On the Optimal Multirate Control of Networked Control Systems

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Abstract: - Optimal multirate control of the networked control systems is investigated in this paper. Under p – to – w_σ communication sequence that sensors access communication medium and m – to – w_ρ communication sequence that actuators access communication medium, the plant of networked control system acts as a linear periodic time-varying system. Using lifting technology, a linear periodic time-varying system is transformed into a linear periodic time-invariant system and the linear quadratic performance index of a linear periodic time-varying system is transformed into that of a linear periodic time-invariant system. Controllability and observability of transformed system are analyzed. An optimal state feedback control and an optimal output feedback control are given. The simulation results show that the optimal multirate control strategies proposed in this paper are feasible.

Key-Words: - optimal multirate control; networked control systems; lifting technology; linear quadratic performance index; optimal state feedback control; optimal output feedback control

1 Introduction

The networked control system (NCS) is a special kind of control system — distributed control system, in which multiple sensors and actuators are connected to a centralized controller via a shared communication medium. The main difference to a conventional control system is that controller, sensors and actuators transmit information and control signal through a shared communication medium[1-4]. Samples of the controlled variable, captured by the sensor device, are encapsulated into a frame, which is sent to the controller after the link is granted. In the same way, control actions calculated by the controller have to be sent across the shared medium, to be applied to the plant. A NCS with m sensors, p actuators and a controller is shown in Fig.1.

In a NCS, communication parameters must be considered. An important problem that has received significant attention in the NCS literature must deal with the joint selection of a policy for managing controller-plant communication and a feedback controller that satisfies performance or stability requirements for the closed loop system. In recent years, communication and control co-design for NCS is studied widely. The communication polices of NCS are classified in two categories: static scheduling and dynamic scheduling. Under static scheduling, the medium access order of different

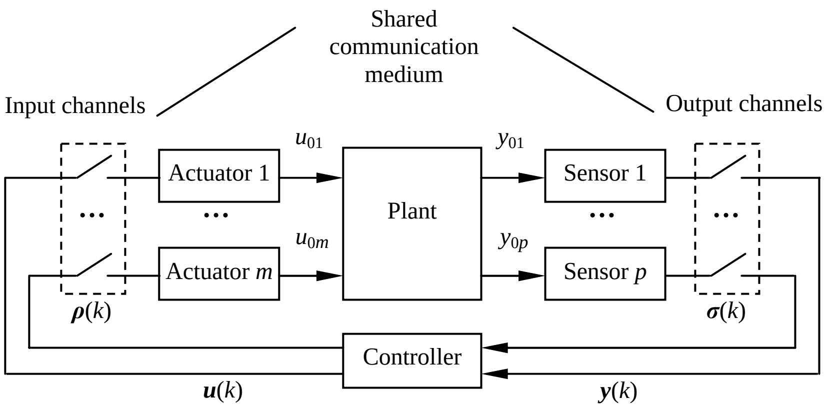

Fig. 1 A NCS with m inputs and p outputs

sensors and actuators is determined off-line by a periodic "communication sequence". Under dynamic scheduling, the medium access order of sensors and actuators is determined on-line, based on state feedback. If a periodic communication sequence has been chosen, the problem that there exists a stabilizing constant feedback controller is NP-hard; If the controller is given in advance, only uncoupled plants exist a stabilizing communication sequence [5-11]. Communication and control co-design could solve the previous problem. Adopting co-design method, the problem of pole-placement output feedback stabilization had been solved in [12]. In this paper, communication and control co-design and multirate method are employed to study optimal control of NCS.

The remainder of this paper is organized as follows: In section 2, we discuss problem formulation that a linear time-invariant (LTI) plant subject to medium access constraints is modeled to a linear time-varying (LTV) system. For discussing in the range of LTI system, we use "lifting" technology to transform a linear period time-varying system into a linear time invariant system, in section 3. Controllability and observability of the linear time invariant system given in section 3 are analyzed in section 4. In section 5, the quadratic performance index transformation is given. With the quadratic performance index given in section 5, an optimal state feedback control law and an optimal output feedback control law are given in section 6 and section 7, respectively. Lastly, a simulation example is given.

2 Problem formulation

Consider the NCS shown in Fig.1, and the dynamics of the plant is given by the continuous time LTI system

$$\dot{\boldsymbol{x}}_0(t) = A_c\boldsymbol{x}_0(t) + B_c\boldsymbol{u}_0(t)$$
$$\boldsymbol{y}_0(t) = C_c\boldsymbol{x}_0(t) \tag{1}$$

where $\boldsymbol{x}_0(t)\in\mathbf{R}^n$, $\boldsymbol{u}_0(t)\in\mathbf{R}^m$ and $\boldsymbol{y}_0(t)\in\mathbf{R}^p$ are the plant's continuous states, inputs and outputs, respectively. Suppose that T is the sample period, then the discrete time description of (1) is represented by

$$\boldsymbol{x}_0[(k+1)T] = A_d\boldsymbol{x}_0(kT) + B_d\boldsymbol{u}_0(kT)$$
$$\boldsymbol{y}_0(kT) = C_d\boldsymbol{x}_0(kT) \tag{2}$$

where

$$A_d = \mathrm{e}^{A_cT},\ B_d = \int_0^T \mathrm{e}^{A_ct}B dt,\ C_d = C_c.$$

$\boldsymbol{x}_0(kT)\in\mathbf{R}^n$, $\boldsymbol{u}_0(kT)\in\mathbf{R}^m$ and $\boldsymbol{y}_0(kT)\in\mathbf{R}^p$ are the plant's discrete states, inputs and outputs, respectively. The communication medium connecting the sensors and the controller has w_σ output channels ($1 \le w_\sigma < p$). At any one time, w_σ of the p sensors can access these channels to communicate with the controller. At the input side, the m actuators share w_p input channels ($1 \le w_p < m$), i.e., the controller can only communicate with w_p of the m actuators at any one time.

