A Behavioral Approach to GNSS Positioning and DOP Determination

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Abstract - In this paper, a behavioral framework is proposed to solve dynamic GNSS positioning problems, which in the mean time may also provide a way to determine the DOP (dilution of precision) of a moving object. The concept of behavioral framework was first proposed by Jan C. Willems in a series of papers (J.C. Willems, "From time series to linear system - part I, II, and III," Automatica Vol. 22, 1986.) as a generic tool for mathematical modeling of dynamical systems. In the proposed approach, a GNSS positioning problem is firstly described by a kernel representation and then the problem can be solved by a structured total least-squares (STLS) algorithm. STLS algorithm is a modified version of the traditional total least-squares (TLS) method. It can be shown that the STLS algorithm is able to provide better performance than the TLS algorithm for the problems that possess a particular structure. In the case of the present paper, it is shown that the GNSS positioning problem has a Hankel structure (i.e., the geometric matrix of the pseudorange equation is Hankel), therefore the problem can be solved by an STLS algorithm subject to a Hankel structure. On the other hand, a formula for calculating DOP of a dynamic positioning problem is also provided. The proposed method is able to calculate the DOP value for multiple time epochs, in compared with the conventional DOP formula which can only be derived at a single time epoch, so as to reflect the inter-dependency between consecutive time epochs.

Keywords – Global Navigation Satellite System (GNSS); Behavioral Framework; Structured Total Least-Squares; Dilution of Precision; Kernel Representation; Total Least-Squares; Least-Squares.

1. INTRODUCTION

Global Navigation Satellite System (GNSS) is a satellite-based utility which provides users with accurate navigation and timing services worldwide. Because of its accuracy, ubiquity, and low cost of user equipment, it has become the navigation and timing system of choice for many users [9][14][15]. Traditionally, in a GNSS receiver, the technique of least-squares optimization is utilized to find a position fix. In this approach, a nonlinear range relationship between the satellites and the receiver has to be obtained at first, and then this nonlinear relationship is linearized around a neighborhood of the equilibrium point to obtain a set of linear pseudorange measurements equations. The method of least-squares estimation can then be applied to the linearized pesudorange equations to acquire an optimal solution in the sense that the error covariance between the true position and the estimated position is minimized[10]. Specifically, consider for example a linear static model represented by an over-determined system of equations $\mathbf{Ax} \cong \mathbf{B}$, where **A**, **B** are given measurements, and the classical least-squares (LS) method, which minimizes the Frobenius norm of the residual **E**:=**Ax**-**B**, i.e.,

$$\min_{\mathbf{E},\mathbf{x}} \|\mathbf{E}\|_F \quad \text{subject to } \mathbf{Ax} = \mathbf{B} + \mathbf{E}.$$

The residual **E** in the LS problem formulation can be viewed as an unobserved, latent variable that allows us to resolve the data-model mismatch. An approximate model for the data is obtained by minimizing some norm (e.g., the Frobenius norm) of **E**. This cost function is called latency, and equation error based methods are called latency oriented.

An alternative approach, called misfit, commonly seen in the literature is to find the smallest correction on the data that makes the corrected data compatible with the model. Then a norm of the lack of fit between the data and the model can be taken to be a quantitative measure, namely the misfit, of the correction. Applied the above-mentioned concept again to the linear static model, represented by the equation $\mathbf{Ax} \cong \mathbf{B}$, the misfit approach leads to the classical total least squares (TLS) method[3]:

$$\min_{\Delta\mathbf{A},\Delta\mathbf{B},\mathbf{x}} \left\| \Delta\mathbf{A} \quad \Delta\mathbf{B} \right\|_F \text{ subject to } (\Delta\mathbf{A} + \Delta\mathbf{A})\mathbf{x} = \mathbf{B} + \Delta\mathbf{B}.$$

Here $\Delta\mathbf{A}$, $\Delta\mathbf{B}$ are corrections on the data **A**, **B**; and **x** is a model parameter.

The classical LS and TLS approximation methods minimize, respectively, the latency and misfit for a linear static model class. However, for the linear time-invariant dynamic model class, these two methods cannot be used directly. Since only one single time epoch is considered in both the LS and TLS algorithm (snapshot algorithm), the correlation between consecutive time epochs is not able to be included in both algorithms. In this paper, to overcome this difficulty, we adopt a behavioral approach to model a dynamical system. The behavioral approach, introduced in a series of papers by J. C. Willems [18][19][20], is able to provide a rigorous framework for deriving mathematical models that is suitable for dynamic model identification (or model approximation). In this approach, a dynamical system can be modeled in a kernel representation and then the kernel representation can be transformed to an equivalent structured TLS problem, which can then be solved by a structured TLS algorithm. In this paper, a system identification problem for a flight dynamical trajectory is served as an example to illustrate the idea. After a dynamical system (in kernel representation) is identified by the structured TLS method, we can then use the derived model to predict the trajectory in arbitrary time epoch.

On the other hand, for a classical GNSS receiver, the DOP (dilution of precision) values can only be calculated at a single time epoch even for a dynamical system that is highly time correlated[8][10]. As we shall see in this paper, the behavioral framework provides a natural way to derive the geometric matrix for multiple time epochs so that the geometric DOP of a kinematic positioning problem can be determined. In this way, the geometric diversity of the positioning problem can be increased so that the DOP of the corresponding problem can be reduced.

The rest of the paper is organized as follows: In Section 2, we will briefly review the concept of behavioral framework as a tool for the optimal approximate modeling of dynamical systems as well as some of the useful results and theorem. In Section 3, we shall use the concept of DOP to give a brief analysis on different optimization algorithms. In Section 4, some experimental results will be given to illustrate the usefulness of the proposed method. Finally, some concluding remarks will be given in Section 5.

2. BEHAVIORAL FRAMEWORK

Dynamical systems describe variables that are functions of one independent variable, usually referred to as "time". In the behavioral setting, a system was defined as a subset $\mathfrak{B}$ of a universum set $\mathfrak{U}$. In the context of dynamical system, $\mathfrak{U}$ is a set of functions $w: \mathbb{T} \to \mathbb{W}$, denoted by $\mathbb{W}^{\mathbb{T}}$. The sets $\mathbb{W}$ and $\mathbb{T} \subseteq \mathbb{R}$ are called signal space and time axis, respectively. The signal space is the set where the system variables take on their values and the time axis is the set where the time variable takes on its values. We use the following definition of a dynamical system [18].

Definition 1. A dynamical system Σ is a 3-tuple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ with $\mathbb{T} \subseteq \mathbb{R}$ the time axis, $\mathbb{W}$ the signal space, and $\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ the behavior. ■

The behavior $\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ is the set of all legitimate functions, according to the system Σ, from the universum set $\mathfrak{U} = \mathbb{W}^{\mathbb{T}}$. The behavior can be described in many ways, while in the context of dynamical systems, the most often used are represented by the mapping $f: \mathbb{W}^{\mathbb{T}} \to \mathbb{R}^g$, i.e.,

$$\mathfrak{B} = \left\{ w \in \mathbb{W}^{\mathbb{T}} \mid f(w) = 0 \right\}.$$

The equations $f(w) = 0$ are called annihilating behavioral equations. A dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ is linear when the signal space $\mathbb{W}$ is a vector space and $\mathfrak{B}$ is a linear subspace of $\mathbb{W}^{\mathbb{T}}$ (viewed as a vector space in the natural way). In this paper, we restrict ourselves to the case when the time axis is either $\mathbb{T} = \mathbb{N}$ or $\mathbb{T} = \mathbb{Z}$, namely the discrete-time case. A system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ is time-invariant if $\mathfrak{B} \subseteq \sigma\mathfrak{B}$, where σ is the backward shift operator $(\sigma w) := w(t+1)$ and $\sigma\mathfrak{B} := \{\sigma W \mid W \in \mathfrak{B}\}$. In the case $\mathbb{T} = \mathbb{Z}$, a system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ is time-invariant if $\mathfrak{B} = \sigma\mathfrak{B}$. Time-invariance requires that if a time series w is a trajectory of a time-invariant system, then all its backward shifts $\sigma^t w$, $t > 0$, are also trajectories of that system.

