1 \\ 0 & 1-\mathcal{C}_{2,2} & 0 \\ 1 & -\mathcal{C}_{3,2} & 0 \end{bmatrix}}_{\mathcal{E}_2} \dot{\boldsymbol{Q}} = \underbrace{\begin{bmatrix} \mathcal{A}_{1,2} & \mathcal{B}_{1,2} \\ \mathcal{A}_{2,2} & \mathcal{B}_{2,2} \\ \mathcal{A}_{3,2} & \mathcal{B}_{3,2} \end{bmatrix}}_{\mathcal{D}_2} \begin{bmatrix} \dot{\gamma}_{p_2} \\ \dot{\gamma}_{b_2} \end{bmatrix} \tag{43}$$

Two solutions with the calculation of this matrix $\Pi_2$ can then be considered:

1. $$\begin{bmatrix} \pi_{2_{11}} & \pi_{2_{12}} & \pi_{2_{13}} \\ \pi_{2_{31}} & \pi_{2_{32}} & \pi_{2_{33}} \end{bmatrix} = \mathcal{D}_2^+\,\mathcal{E}_2 \tag{44}$$

2. $$\begin{bmatrix} \pi_{2_{11}} & \pi_{2_{12}} & \pi_{2_{13}} \\ \pi_{2_{31}} & \pi_{2_{32}} & \pi_{2_{33}} \end{bmatrix} = \left(\mathcal{E}_2^{-1}\,\mathcal{D}_2\right)^+ \tag{45}$$

**$\Pi_3$ matrix computation** The matrix $\Pi_3$ is obtained by:

$$\Pi_3 = \begin{bmatrix} \pi_{3_{11}} & \pi_{3_{12}} & \pi_{3_{13}} \\ 1 & 0 & 0 \\ \pi_{3_{31}} & \pi_{3_{32}} & \pi_{3_{33}} \end{bmatrix} \tag{46}$$

The equation (34) is written for $i = 3$ and $j = 1$ to 3 in a matrix form as:

$$\underbrace{\begin{bmatrix} -\mathcal{C}_{1,3} & 0 & 1 \\ 1 & \mathcal{C}_{2,3} & 0 \\ 1-\mathcal{C}_{3,3} & 0 & 0 \end{bmatrix}}_{\mathcal{E}_3} \dot{\boldsymbol{Q}} = \underbrace{\begin{bmatrix} \mathcal{A}_{1,3} & \mathcal{B}_{1,3} \\ \mathcal{A}_{2,3} & \mathcal{B}_{2,3} \\ \mathcal{A}_{3,3} & \mathcal{B}_{3,3} \end{bmatrix}}_{\mathcal{D}_3} \begin{bmatrix} \dot{\gamma}_{p_3} \\ \dot{\gamma}_{b_3} \end{bmatrix} \tag{47}$$

Two solutions with the calculation of this matrix $\Pi_3$ can then be considered:

1. $$\begin{bmatrix} \pi_{3_{11}} & \pi_{3_{12}} & \pi_{3_{13}} \\ \pi_{3_{31}} & \pi_{3_{32}} & \pi_{3_{33}} \end{bmatrix} = \mathcal{D}_3^+\,\mathcal{E}_3 \tag{48}$$

2. $$\begin{bmatrix} \pi_{3_{11}} & \pi_{3_{12}} & \pi_{3_{13}} \\ \pi_{3_{31}} & \pi_{3_{32}} & \pi_{3_{33}} \end{bmatrix} = \left(\mathcal{E}_3^{-1}\,\mathcal{D}_3\right)^+ \tag{49}$$