For $i = 1, \ldots, p$, use the binary-valued function $\sigma_i(kT)$ denote the medium access status of sensor i at discrete time kT, i.e., $\sigma_i(kT)$: $\boldsymbol{Z} \mapsto \{0, 1\}$, where 1 means "accessing" and 0 means "not accessing". The medium access status of the p sensors over time can be represented by the "p – to – w_σ communication sequence" varied with period NT, $\boldsymbol{\sigma}(kT) = [\sigma_1(kT), \ldots, \sigma_p(kT)]^{\mathrm{T}}$. We have

$$y_i(kT) = \sigma_i(kT)y_{0i}(kT),\ \forall i.$$

For $i = 1, \ldots, m$, use the binary-valued function $\rho_i(kT)$ denote the medium access status of sensor i at discrete time kT, i.e., $\rho_i(kT)$: $\boldsymbol{Z} \mapsto \{0, 1\}$, where 1 means "accessing" and 0 means "not accessing". The medium access status of the m sensors over time can be represented by the "m – to – w_ρ communication sequence" varied with period MT, $\boldsymbol{\rho}(kT) = [\rho_1(kT), \ldots, \rho_m(kT)]^{\mathrm{T}}$. We have

$$u_{0i}(kT) = \rho_i(kT)u_i(kT),\ \forall i.$$

Given a communication sequence $\eta(kT)$, $M_\eta(kT)$ is defined as follow

$$M_\eta(kT) \triangleq \mathrm{diag}(\eta(kT)) \tag{3}$$

At the sensors side, if $\boldsymbol{y}(kT)$ denote the input signal used by the controller and $\boldsymbol{y}_0(kT)$ denote the output signal generated by the plant, at time kT, we have

$$\boldsymbol{y}(kT) = M_\sigma(kT)\boldsymbol{y}_0(kT) \tag{4}$$

At the actuators side, if $\boldsymbol{u}(kT)$ denote the output signal generated by the controller and $\boldsymbol{u}_0(kT)$ denote the input signal used by the plant, at time kT, we have

$$\boldsymbol{u}_0(kT) = M_\rho(kT)\boldsymbol{u}(kT) \tag{5}$$

From the previous discussion, because of introducing network communication, we can see that the plant has been changed and has turned into the original plant plus communication network. The new plant has inputs $\boldsymbol{u}(kT)$ and outputs $\boldsymbol{y}(kT)$. $M_\sigma(kT)$ in (4) and $M_\rho(kT)$ in (5) are time-varying. We can describe the new plant as "extended plant" combining the dynamics of (1) with the medium access status of all sensors and actuators. From (2) – (5), we can describe the extended plant as the LTV system

$$\boldsymbol{x}[(k+1)T] = A\boldsymbol{x}(kT) + B(kT)\boldsymbol{u}(kT)$$
$$\boldsymbol{y}(kT) = C(kT)\boldsymbol{x}(kT) \tag{6}$$

where

$A = A_d$, $B(kT) = B_d M_\rho(kT)$, $C(kT) = M_\sigma(kT)C_d$.

3 System transformation

The input matrix $B(kT)$ in (6) is varied with period MT and the output matrix $C(kT)$ in (6) is varied with period NT. Because the input and the output of the plant given in (6) possess multirate characteristic and we employ T as system sample period, the plant given in (6) will be a LTV system. In other words, If we employ T as basic sample period to control system (6), the digital controller will be a periodically time-varying system. For discussing the previous system in the range of LTI system, we use "lifting" technology and a linear period time-varying system is transformed into a linear time invariant system.

Supposed that the period of $B(kT)$ is $T_{\boldsymbol{u}}$ and the period of $C(kT)$ is T_y, then we have the follows

$$T_{\boldsymbol{u}} = MT,\ T_y = NT$$

Let q be the least common multiple of M and N, i.e.

$$q = \mathrm{LCM}(M, N)$$

and let $T_0 = qT$, then T_0 is the system cycle period, i.e., frame period.