The restriction of the behavior $\mathfrak{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{T}}$ to the time interval $[t_1, t_2]$, where $t_1, t_2 \in \mathbb{T}$ and $t_1 < t_2$, is denoted by

$$\mathfrak{B}|_{[t_1,t_2]} := \{w \in (\mathbb{R}^{\mathtt{w}})^{t_2-t_1+1} \mid \exists w_-, w_+, \ni \mathrm{col}(w_-, w, w_+) \in \mathfrak{B}\}.$$

A system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ is complete if

$w|_{[t_0,t_1]} \in \mathfrak{B}|_{[t_1,t_2]}$ for all $t_0, t_1 \in \mathbb{T}, t_0 \le t_1$
implies that $w \in \mathfrak{B}$;

i.e., by looking at the time series w through a window of finite width t_1-t_0, one can decide if it is in the behavior or not. The class of all complete LTI systems with w variables is denoted by $\mathcal{L}^{\mathtt{w}}$.

The class of dynamical systems that are studied in this paper consists of those that can be described by the following type of behavioral difference equation:

$$R_0 w(t) + R_1 w(t+1) + \cdots R_l w(t+l) = 0, \tag{1}$$

where $R_1, R_1, ..., R_l \in \mathbb{R}^{g\times \mathtt{w}}$. Equation (1) shows the dependence between successive samples of the time series w. Without loss of generality, we can assume that $R_l \neq 0$ so that the equation has the maximum number of shifts l, which is also called the lag of the difference equation. Usually, Equation (1) will be much more analyzed by using the polynomial matrix representation. Define

$$R(z) := R_0 + R_1 z + R_2 z^2 + \cdots + R_l z^l \in \mathbb{R}^{g\times \mathtt{w}}[z],$$

then Equation (1) can be compactly written as

$$R(\sigma)w = 0. \tag{2}$$

Equation (2) (or equivalently, Equation (1)) is usually called a kernel representation in the behavioral context. A kernel representation (2) for a given $\mathfrak{B}$ is not unique. If the polynomial matrix $R(z)$ defines a kernel representation of $\mathfrak{B}$, then for any unimodular matrix $U \in \mathbb{R}^{p\times p}[z]$, $U(z)R(z)$ also defines a kernel representation of $\mathfrak{B}$. The kernel representation is called minimal if R is full row rank. There exists a minimal one, called shortest lag representation, in which the number of equations p = rowdim(R), the maximum lag l, and the total lag

$$n = \sum_{i=1}^{p} l_i,$$

where l_i is the lag of the ith equation, are all minimal. A kernel representation $R(\sigma)w = 0$ is a shortest lag representation if and only if $R(z)$ is row proper. Let l_i be the degree of the ith row of R. The polynomial matrix R is called row proper if the leading row coefficient matrix, i.e., the matrix of which the (i, j)-th entry is the coefficient of the term with power l_i of $R_{ij}(z)$, is full row rank.

In a shortest lag representation the number of equations is equal to the number of outputs in an input/output representation and the total lag is equal to the state dimension in a minimal state-space representation. These numbers are invariants of the system. The maximal lag of $\mathfrak{B}$, denoted by $\mathbf{l}(\mathfrak{B})$, is called the lag of the system, and its total lag, denoted by $\mathbf{n}(\mathfrak{B})$, is called the order of the system.

The behavior (or the system) induced by Equation (1) can be defined as:

$$\mathrm{Ker}(R(\sigma)) := \{w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{N}} \mid R(\sigma)w = 0\}.$$

The following theorem gives a necessary and sufficient condition for the existence of a kernel representation.

Theorem 2[11]. The following statements are equivalent:
(i) $\Sigma = (\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ is linear, time-invariant, and complete.
(ii) $\mathfrak{B}$ is linear, shift-invariant, and closed in the topology of pointwise convergence.
(iii) There is a polynomial matrix $R(z) \in \mathbb{R}^{\bullet\times \mathtt{w}}[z]$, such that $\mathfrak{B} = \mathrm{Ker}(R(\sigma))$. ■

Next, we shall introduce some of the algorithms that can be used to identify a dynamical system in a behavioral setting. Consider a time series $w := (w(1),\ldots, w(T))$. The block-Hankel matrix with l block rows, constructed from the time series w, is denoted by

$$\mathcal{H}_l(w) := \begin{bmatrix} w(1) & w(2) & \cdots & w(T-l+1) \\ w(2) & w(3) & \cdots & w(T-l+2) \\ \vdots & \vdots & & \vdots \\ w(l) & w(l+1) & \cdots & w(T) \end{bmatrix}.$$

The time series w satisfies the set of difference equations

$$R_0 w(t) + R_1 w(t+1) + \cdots + R_l w(t+l) = 0, \text{ for } t = 1,\ldots, T\text{-}l \tag{3}$$

with maximum l lags (i.e., unit delays in time), if and only if

$$\mathbf{R}\mathcal{H}_{l+1}(w) = 0 \text{ where } \mathbf{R} := [R_0\ R_1\ \ldots\ R_l].$$

The above identity shows that there exists a matrix **R** that is not identically zero if and only if the matrix $\mathcal{H}_{l+1}(w)$ does not have a full row rank.

Let $\mathfrak{B}$ be the set of all trajectories of a system Σ, described by (2), i.e.,

$$\mathfrak{B} := \{w : \mathbb{N} \to \mathbb{R}^{\mathtt{w}} \mid R(\sigma)w = 0\}.$$

The global total least-squares (GTLS) problem [1] can be described as follows. Let $\mathcal{M}$ be a user specified model class. Let $\mathfrak{B}$ be defined as above and let w be an observed time series such that $w \in \mathfrak{B}$. It is assumed that $\mathfrak{B} \in \mathcal{M}$. The problem of global total least-squares aims to find the model $\hat{\mathfrak{B}}$ that best fits the data according to the misfit criterion

$$\hat{\mathfrak{B}} := \arg\min_{\mathfrak{B}\in\mathcal{M}} M(w, \mathfrak{B})$$

with

$$M(w, \mathfrak{B}) := \min_{\hat{w}\in\mathfrak{B}} \|w - \hat{w}\|_{\ell_2}^2 .$$

For the solution of the GTLS problem, on the one hand, the approach taken by Roorda and Heij [12][13] is based on solving the inner minimization problem, the misfit computation, by using isometric state representation and subsequently used alternating least-squares or Gauss-Newton type algorithm for the outer minimization problem. Also a state-space representation with driving input is used in [12][13]. On the other hand, in [6], Markovsky *et al*. use a different approach to solve the GTLS problem. They have related the identification problem to the structured total least-squares (STLS) problem and subsequently used solution methods developed for the STLS problem. Also a kernel representation of the system was used in their approach.

The optimization problem considered in this paper is defined as follows.

Problem 1(*GTLS*): For a given time series w and a complexity specification (m, l), where m is the number of inputs and l is the lag of the identified system, solve the optimization problem

$$\hat{\mathfrak{B}} := \arg\min_{\mathfrak{B}\in\mathcal{L}_{m,l}} \left(\min_{\hat{w}\in\mathfrak{B}} \|w - \hat{w}\|_{\ell_2}^2\right). \tag{4}$$

The optimal approximating time series is $\hat{w}$, corresponding to a global minimum point of (4), and the optimal approximating system is $\hat{\mathfrak{B}}$.

Problem 1 is a GTLS problem for the model class $\mathcal{M} = \mathcal{L}_{m,l}$. The first step here to solve the GTLS problem is to express (3) as an STLS problem. The STLS problem is defined as follows [1][4]: Given a time series w and a structure specification $\mathcal{S}$, find the global minimum point of the optimization problem

$$\min_{X}\left(\min_{\hat{w}} \|w - \hat{w}\|_{\ell_2}^2 \text{ subject to } \mathcal{S}(\hat{w})\begin{bmatrix} X \\ -I \end{bmatrix} = 0\right). \tag{5}$$

The constraint of (5) enforces the structured matrix $\mathcal{S}(\hat{w})$ to be rank deficient, with rank at most rowdim(X). The cost function measures the distance of the given data w to its approximation $\hat{w}$. Thus the STLS problem aims at optimal structured low rank approximation of $\mathcal{S}(w)$ by $\mathcal{S}(\hat{w})$. To express (4) as an STLS problem (5), we need to ensure that the constraint

$$\mathcal{S}(\hat{w})\begin{bmatrix} X \\ -I \end{bmatrix} = 0$$

is equivalent to $\hat{w} \in \mathfrak{B} \in \mathcal{L}_{m,l}$. The following results are taken from [7].