## 7 Forward kinematics

The Forward Kinematic Problem (FKP) may be stated as: given the current active joint:

$$\boldsymbol{Q} = \begin{bmatrix} \theta_3^a & \theta_2^a & \theta_1^a \end{bmatrix}^t$$

calculate the Cartesian pose:

$$\boldsymbol{X} = \begin{bmatrix} \gamma & x_c & y_c \end{bmatrix}^t$$

FKP for parallel manipulators is a classical problem in robotics and it has been and continue to be addressed by several authors. Thus Merlet [12] proposes an approach based on interval analysis for solving the forward kinematics of a Gough-Stewart platform, after

an analysis of the traditional approaches proposed in the literature. As stated by Merlet [12], the forward kinematics problem (FKP) has been largely addressed in the literature, but it has never been fully solved.
As known the FKP exhibits numerous solutions. Generally the following problem is addressed first:
given the current active joint vector $\boldsymbol{Q}$, find all possible poses of the platform.
Then, a second problem is considered and which caters with finding the right solution corresponding to the real pose of the platform.
Regarding the FKP, solving methods may be classified as [12]:

- the elimination method
- the continuation method
- the Gröebner basis method
- and the interval analysis proposed by Merlet.

Main drawback of the first proposed methods, concerns computational efficiency and have been analysed by Merlet [12].
Based on the factrorization form of the jacobian matrix, we propose a numerical algorithm for solving the FKP:

- One sets $\boldsymbol{X_0}$
- We compute $\boldsymbol{Q_0}$ with de inverse kinematic model given by Eq. (6)
- We compute the error : $\boldsymbol{\epsilon} = \boldsymbol{Q} - \boldsymbol{Q_0}$
- We compute $\boldsymbol{X} = \boldsymbol{X_0} + \mathcal{J}_i \, \Pi_i \, \boldsymbol{\epsilon}$

Depending on the right solutions for the matrix $\Pi_i$, the expression $\mathcal{J}_i \, \Pi_i$ may have 6 solutions at least (see Eq. (40), Eq. (41), Eq. (44), Eq. (45), Eq. (48), Eq. (49)).

Our algorithm has been implemented on Matlab and a simulation example is given figure (3) where the four solutions found for the platform's pose, corresponding to the active joints $\boldsymbol{Q} = \begin{bmatrix} 2 & 1.5 & 1 \end{bmatrix}^t$, are given.

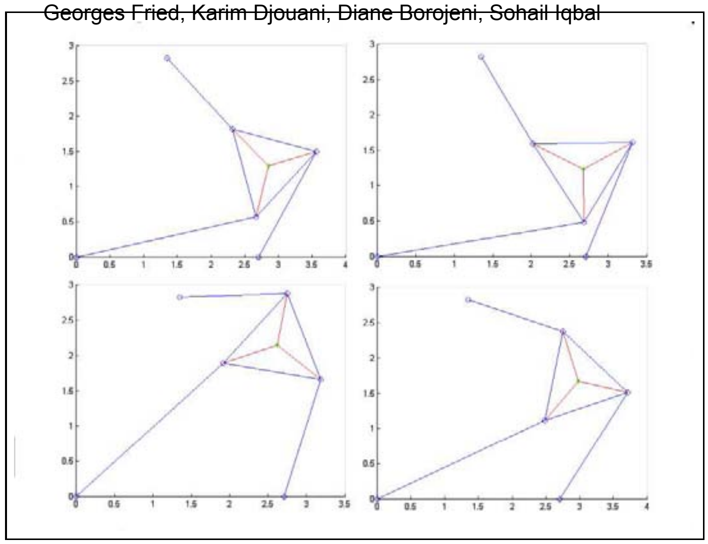

Figure 3: Solutions of the forward kinematics pose

# 8 Conclusion

This paper introduces an approach to determine the assembly modes for a planar parallel robot. The proposed approach is based on a global formalism which allows the determination of a Jacobian matrix factorized expression. This factorization is used in previous work to find the singular configurations of a spatial $C5$ parallel robot [23]. Another interest of our approach is within parallel robot simulation, design and operational space control. The dynamic modeling, based on this formalism is under investigation for the factorization of the inertia matrices (joint and operational spaces) and their inverses, leading to the modeling algebra for robot modeling and control [24].

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