We define extended input vector $\boldsymbol{u}_e(kT_0)$ and another form $\boldsymbol{u}_e^*(kT_0)$, extended output vector $\boldsymbol{y}_e(kT_0)$, extended state vector $\boldsymbol{x}_e(kT_0)$ as follows, respectively.

$$\boldsymbol{u}_e(kT_0) = \begin{bmatrix} \boldsymbol{u}(kT_0) \\ \boldsymbol{u}(kT_0+T) \\ \vdots \\ \boldsymbol{u}[kT_0+(q-1)T] \end{bmatrix},$$

$$\boldsymbol{u}_e^*(kT_0) = \begin{bmatrix} u_1(kT_0) \\ u_1(kT_0+T) \\ \vdots \\ u_1[kT_0+(q-1)T] \\ \vdots \\ u_m(kT_0) \\ u_m(kT_0+T) \\ \vdots \\ u_m[kT_0+(q-1)T] \end{bmatrix},$$

$$\boldsymbol{y}_e(kT_0) = \begin{bmatrix} \boldsymbol{y}(kT_0) \\ \boldsymbol{y}(kT_0+T) \\ \vdots \\ \boldsymbol{y}[kT_0+(q-1)T] \end{bmatrix},$$

$$\boldsymbol{x}_e(kT_0) = \begin{bmatrix} \boldsymbol{x}[(k-1)T_0+T] \\ \boldsymbol{x}[(k-1)T_0+2T] \\ \vdots \\ \boldsymbol{x}[(k-1)T_0+(q-1)T] \\ \boldsymbol{x}(kT_0) \end{bmatrix}.$$

By appropriate elementary row transformation, we can transform $\boldsymbol{u}_e(kT_0)$ into $\boldsymbol{u}_e^*(kT_0)$. Making use of $\boldsymbol{x}_e(kT_0)$, $\boldsymbol{u}_e^*(kT_0)$, $\boldsymbol{y}_e(kT_0)$ and supposed that T_0 is the sample period, the discrete time model of the extended plant is represented by

$$\boldsymbol{x}_e[(k+1)T_0] = A_e\boldsymbol{x}_e(kT_0) + B_e^*\boldsymbol{u}_e^*(kT_0)$$

$$\boldsymbol{y}_e(kT_0) = C_{e1}\boldsymbol{x}_e(kT_0) + C_{e2}\boldsymbol{x}_e[(k+1)T_0] \qquad (7)$$

where

$$A_e = \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & A \\ \mathbf{0} & \cdots & \mathbf{0} & A^2 \\ \vdots & & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & A^q \end{bmatrix},$$

$$B_e^* = [B_1^*\ \ B_2^*\ \ \cdots\ \ B_m^*],$$

$$B_j^* = \begin{bmatrix} B_{j1}^* & \mathbf{0} & \cdots & \mathbf{0} \\ B_{j2}^* & B_{j1}^* & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \\ B_{jq/M}^* & B_{j(q/M-1)}^* & \cdots & B_{j1}^* \end{bmatrix},$$

$$B_{j1}^* = \begin{bmatrix} b_j & \mathbf{0} & \cdots & \mathbf{0} \\ b_j + Ab_j & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \xi_j & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix},$$

$$B_{jk}^* = \begin{bmatrix} A^{(k-2)M+1}\xi_j & \mathbf{0} & \cdots & \mathbf{0} \\ A^{(k-2)M+2}\xi_j & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ A^{(k-1)M}\xi_j & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}.$$

with $k = 2, \ldots, q/M$,

$\boldsymbol{\xi}_j = \boldsymbol{b}_j + A\boldsymbol{b}_j + \cdots + A^{M-1}\boldsymbol{b}_j$,

$\boldsymbol{b}_j$ is the j-th element of B,

$$C_{e1} = \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & C \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$C_{e2} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} \\ C & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & C & \mathbf{0} & & & \vdots \\ \mathbf{0} & \mathbf{0} & C & \mathbf{0} & & \vdots \\ \vdots & \vdots & & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & C & \mathbf{0} \end{bmatrix}.$$

By the elementary row transformation transforming $\boldsymbol{u}_e(kT_0)$ into $\boldsymbol{u}_e^*(kT_0)$, we rearrange B_e^*'s columns order and transform B_e^* into B_e. Based on the previous transformation, the following model is derived

$$\boldsymbol{x}_e[(k+1)T_0] = A_e\boldsymbol{x}_e(kT_0) + B_e\boldsymbol{u}_e(kT_0)$$

$$\boldsymbol{y}_e(kT_0) = C_{e1}\boldsymbol{x}_e(kT_0) + C_{e2}\boldsymbol{x}_e[(k+1)T_0] \qquad (8)$$

Obviously, because $\boldsymbol{x}(kT_0 + jT)$ is only related to the input $\boldsymbol{u}(kT_0 + iT)$ ($i \leqslant j$), B_e in (8) is a block lower triangle matrix. There exists the following form

$$B_e = \begin{bmatrix} B_{11} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ B_{21} & B_{22} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots \\ B_{q1} & B_{q2} & B_{q3} & \cdots & B_{qq} \end{bmatrix}.$$

Although (8) using frame period T_0 gives a LTI space state model of the plant given by (6), it is not convenient in use. Because the dimension of the state $\boldsymbol{x}_e(kT_0)$ in (8) is high, it is very difficult to analyze system and design controller. We discuss how to reduce the dimension of the state $\boldsymbol{x}_e(kT_0)$ in (8).