Lemma 3. Consider a time series $w := (w(1),\ldots,w(T))$, $w(t) \in \mathbb{R}^{\mathtt{w}}$, and natural numbers $m \le \mathtt{w}$ and $l \le T-1$. Assume that $\mathbf{R}\mathcal{H}_{l+1}(w) = 0$ for certain matrix $\mathbf{R} := [R_0\ R_1\ \ldots\ R_l]$, $R_1, R_1, \ldots, R_l \in \mathbb{R}^{p\times\mathtt{w}}$, where $p := \mathtt{w} - m$, with R_l being full row rank. Then the system $\mathfrak{B}$, defined by the kernel representation $R(\sigma)w = 0$ with $R(z) = \sum_{i=0}^{l} R_i z^i$, is such that $w \in \mathfrak{B}|_{[1,T]}, \mathfrak{B} \in \mathcal{L}_{m,l}$, and the order of $\mathfrak{B}$ is $\mathbf{n}(\mathfrak{B}) = pl$. ■

Lemma 4. Consider a time series

$$w := (w(1),\ldots,w(T)),\ w(t) \in \mathbb{R}^{\mathtt{w}},$$

and natural numbers $m \le \mathtt{w}$ and $l \le T-1$. Assume that there is a system $\mathfrak{B} \in \mathcal{L}_{m,l}$ with order $\mathbf{n}(\mathfrak{B}) = pl$, such that $w \in \mathfrak{B}|_{[1,T]}$. Let $R(\sigma)w = 0$, where $R(z) = \sum_{i=0}^{l} R_i z^i$, be a shortest lag kernel representation of $\mathfrak{B}$. Then R_l is full-row rank and the matrix $\mathbf{R} := [R_0\ R_1\ \ldots\ R_l]$ annihilates the Hankel matrix $\mathcal{H}_{l+1}(w)$, i.e., $\mathbf{R}\mathcal{H}_{l+1}(w) = 0$. ■

Theorem 5. Assume that $\mathfrak{B} \in \mathcal{L}_{m,l}$ is a system that admits a kernel representation $R(\sigma)w = 0$, $R(z) = \sum_{i=0}^{l} R_i z^i$ with $R_l =: [Q_l\ \ -P_l]$, $P_l \in \mathbb{R}^{p\times p}$ of full-row rank. Then the constraint $w \in \mathfrak{B}|_{[1,T]}$ is equivalent to the constraint

$$\mathcal{H}_{l+1}^T(w)\begin{bmatrix} X \\ -I \end{bmatrix} = 0, \tag{6}$$

where $X^T = P_l^{-1}[R_0 \ \cdots \ R_{l-1}]Q_l$. ■

Before the end of this section, we will present a simple way to model a dynamic positioning problem in the kernel representation. For the purpose of navigation, usually a PVA (Position, Velocity, and Acceleration) model is used to describe a system with relatively higher dynamics:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t) \tag{7}$$

$$y(t) = Hx(t) \tag{8}$$

where

$$A = \begin{bmatrix} 0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix},$$

$H \in \mathbb{R}^{k\times 11}$ (k denotes the number of satellites in view) and

$$x(t) = [x_1(t) \ \dot{x}_1(t) \ \ddot{x}_1(t) \ x_2(t) \ \dot{x}_2(t) \ \ddot{x}_2(t) \ x_3(t) \ \dot{x}_3(t) \ \ddot{x}_3(t) \ b(t) \ \dot{b}(t)]^T$$

representing the three-dimensional position, velocity, and acceleration ($x_1(t)$, $x_2(t)$, and $x_3(t)$) and the receiver clock bias ($b(t)$) and clock drift ($\dot{b}(t)$), respectively. Equation (8) is a standard linearized GNSS observable equation, which will be described in a greater detail in the next section. In these two equations, we have omitted the noise terms. Now, define a polynomial $p(s) \in \mathbb{R}[s]$ and a matrix polynomial $r(s) \in \mathbb{R}^{k\times 11}[s]$ by

$$p(s) := \det(sI - A), \quad r(s) = p(s)H(sI-A)^{-1}.$$

Let $g(s)$ be the greatest common right divisor of $p(s)I_{11}$ and $r(s)$, then the kernel representation of (7)-(8) is given by

$$R(\frac{\mathrm{d}}{\mathrm{d}t})y := p(\frac{\mathrm{d}}{\mathrm{d}t})g^{-1}(\frac{\mathrm{d}}{\mathrm{d}t})y = 0$$

3. DILUTION OF PRECISION

The concept of dilution of precision (DOP) provides a simple way to evaluate the performance of a navigation algorithm. In this section, we shall let $\mathbf{u} = [x \ y \ z]^T$ be the user's position and $\mathbf{s}^i = [x^i \ y^i \ z^i]$ be the broadcast position of the ith GPS satellite. The pseudo-range measurement ρ^i between the user and the ith GPS satellite is given by

$$\rho^i = \left\| \mathbf{u} - \mathbf{s}^i - \Delta\mathbf{s}^i \right\| + b + \varepsilon^i, \tag{9}$$

where b represents the error associated with the GPS receiver, $\Delta\mathbf{s}^i = [\Delta x^i \ \Delta y^i \ \Delta z^i]$ is the difference between the broadcast position and true position of the ith GPS satellite, and ε^i stands for the errors due to the ith satellite. Typically in GPS positioning, b is assumed to be an offset term and is ε^i treated as a zero mean noise. The linearized equation of (9) around the nominal point $[x_0 \ y_0 \ z_0]^T$ can be written as:

$$\rho_0^i - \rho^i + e^i = \begin{bmatrix} h'_{11} & h'_{12} & h'_{13} & 1 \\ h'_{21} & h'_{22} & h'_{23} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ h'_{i1} & h'_{i2} & h'_{i3} & 1 \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \\ -b + b_0 \end{bmatrix}$$

where

$$\rho_0 = \sqrt{x_0^2 + y_0^2 + z_0^2}$$

$$h'_{i1} = \frac{x^i + \Delta x^i - x_0}{\sqrt{(x^i + \Delta x^i - x_0)^2 + (y^i + \Delta y^i - y_0)^2 + (z^i + \Delta z^i - z_0)^2}}$$

$$h'_{i2} = \frac{y^i + \Delta y^i - y_0}{\sqrt{(x^i + \Delta x^i - x_0)^2 + (y^i + \Delta y^i - y_0)^2 + (z^i + \Delta z^i - z_0)^2}}$$

$$h'_{i3} = \frac{z^i + \Delta z^i - z_0}{\sqrt{(x^i + \Delta x^i - x_0)^2 + (y^i + \Delta y^i - y_0)^2 + (z^i + \Delta z^i - z_0)^2}}$$

When there are n observable satellites, in terms of matrix notations, we then have

$$H'\mathbf{p} = \mathbf{q} + \mathbf{e} \tag{10}$$

where

$$\mathbf{e} = [\varepsilon^1 \ \varepsilon^1 \cdots \varepsilon^n]^T$$

$$H' = \begin{bmatrix} h'_{11} & h'_{12} & h'_{13} & 1 \\ h'_{21} & h'_{22} & h'_{23} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ h'_{i1} & h'_{i2} & h'_{i3} & 1 \end{bmatrix}$$

$$\mathbf{p} = \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \\ -b + b_0 \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} \rho_0^1 - \rho^1 \\ \rho_0^2 - \rho^2 \\ \vdots \\ \rho_0^i - \rho^i \end{bmatrix}$$

Clearly, the observation matrix H' is subject to uncertainty mainly due to ephemeris error in this more realistic model. A positioning method that accounts for the error associated with observation matrix is worth investigating.