We substitute the state equations of (8) with respect to $\boldsymbol{x}_e[(k+1)T_0]$ in the output equations of (8) and derive

$$\begin{aligned}\boldsymbol{y}_e(kT_0) &= C_{e1}\boldsymbol{x}_e(kT_0) + C_{e2}\boldsymbol{x}_e(kT_0) + C_{e2}B_e\boldsymbol{u}_e(kT_0) \\ &= (C_{e1} + C_{e2})\boldsymbol{x}_e(kT_0) + C_{e2}B_e\boldsymbol{u}_e(kT_0) \\ &= C_m\boldsymbol{x}_e(kT_0) + D_m\boldsymbol{u}_e(kT_0)\end{aligned}$$

We define C_m and D_m as follows

$$C_m = C_{e1} + C_{e2}A_e,$$
$$D_m = C_{e2}B_e.$$

C_m and D_m have the following form

$$C_m = \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & C \\ \mathbf{0} & \cdots & \mathbf{0} & CA \\ \vdots & & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & CA^{q-1} \end{bmatrix},$$

$$D_m = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ CB_{11} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ CB_{21} & CB_{22} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots \\ CB_{(q-1)1} & CB_{(q-1)2} & CB_{(q-1)3} & \cdots & \mathbf{0} \end{bmatrix}.$$

Thus, the state space model given by (8) can be rewritten as follows

$$\begin{aligned}&\boldsymbol{x}_e[(k+1)T_0] = A_e\boldsymbol{x}_e(kT_0) + B_e\boldsymbol{u}_e(kT_0) \\ &\boldsymbol{y}_e(kT_0) = C_m\boldsymbol{x}_e(kT_0) + D_m\boldsymbol{u}_e(kT_0) \qquad (9)\end{aligned}$$

The matrix A_e in (8) has $n(q-1)$ zero eigen-values and these eigenvalues correspond to the no observable modes. These modes correspond to the previous $n(q-1)$ elements of the extended state vector $\boldsymbol{x}_e(kT_0)$. The extended output vector $\boldsymbol{y}_e(kT_0)$ is not related to the previous $n(q-1)$ elements of the extended state vector $\boldsymbol{x}_e(kT_0)$, too. For reducing system dimension, we delete these elements from $\boldsymbol{x}_e(kT_0)$. The state space model given by (9) can be simplified as follows

$$\begin{aligned}&\boldsymbol{x}[(k+1)T_0] = A_r\boldsymbol{x}(kT_0) + B_M\boldsymbol{u}_e(kT_0) \\ &\boldsymbol{y}_e(kT_0) = C_M\boldsymbol{x}(kT_0) + D_M\boldsymbol{u}_e(kT_0) \qquad (10)\end{aligned}$$

where

$$A_r = A^q,\ B_M = [B_{q1}\ B_{q2}\ \cdots\ B_{qq}],$$

$$C_M = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix},\ D_M = D_m.$$

B_M is the last n rows of B_e and C_M is the last n columns of C_m.

Obviously, the dimension of the state space model described by (10) is n, and it is same as the dimension of the original plant described by (2) using T as the sample period.

In the state space model described by (10), it is not all elements of $\boldsymbol{u}_e(kT_0)$ that correspond to the input vector $\boldsymbol{u}$ at the sample time, and it is not all elements of $\boldsymbol{y}_e(kT_0)$ that correspond to the input vector $\boldsymbol{y}$ at the sample time, too. For simplifying the state space model given by (10), We delete all elements not corresponding to at the sample time from $\boldsymbol{u}_e(kT_0)$ and obtain $\boldsymbol{u}^*_r(kT_0)$, i.e.

$$\boldsymbol{u}^*_r(kT_0) = \begin{bmatrix} u_1(kT_0) \\ u_1(kT_0+T) \\ \vdots \\ u_1[kT_0+(\mu_1-1)T] \\ \vdots \\ u_m(kT_0) \\ u_m(kT_0+T) \\ \vdots \\ u_m[kT_0+(\mu_m-1)T] \end{bmatrix}.$$

According to the order of sample instant, we rearrange $\boldsymbol{u}^*_r(kT_0)$ as $\boldsymbol{u}_r(kT_0)$. If $T_{u1} \leqslant T_{u2} \leqslant \cdots \leqslant T_{um} \leqslant 2T_{u1}$, $\boldsymbol{u}_r(kT_0)$ is as follow

$$\boldsymbol{u}_r(kT_0) = \begin{bmatrix} u_1(kT_0) \\ \vdots \\ u_m(kT_0) \\ u_1(kT_0+T_{u1}) \\ \vdots \\ u_1(kT_0+T_{um}) \\ \vdots \\ u_m[kT_0+(\mu_1-1)T] \\ \vdots \\ u_m[kT_0+(\mu_m-1)T] \end{bmatrix}.$$

In matrix B_M and D_M, we delete the columns corresponding to the elements deleted from $\boldsymbol{u}_e(kT_0)$ and obtain matrix B_r and D_R, respectively. We rewrite (10) as the following form

$$\begin{aligned}&\boldsymbol{x}[(k+1)T_0] = A_r\boldsymbol{x}(kT_0) + B_r\boldsymbol{u}_r(kT_0) \\ &\boldsymbol{y}_e(kT_0) = C_M\boldsymbol{x}(kT_0) + D_r\boldsymbol{u}_r(kT_0)\end{aligned}$$

We delete all elements not corresponding to at the sample time from $\boldsymbol{y}_e(kT_0)$ and obtain $\boldsymbol{y}_r(kT_0)$, too. In matrix C_M and D_R, we delete the rows corresponding to the elements deleted from $\boldsymbol{y}_e(kT_0)$ and obtain matrix C_r and D_r, respectively. We can simplify the state space model given by (10) as follow

$$\begin{aligned}&\boldsymbol{x}[(k+1)T_0] = A_r\boldsymbol{x}(kT_0) + B_r\boldsymbol{u}_r(kT_0) \\ &\boldsymbol{y}_r(kT_0) = C_r\boldsymbol{x}(kT_0) + D_r\boldsymbol{u}_r(kT_0) \qquad (11)\end{aligned}$$

So, the extended plant, described by (6), whose inputs varied with period MT and outputs varied with period NT is transformed into the discrete time model described by (11), sampling with frame period T_0.