Let $H' = H + \Delta H$, where ΔH is the perturbation matrix, the resulting matrix equation becomes

$$(H + \Delta H)\mathbf{p} = \mathbf{q} + \mathbf{e}. \tag{11}$$

This model is termed as the unstructured error-in-variable model as the matrix ΔH represents the errors associated with the observation matrix and no *a priori* structure on ΔH is being imposed. The traditional least-squares solution of Equation (10) can be derived as:

$$\hat{\mathbf{p}}_{LS} = (H'^T H')^{-1} H'^T \mathbf{q};$$

and the geometric dilution of precision (GDOP) for the LS algorithm is given by

$$\mathrm{GDOP}_{LS} = \sqrt{trace(H'^T H')^{-1}}$$

On the other hand, for the TLS algorithm, the GDOP can be derived as follows. The TLS optimization has a solution if and only if the matrix [H' **q**] is rank deficient. A general procedure for solving the TLS problem can be found in [17] and we shall only give a brief outline here. Let

$$[H' \quad \mathbf{q}] = U_{TLS} S_{TLS} V_{TLS}^T$$

be a singular value decomposition of the augmentated matrix [H' **q**], where U_{TLS} is an $n \times n$ orthogonal matrix,

$$S_{TLS} = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 & 0 \\ 0 & 0 & 0 & \sigma_4 & 0 \\ 0 & 0 & 0 & 0 & \sigma_5 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{n\times 5}$$

and V_{TLS} is a 5 × 5 orthogonal matrix. Then the optimal TLS estimator is given by

$$\hat{\mathbf{p}}_{TLS} = (H'^T H' - \sigma_5{}^2 I)^{-1} H'^T \mathbf{q},$$

and the GDOP in this case is given by

$$\mathrm{GDOP}_{TLS} = \sqrt{trace(H'^T H' - \sigma_5{}^2 I)^{-1} H'^T H'(H'^T H' - \sigma_5{}^2 I)^{-1}}$$

To calculate GDOP for the STLS algorithm, again we have to firstly derive a solution of the STLS optimization problem, which can be obtained by the subspace identification method. We will not present the detail of the procedure here. Only the results will be presented here, which can be obtained by using the method shown in [7][16]. By using the subspace identification method, the solution to the STLS problem (or equivalently, solution to (6)) can be shown to be

$$\begin{bmatrix} X \\ I \end{bmatrix}_{STLS} := Y_0$$
$$= \mathcal{H}_{l+1}(w)\left(I - \mathcal{H}_{l+1}^T(w)(\mathcal{H}_{l+1}(w)\mathcal{H}_{l+1}^T(w))^{-1}\mathcal{H}_{l+1}(w)\right);$$

Note that the similarity between the solution of the TLS problem and STLS problem; however, in the case of the STLS optimization, the dimension of $\mathcal{H}_{l+1}(w)$ is $n \times (l+1)$ times $(l - T)$, therefore the solution Y_0 would be a $n \times (l+1)$ times n matrix rather than a single vector as in the case of the TLS problem. With the above identity in hand, the GDOP for the STLS problem can be easily computed as follows:

$$\mathrm{GDOP}_{STLS}$$
$$= \sqrt{trace\left(\mathcal{H}_{l+1}(w)\left(I - \mathcal{H}_{l+1}^T(w)(\mathcal{H}_{l+1}(w)\mathcal{H}_{l+1}^T(w))^{-1}\mathcal{H}_{l+1}(w)\right)\mathcal{H}_{l+1}^T(w)\right)}.$$

4. EXPERIMENTAL AND SIMULATION RESULTS

In order to verify the performance of proposed methods, we shall conduct several experiments in this section. Firstly, a static positioning problem is investigated. We will compare the results of different optimization algorithms. Then a kinematic positioning problem is investigated. To do this, we first construct a flight trajectory and then analyze the resulting error statistics. Finally, we will consider a model identification problem which may be solved by using the STLS algorithm in the behavioral setting.

For the first experiment, the results shown in Figures 1-6 are based on real collected data. In those figures, DLS, MTLS, and STLS stand for data least-squares, mix least-squares-total least squares, and structured total least squares, respectively. DLS is a modified version of the TLS algorithm, and MTLS can be viewed as a special case for the STLS algorithm. Here we shall compare the results of different algorithm. For detailed discussion of those methods, please see [1][2][12][17]. Figure 1 shows the estimated ECEF (Earth-Centered, Earth-Fixed) positions by using different algorithms. Figure 2 and 3 are the estimated position errors for all the algorithms. Figure 4 and Figure 5 show the mean values and error covariances, respectively, for different algorithms, and finally Figure 6 shows the GDOP values for all the algorithms used in this experiment. From the results shown in those figures, we can easily see that the proposed method is able to provide better performance than all other methods. Note that in this experiment only one single epoch is considered at one time (hence the name: static positioning). In the following, the attention will be made on the dynamic position problem. In this case, only the proposed method in this paper is able to treat multiple epochs at one time.

To do this, we first construct a simulated flight trajectory and then use suitable software to generate simulated GPS observable data. The analysis results are based on this generated data. The flight trajectory in ECEF coordinate is shown in Figure 7. The estimation error is shown in Figure 8. Means and error covariances for all cases are shown in Figure 9 and Figure 10, respectively, and the GDOP values for each case are shown in Figure 10. The results also show that the STLS algorithm has the lowest GDOP. As we have explained before, this is because the geometric diversity can be increased in the STLS algorithm.

Next, we shall consider a model identification problem for dynamical systems. The method introduced in Section 2 will be used. Again, we first construct a flight trajectory as shown in the Figure 12 on top of the page, which is simply a semi-circular. It is noted that we choose such a simple trajectory is merely for illustration purpose. As a matter of fact, a much more complicated model can also be used. In this case, we only choose a behavioral model with a larger time lag. The results of the simulation are shown in Figures 13 to 18. For the matter of comparison, we also use the WTLS (weighted TLS)[1][17] method to predict the flight trajectory. In the present simulation, we use a behavioral model of time lag 3 (i.e., $l = 3$ in Equation (3)); therefore we can use the first 4 epoch to predict the flight trajectory. Figure 13 shows that we use the first 7 epoch to construct a behavioral model and then use the model to construct the whole trajectory. For a comparison of the WTLS method and the STLS method with the true trajectory, the results are shown in Figure 14, 16 and 17.

5. CONCLUSION

In this paper, on the one hand, we have presented a behavioral approach to solve GNSS positioning problems. In the proposed approach, a GNSS positioning problem is firstly described by a kernel representation and then the problem can be solved by a structured total least-squares (STLS) algorithm. It has been shown that the STLS algorithm is able to provide better performance than the TLS algorithm for the problems that possess a particular structure. We can also use the proposed approach to perform the model identification. On the other hand, we also show a possible way to derive GDOP for a dynamic positioning problem. By considering multiple time epochs in the same time, the geometric diversity of the positioning problem can be increased so as to reduce the GDOP values.

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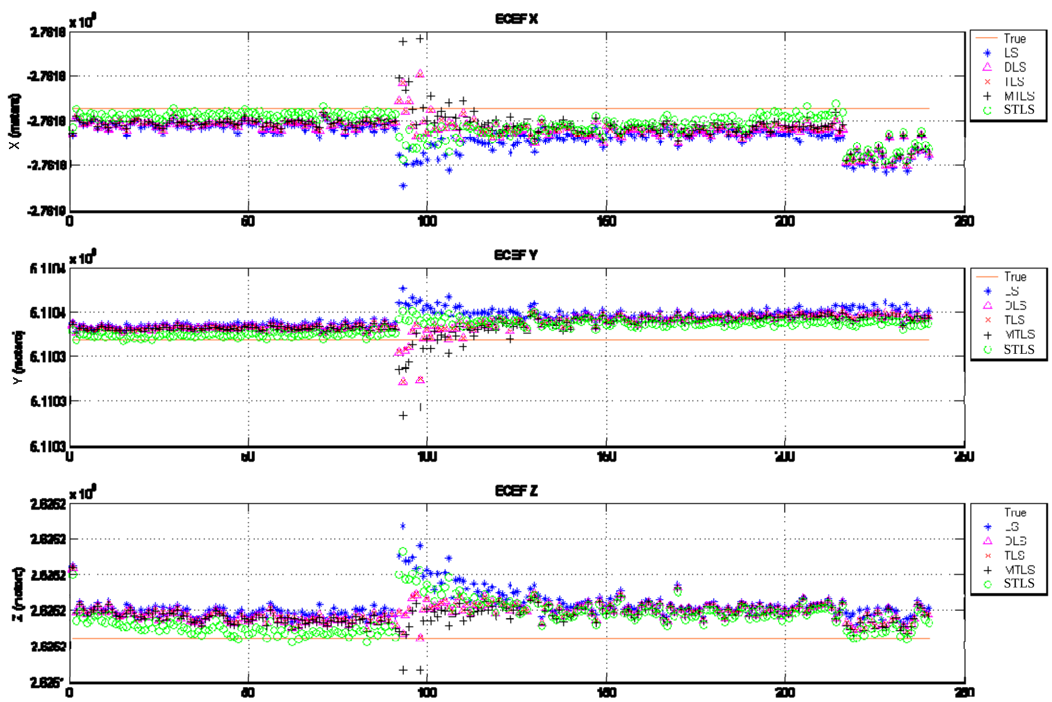