4 Controllability and observability

Before introducing optimal control of the discrete time model given by (11), it is necessary to analyze controllability and observability of model (11), in this section.

Firstly, we discuss controllability and reachability of the state space model given by (9). Because the matrix A_e in (9) is singular, if reachability of system (9) holds, controllability of system (9) holds, but, if its controllability holds, its reachability do not completely holds. The dimension of state space model described by (9) is n_e, which is equal to qn. According to the related conclusions in linear system theory, we have the following conclusion.

Conclusion 1 The sufficiency and necessity condition that system (9) is not completely reachable is that there exists a non-zero row vector $\boldsymbol{\theta}^{\mathrm{T}}$ which satisfies

$$\boldsymbol{\theta}^{\mathrm{T}}A_e = \lambda\boldsymbol{\theta}^{\mathrm{T}},\ \boldsymbol{\theta}^{\mathrm{T}}B_e = \mathbf{0} \quad (12)$$

Proof: Sufficiency

If $\boldsymbol{\theta}^{\mathrm{T}}$ satisfies (12), then

$$\boldsymbol{\theta}^{\mathrm{T}}A_eB_e = \lambda\boldsymbol{\theta}^{\mathrm{T}}B_e = \mathbf{0}$$

$$\vdots$$

$$\boldsymbol{\theta}^{\mathrm{T}}A_e^{n_e-1}B_e = \boldsymbol{\theta}^{\mathrm{T}}A_e^{n_e-2}B_e = \cdots = \lambda^{n_e-1}\boldsymbol{\theta}^{\mathrm{T}}B_e = \mathbf{0}$$

i.e., $\boldsymbol{\theta}^{\mathrm{T}}[B_e, A_eB_e, \ldots, A_e^{n_e-1}B_e] = \mathbf{0}$,

thus, system (9) is not completely reachable.

Necessity

If system (9) is not completely reachable, it can be decomposed by controllability as follows

$$\hat{A}_e = \begin{bmatrix} A_c & A_{c\bar{c}} \\ \mathbf{0} & A_{\bar{c}} \end{bmatrix},\quad \hat{B}_e = \begin{bmatrix} B_c \\ \mathbf{0} \end{bmatrix}$$

where $\hat{A}_e = PA_eP^{-1}$, $\hat{B}_e = PB_e$, A_c is the reachable part of A_e and $A_{\bar{c}}$ is the unreachable part of A_e. Let $\hat{\boldsymbol{\theta}}^{\mathrm{T}} = [\mathbf{0}\ \boldsymbol{z}^{\mathrm{T}}]$, and $\boldsymbol{z}^{\mathrm{T}}$ satisfies

$$\boldsymbol{z}^{\mathrm{T}}A_{\bar{c}} = \lambda\boldsymbol{z}^{\mathrm{T}}$$

i.e., $\boldsymbol{z}^{\mathrm{T}}$ is the left eigenvector corresponding to the eigenvalue of $A_{\bar{c}}$, then

$$\hat{\boldsymbol{\theta}}^{\mathrm{T}}\hat{A}_e = [\mathbf{0}\ \lambda\boldsymbol{z}^{\mathrm{T}}] = \lambda\hat{\boldsymbol{\theta}}^{\mathrm{T}},$$

$$\hat{\boldsymbol{\theta}}^{\mathrm{T}}\hat{B}_e = \mathbf{0}.$$

Let $\boldsymbol{\theta}^{\mathrm{T}} = \hat{\boldsymbol{\theta}}^{\mathrm{T}}P$, then $\boldsymbol{\theta}^{\mathrm{T}}$ satisfies (12). □

According to the above discussion, it is easy to derive the following criterion relating to reachability of system (9).

Criterion 1 The sufficiency and necessity condition that system (9) is completely reachable is

$$\operatorname{rank}[\boldsymbol{z}\mathrm{I} - A_e, B_e] = n_e,\quad \forall z \in C \quad (13)$$

Proof: If (13) holds, then there exist not a non-zero row vector $\boldsymbol{\theta}^{\mathrm{T}}$ and a scalar λ that (12) holds. Oppositely, if there exists not a non-zero row vector $\boldsymbol{\theta}^{\mathrm{T}}$ that (12) holds, then (13) must hold. □

Obviously, (12) holds, then, λ corresponding to $\boldsymbol{\theta}^{\mathrm{T}}$ is a no controllable mode of A_e.

Then, we discuss controllability of (11) which system (9) is simplified to. According to the above discussion, we have the following theorem.

Theorem 1 Consider the discrete time plants given by (2) and controllability of the discrete time model given by (11). If λ is a no controllable mode of the discrete time plants given by (2), λ^q must be a no controllable mode of (11). If system (2) is completely controllable and at least there exists a subscript j that $\boldsymbol{\theta}_i^{\mathrm{T}} b_j \neq 0$ and

$$1 + \lambda_i + \lambda_i^2 + \cdots + \lambda_i^{q_{uj}-1} \neq 0 \quad (14)$$

hold, with respect to the eigenvalues λ_i ($i = 1, 2, \ldots, n$) of the system matrix A of (2) and the left eigenvectors λ_i corresponding to, then the discrete time model described by (11) is completely controllable.

Proof: If λ is a no controllable mode of (2), there exists a non-zero row vector $\boldsymbol{\theta}^{\mathrm{T}}$ that

$$\boldsymbol{\theta}^{\mathrm{T}}A = \lambda\boldsymbol{\theta}^{\mathrm{T}},\ \boldsymbol{\theta}^{\mathrm{T}}b_j = 0,\quad j = 1, 2, \ldots, m \quad (15)$$

hold. Obviously, from (10),

$$\boldsymbol{\theta}^{\mathrm{T}}A_r = \boldsymbol{\theta}^{\mathrm{T}}A^q = \lambda^q\boldsymbol{\theta}^{\mathrm{T}} \quad (16)$$

holds.

On the other hand, according to the above discussion in section 3, each non-zero column in B_e has the following forms

$$A^{q-q_{uj}}\xi_j,\ A^{q-2q_{uj}}\xi_j,\ \ldots,\ \zeta_j,$$

$$\zeta_j = b_j + Ab_j + \cdots + A^{q_{uj}-1}b_j,\quad j = 1, 2, \ldots, m.$$

Thus, each non-zero element in the row vector $\boldsymbol{\theta}^{\mathrm{T}}B_e$ has the following forms

$$\boldsymbol{\theta}^{\mathrm{T}}A^{q-q_{uj}}\xi_j,\ \boldsymbol{\theta}^{\mathrm{T}}A^{q-2q_{uj}}\xi_j,\ \ldots,\ \boldsymbol{\theta}^{\mathrm{T}}\zeta_j,$$

$$j = 1, 2, \ldots, m.$$

From (15), we have

$$\boldsymbol{\theta}^{\mathrm{T}}\zeta_j = \boldsymbol{\theta}^{\mathrm{T}}(b_j + Ab_j + \cdots + A^{q_{uj}-1}b_j)$$

$$= (1 + \lambda + \lambda^2 + \cdots + \lambda^{q_{uj}-1})\boldsymbol{\theta}^{\mathrm{T}}b_j,$$

$$\vdots$$

$$\boldsymbol{\theta}^{\mathrm{T}}A^{q-q_{uj}}\xi_j = \lambda^{q-q_{uj}}(1 + \lambda + \lambda^2 + \cdots + \lambda^{q_{uj}-1})\boldsymbol{\theta}^{\mathrm{T}}b_j,\quad j = 1, 2, \ldots, m \quad (17)$$

According to (17), if (15) holds, then

$$\boldsymbol{\theta}^{\mathrm{T}}B_e = \mathbf{0}$$

holds.

From(16), λ^q is a no controllable mode of system (10). As the no controllable modes of system (11) are same to those of system (10), λ^q is a no controllable mode of system (11), too. On the other hand, If system (2) is completely controllable and supposed that λ_i is a arbitrary eigenvalue of A and θ^{T} is the left eigenvector λ_i corresponding to, then there exist several subscripts j that $\theta_i^{\mathrm{T}} b_j \neq 0$ holds. Furthermore, if (14) holds, then

$$\theta^{\mathrm{T}} B_e \neq \mathbf{0}$$

holds.

Because λ_i is arbitrary, system (10) is completely controllable, i.e., system (11) is completely controllable. □

Lastly, we discuss observability and constructibility of the state space model given by (9). Similar to controllability and reachability, because the matrix A_e in (9) is singular, observability and constructibility of system (11) ought to be investigated, respectively. Similar to reachability, we have the following conclusion relating to constructibility of (9).

Conclusion 2 The sufficiency and necessity condition that system (9) is not completely constructible is that there exists a non-zero row vector $\boldsymbol{\theta}^{\mathrm{T}}$ which satisfies

$$A_e\boldsymbol{\theta} = \lambda\boldsymbol{\theta},\ C_e\boldsymbol{\theta} = \mathbf{0} \tag{18}$$

Proof: The proof of conclusion 2 is completely similar to the proof of conclusion 1. □

Using conclusion 2, it is easy to derive the following criterion relating to observability of (9).

Criterion 2 The sufficiency and necessity condition that system (9) is completely observable is

$$\operatorname{rank}\begin{bmatrix} z\mathrm{I} - A_e \\ C_m \end{bmatrix} = n_e, \quad \forall z \in C \tag{19}$$

Proof: The proof of criterion 2 is completely similar to the proof of criterion 1. □

Using criterion 2, it is easy to derive the following theorem.

Theorem 2 System (9) isn't completely observable, and 0 must be a no observable mode of (9).