Figure 1. Estimated *x*-, *y*-, *z*-positions in ECEF coordinate

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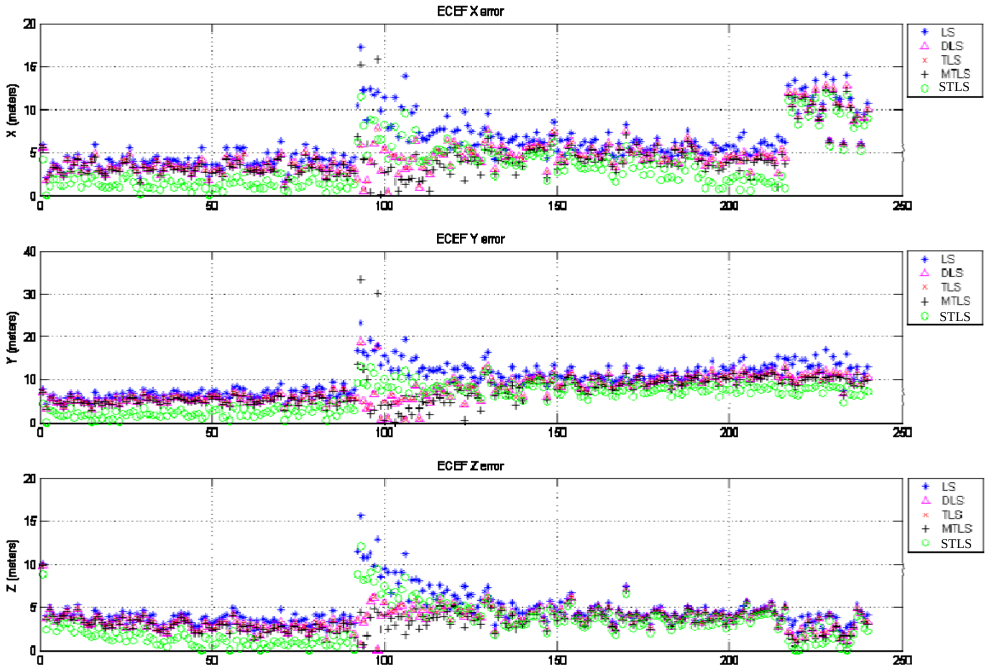