Proof: According to criterion 2, we can construct the following matrix.

$$\begin{bmatrix} z\mathrm{I} - A_e \\ C_m \end{bmatrix} = \begin{bmatrix} z\mathrm{I} & \mathbf{0} & \mathbf{0} & \cdots & -A \\ \mathbf{0} & z\mathrm{I} & \mathbf{0} & \cdots & -A^2 \\ \vdots & \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & z\mathrm{I} - A^q \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & C \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & CA \\ \vdots & \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & CA^{q-1} \end{bmatrix} \tag{20}$$

Obviously, if $z = 0$, the rank of the matrix shown in (20) is less than n_e. □

Besides eigenvalue A^q, matrix A_e in (9) includes $n_e - n = (q - 1)n$ zero eigenvalues, which are all the no observable modes of system (9). Then, we analyze observability of simplified system given by (11). Using criterion 2, it is easy to derive the following theorem.

Theorem 3 Consider the discrete time plants given by (2) and observability of the discrete time model given by (11). The following conclusions hold.

If system (2) has a no observable mode λ, then λ^q must be a no observable mode of system (11).

If system (2) is completely observable, then system (11) is completely observable.

Proof: According to the above discussion in section 3 and the definition of C_M in (10), C_r in (11) has the following form.

$$C_r = \begin{bmatrix} C \\ H \end{bmatrix}$$

where the submatrix C is compose of the rows in C_M corresponding to the elements of the output sample instants of $\boldsymbol{y}_e(kT_0)$ and the submatrix H is compose of CA, CA^2, ···, CA^{q-1} being appropriately deleted some rows.

Suppose that λ is a no observable mode of system (2), there must exists a non-zero vector $\boldsymbol{\theta}$ which satisfies

$$A\boldsymbol{\theta} = \lambda\boldsymbol{\theta},\ C\boldsymbol{\theta} = \mathbf{0} \tag{21}$$

So, we have

$$A_r\boldsymbol{\theta} = A^q\boldsymbol{\theta} = \lambda^q\boldsymbol{\theta} \tag{22}$$

On the other hand, if (21) holds, then

$$CA\boldsymbol{\theta} = \lambda C\boldsymbol{\theta} = \mathbf{0},$$
$$CA^2\boldsymbol{\theta} = \lambda^2 C\boldsymbol{\theta} = \mathbf{0},$$
$$\vdots$$
$$CA^{q-1}\boldsymbol{\theta} = \lambda^{q-1} C\boldsymbol{\theta} = \mathbf{0}$$

hold. Consequently,

$$H\boldsymbol{\theta} = \mathbf{0}$$

hold. Thus, we have

$$C_r\boldsymbol{\theta} = \begin{bmatrix} C\boldsymbol{\theta} \\ H\boldsymbol{\theta} \end{bmatrix} = \mathbf{0} \tag{23}$$

i.e., λ^q is a no observable mode of system (11).

On the other hand, if there exists not a vector θ that (23) holds, there exists not a vector θ that (22) and (23) hold simultaneously. Thus, if system (2) is completely observable, system (11) is completely observable. □

5 Quadratic performance index transformation

Optimal control is a main kind of control method. The center of optimal control is: with given conditions and for certain plants, a control law is decided by some predefined performance indexes. Under this control law, the closed-loop system has optimal value. Appropriately selecting each weight matrix in the performance indexes given by system, several requirements on system, for example, rapidity, precision, stability, sensitivity, and so on, are satisfied. In this section, according to the multi-rate characteristic of the extended plan described by (6) and based on the state space model given by (11), we give a discrete form of integrator performance index.

Consider linear quadratic optimal control problem of the continuous time plant given by (1). Supposed that the quadratic integrator index of the system represented by (1) is given as follow

$$J_c = \int_0^{t_f} [\boldsymbol{x}_c^{\mathrm{T}}(t) Q\, \boldsymbol{x}_c(t) + \boldsymbol{u}_c^{\mathrm{T}}(t) R\, \boldsymbol{u}_c(t)] dt \tag{24}$$

where Q is a semi-definite constant matrix and R is a definite constant matrix. If a zero-order-holder is employed and the sample period is the basic sample period T, the discrete form for (24) is given as follow

$$J_T = \sum_{kT=0}^{N_T} [\boldsymbol{x}^{\mathrm{T}}(kT) Q_J\ \boldsymbol{x}(kT) + 2\,\boldsymbol{x}^{\mathrm{T}}(kT) M_J\ \boldsymbol{u}(kT) + \boldsymbol{u}^{\mathrm{T}}(kT) R_J\ \boldsymbol{u}(kT)] \tag{25}$$

where, weight matrix Q_J , M_J and R_J are given as follows, respectively.

$$Q_J = \int_0^T \boldsymbol{\alpha}^{\mathrm{T}}(t) Q\, \boldsymbol{\alpha}(t) dt,$$

$$M_J = \int_0^T \boldsymbol{\alpha}^{\mathrm{T}}(t) Q\, \boldsymbol{\beta}(t) dt,$$

$$R_J = \int_0^T \boldsymbol{\beta}^{\mathrm{T}}(t) Q\, \boldsymbol{\beta}(t) dt + TR,$$

$$\boldsymbol{\alpha}(t) = \mathrm{e}^{At},\ \boldsymbol{\beta}(t) = \int_0^t \boldsymbol{\alpha}(\tau) B d\tau.$$