Figure 2. Estimated Positioning Errors

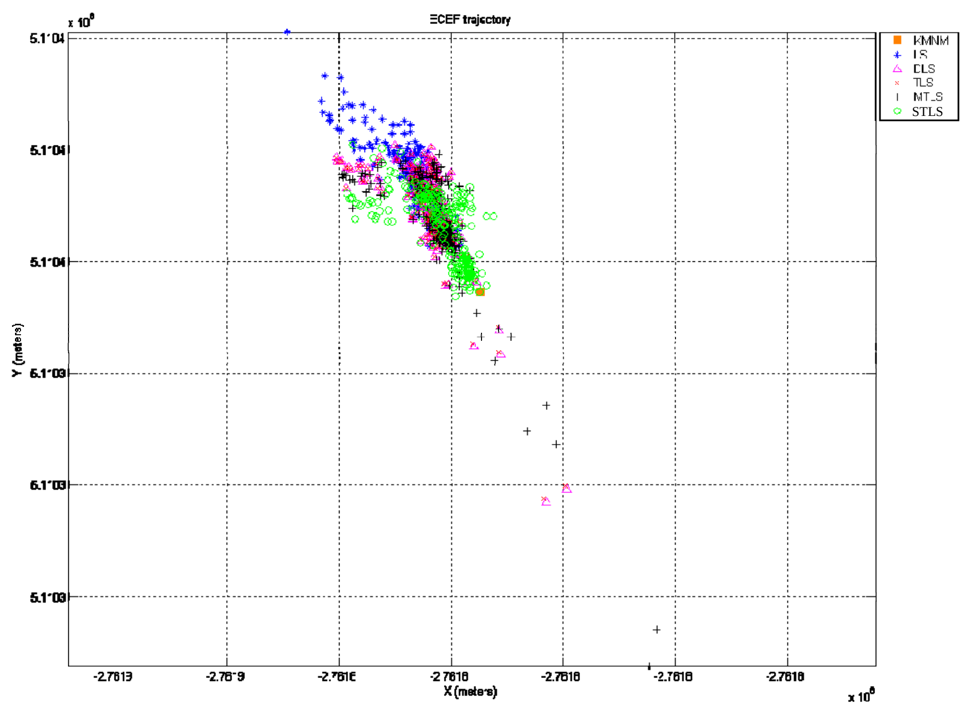

Figure 3. Estimated User Position Using Different Algorithms

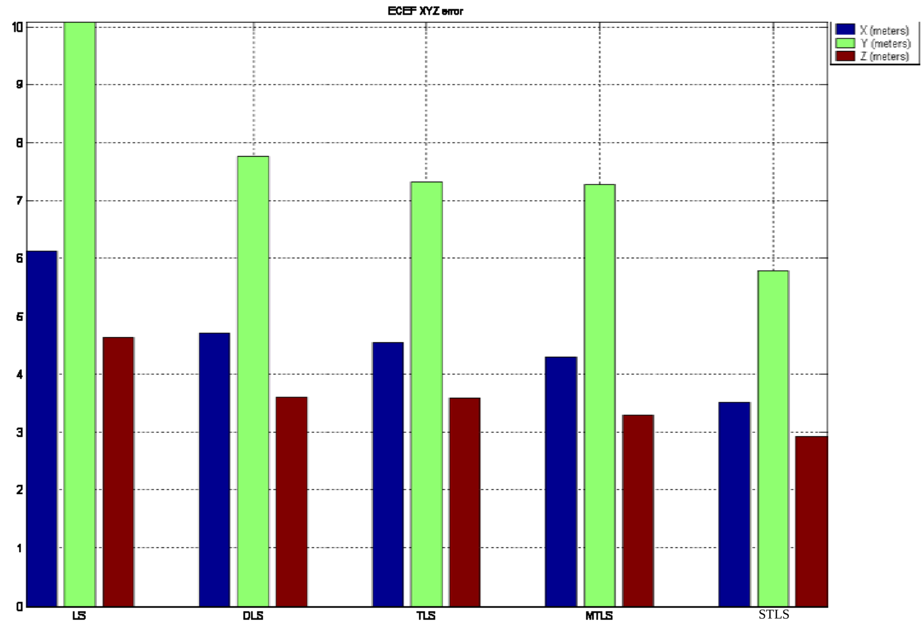

Figure 4. Mean Estimation Errors for Different Algorithms

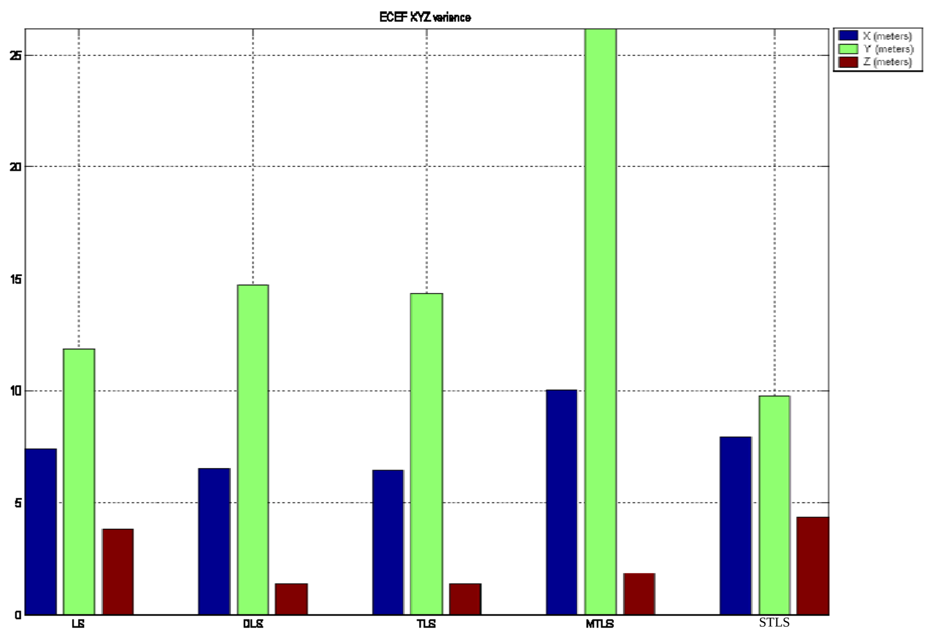

Figure 5. Error Covariances for Different Algorithms

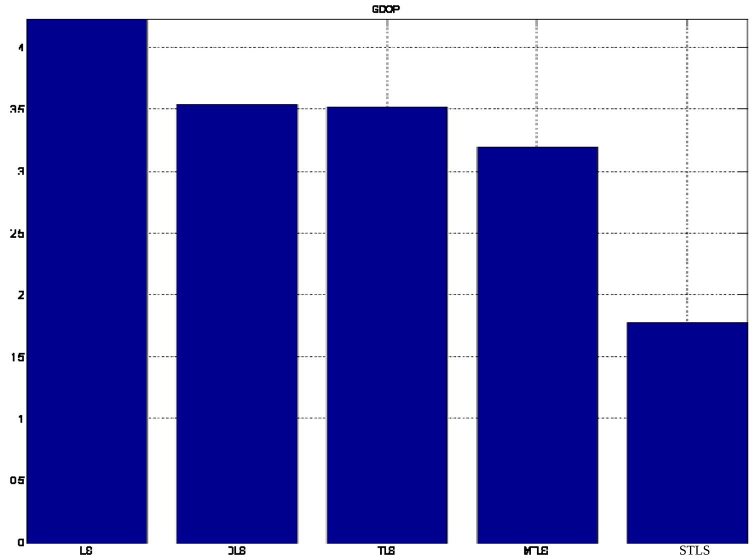

Figure 6. Comparison of Geometric Dilution of Precision

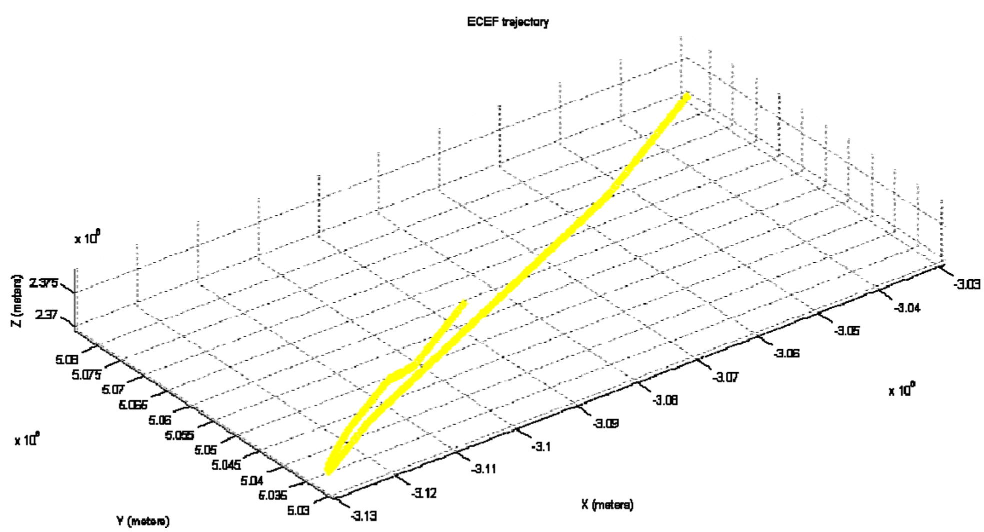

Figure 7. Flight Trajectory in ECEF Coordinate

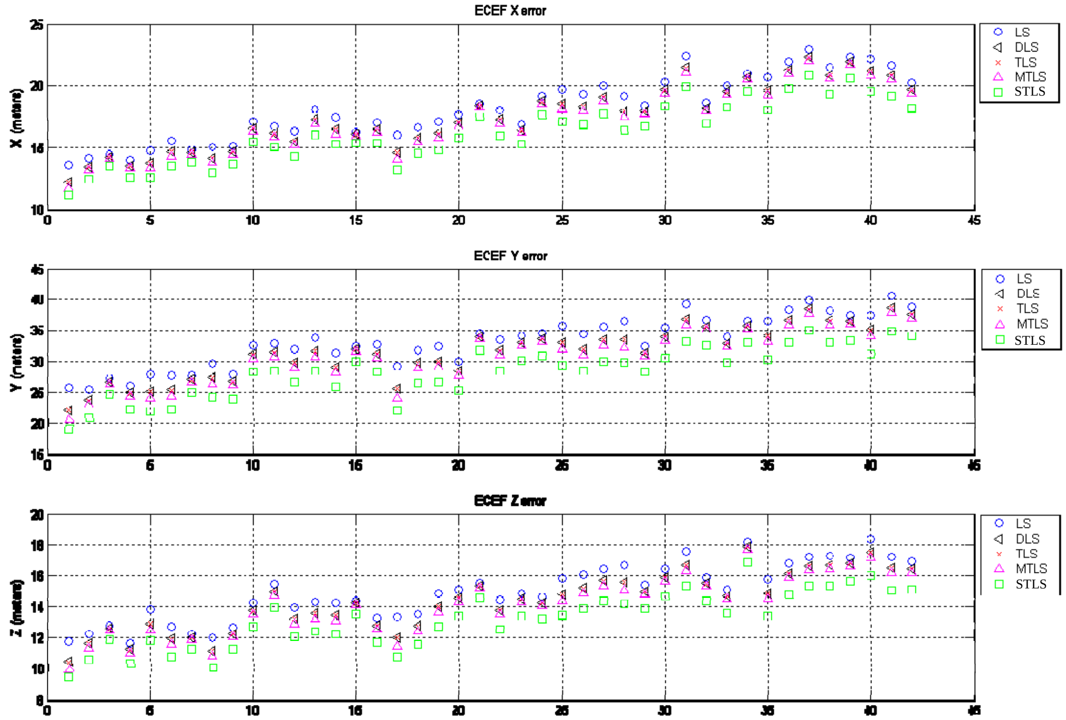

Figure 8. Estimated Errors in *x*-, *y*-, *z*-directions

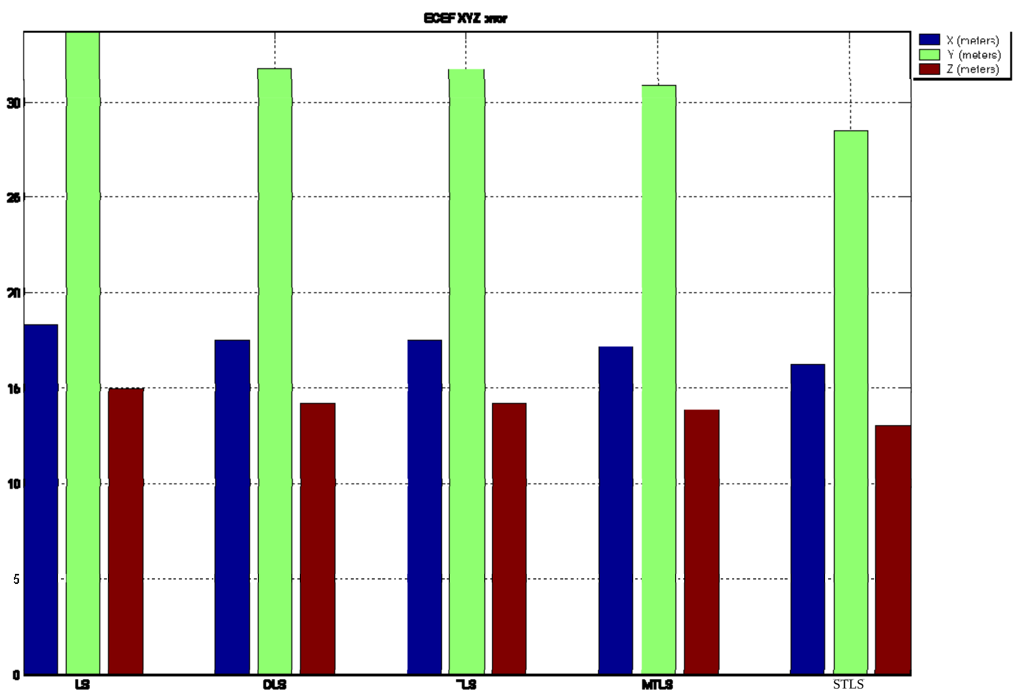

Figure 9. Expectation Values of Estimated Errors

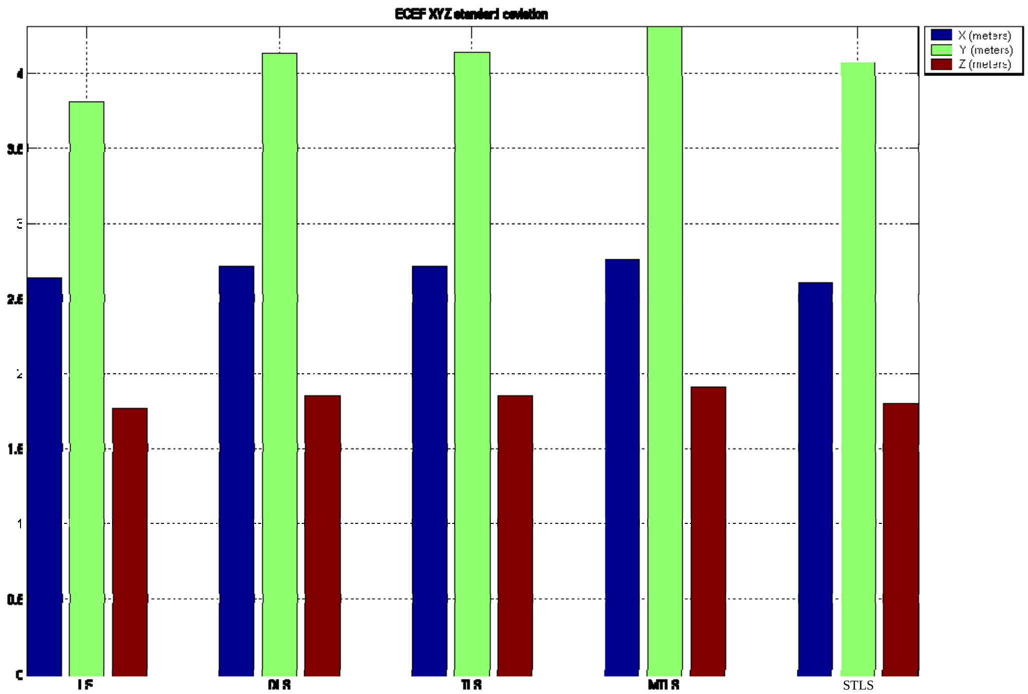

Figure 10. Error Covariances for Different Algorithms

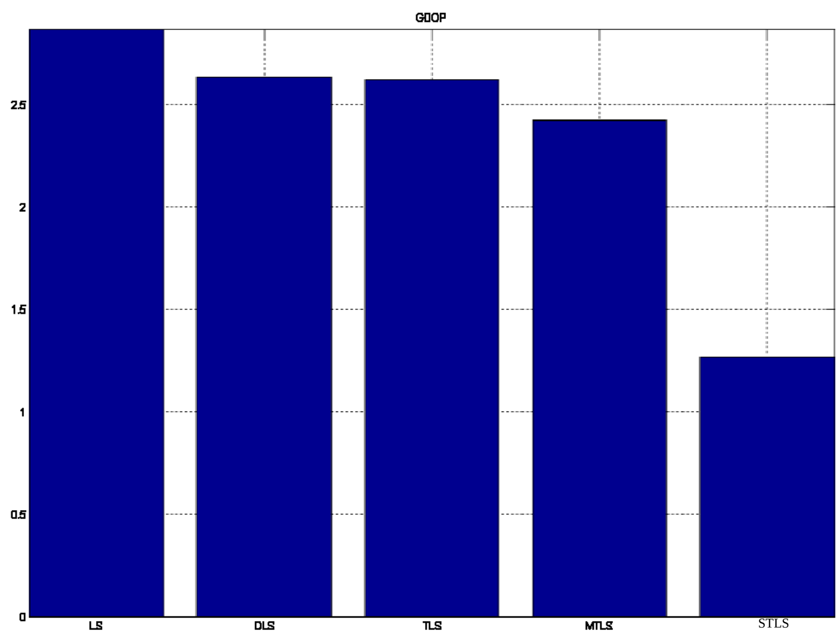

Figure 11. Comparison of Geometric Dilution of Precision

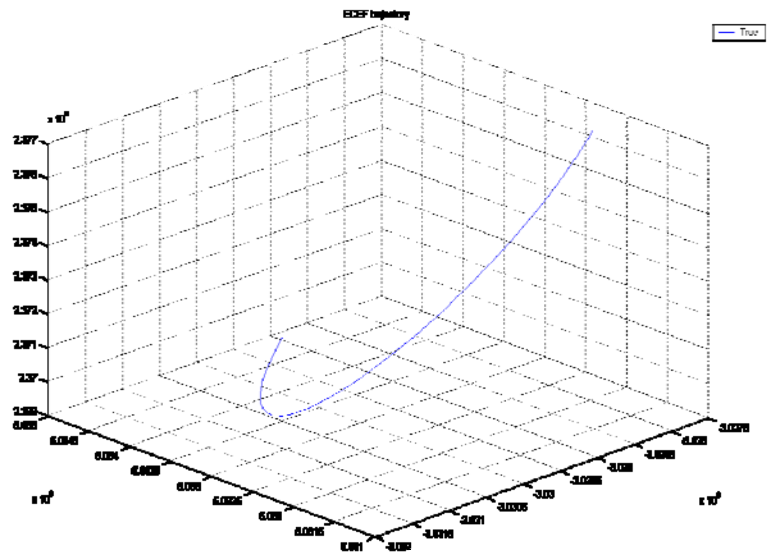

Figure 12. Flight Trajectory in ECEF Coordinate

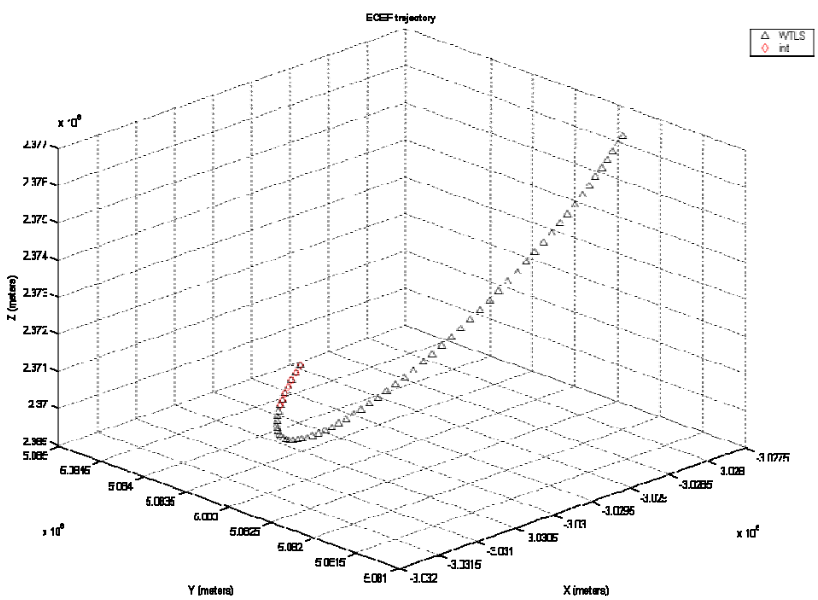

Figure 13. Initial Trajectory and Predicted Trajectory in ECEF Coordinate Using WTLS Method

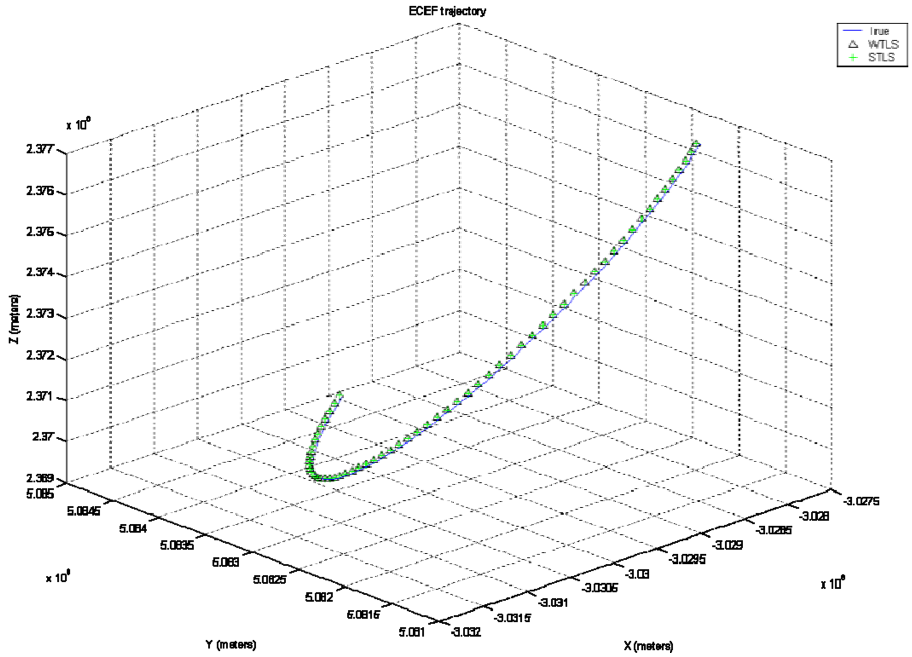

Figure 14. Comparison of Estimated and True Trajectory

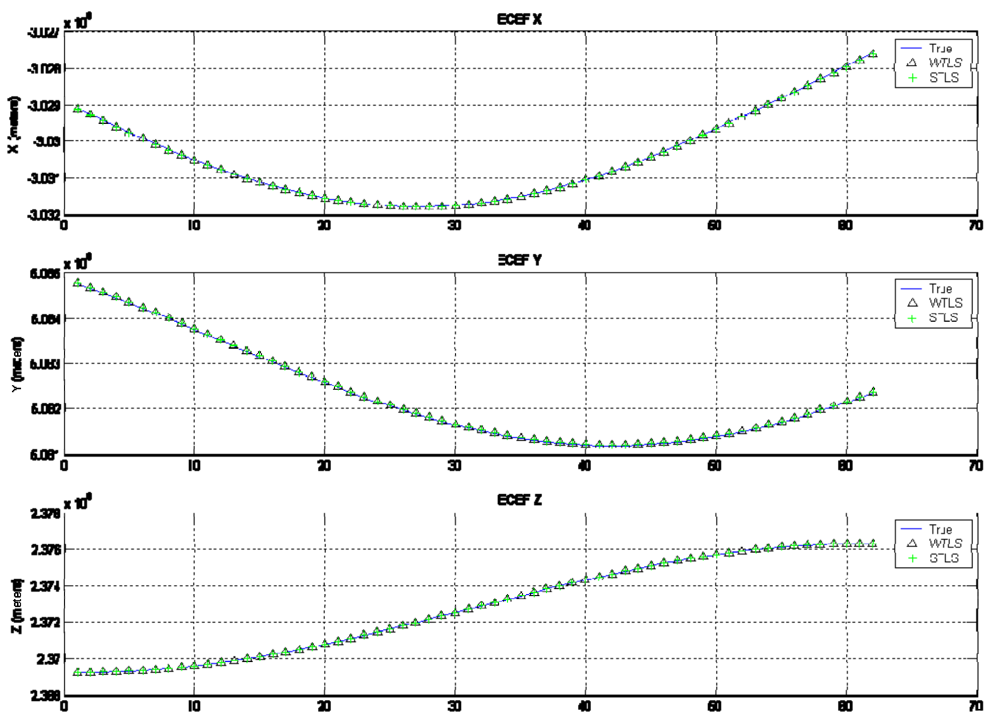

Figure 15. Estimated Trajectories in *x*-, *y*-, *z*-directions

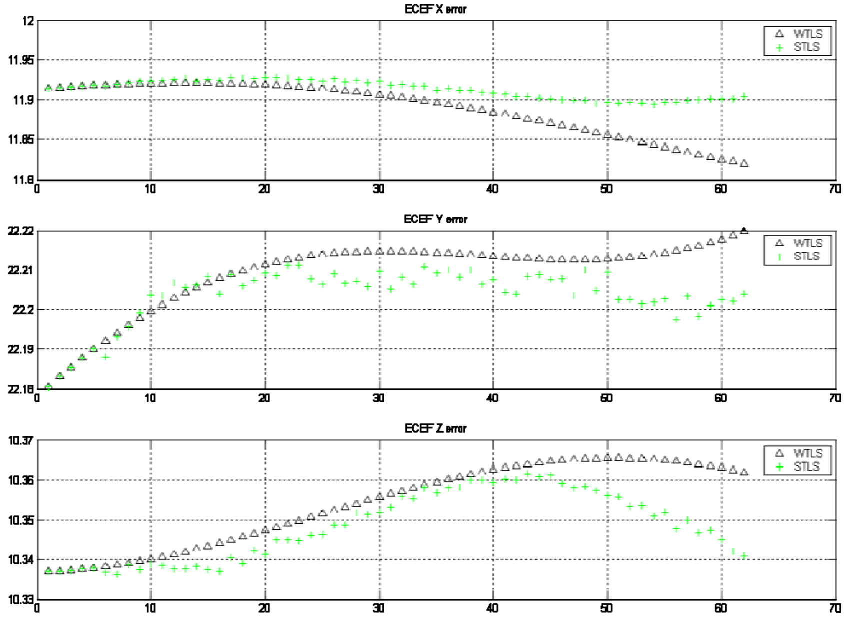

Figure 16. Estimated Errors in *x*-, *y*-, *z*-directions

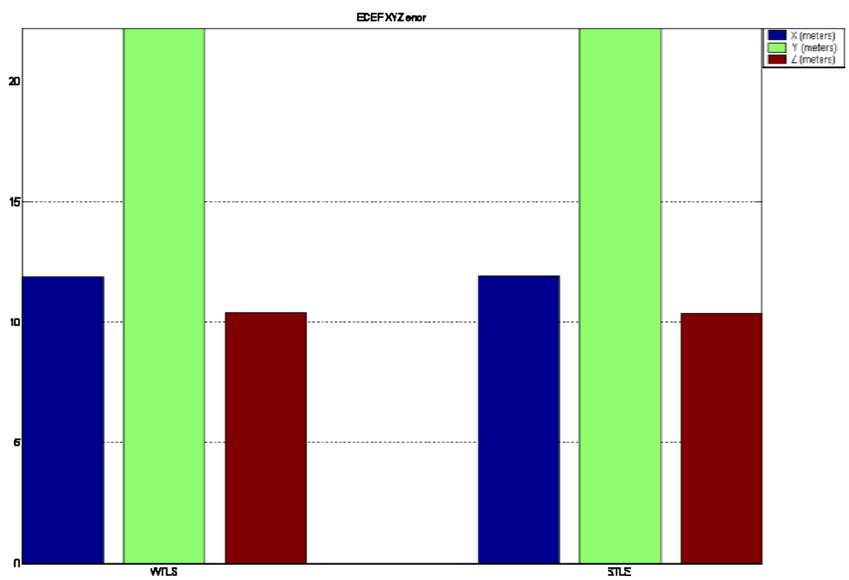

Figure 17. Mean Estimation Errors for Different Algorithms