Using two extended vector $\boldsymbol{u}_e(kT_0)$ and $\boldsymbol{x}_e(kT_0)$, the quadratic integrator index (25) can be equivalently transformed into the following form with sample period T_0 (frame period)

$$J_M = \sum_{kT_0=0}^{N_M} \{\boldsymbol{x}_e^{\mathrm{T}}[(k+1)T_0] Q_e\ \boldsymbol{x}_e[(k+1)T_0] + 2\,\boldsymbol{x}_e^{\mathrm{T}}[(k+1)T_0] M_e\ \boldsymbol{u}_e(kT_0) + \boldsymbol{u}_e^{T}(kT_0) R_e\ \boldsymbol{u}_e(kT_0)\} \tag{26}$$

where

$$Q_e = \begin{bmatrix} Q_J & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & Q_J & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & Q_J \end{bmatrix},$$

$$M_e = \begin{bmatrix} \mathbf{0} & M_J & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & M_J & \cdots & \mathbf{0} \\ \vdots & \mathbf{0} & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & M_J \\ (A_r^{-1})^{\mathrm{T}} M_J & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix},$$

$$R_e = \begin{bmatrix} R_J - 2P_1 & -P_2 & -P_3 & \cdots & -P_{q-1} & -P_q \\ -P_2 & R_J & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -P_{q-1} & \mathbf{0} & \mathbf{0} & & R_J & \mathbf{0} \\ -P_q & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & R_J \end{bmatrix},$$

with

$$P_k = B_{qk}^{\mathrm{T}} (A_r^{-1})^{\mathrm{T}} M_J, \quad k = 1, \ldots, q$$

A_r and B_{qk} stated by (10).

The special structure of the first column in weight matrix M_e and the special structure of the first row and the first column in weight matrix R_e are produced by a cross-term $\boldsymbol{x}_{\boldsymbol{e}}^{\mathrm{T}}(kT_0) M_J \boldsymbol{u}_e(kT_0)$. Employing the state space model given by (10), we have the following expression

$$\boldsymbol{x}(kT_0) = A_r^{-1}\{\boldsymbol{x}_e[(k+1)T_0] - B_M \boldsymbol{u}_e(kT_0)\}$$

Then, there exsits the following equation

$$\begin{aligned} &\boldsymbol{x}_{\boldsymbol{e}}^{\mathrm{T}}(kT_0) M_J \boldsymbol{u}_e(kT_0) \\ &= \{\boldsymbol{x}_e[(k+1)T_0] - B_M \boldsymbol{u}_e(kT_0)\}^{\mathrm{T}} (A_r^{-1})^{\mathrm{T}} M_J \boldsymbol{u}_e(kT_0) \\ &= \boldsymbol{x}_{\boldsymbol{e}}^{\mathrm{T}}[(k+1)T_0](A_r^{-1})^{\mathrm{T}} M_J \boldsymbol{u}_e(kT_0) - \boldsymbol{u}_{\boldsymbol{e}}^{\mathrm{T}}(kT_0) B_M^{\mathrm{T}} (A_r^{-1})^{\mathrm{T}} M_J \boldsymbol{u}_e(kT_0) \end{aligned}$$

According to the above expression, the special structure of the first column in weight matrix M_e and the special structure of the first row and the first column in weight matrix R_e are produced. The performance index J_M defined in (26) corresponds to the multirate model described by (8).

Because the dimension of state vector $\boldsymbol{x}_e(kT_0)$ in (8) is high, computation is high. Thus, we simplify the model described by (8) and derive simplified model given by (10). For utilizing (11), $\boldsymbol{u}(kT_0)$ and $\boldsymbol{x}(kT_0)$ substitute $\boldsymbol{u}_e(kT_0)$ and $\boldsymbol{x}_e(kT_0)$ in (25), respectively. We discuss how to implement this transformation. From (10), we can derive as follow

$$\begin{aligned} \boldsymbol{x}_e[(k+1)T_0] &= A_e \boldsymbol{x}_e(kT_0) + B_e \boldsymbol{u}_e(kT_0) \\ &= A_e \boldsymbol{x}_e(kT_0) + B_r \boldsymbol{u}_r(kT_0) \\ &= \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^q \end{bmatrix} \boldsymbol{x}_e(kT_0) + B_r \boldsymbol{u}_r(kT_0) \\ &= A_E \boldsymbol{x}(kT_0) + B_r \boldsymbol{u}_r(kT_0) \end{aligned} \tag{27}$$

where $A_E = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^q \end{bmatrix}$.

Making use of (27), we can derive the following form

$$x_e^{\mathrm{T}}[(k+1)T_0]Q_e x_e[(k+1)T_0]$$
$$= x^{\mathrm{T}}(kT_0)A_E^{\mathrm{T}} Q_e A_E x(kT_0) + 2x^{\mathrm{T}}(kT_0)A_E^{\mathrm{T}} Q_e B_r u_r(kT_0) + u_r^{\mathrm{T}}(kT_0)B_r^{\mathrm{T}} Q_e B_r u_r(kT_0) \quad (28)$$

$$x_e^{\mathrm{T}}[(k+1)T_0]M_e u_e(kT_0)$$
$$= x_e^{\mathrm{T}}[(k+1)T_0]\overline{M}_r u_r(kT_0)$$
$$= x^{\mathrm{T}}(kT_0)A_E^{\mathrm{T}}\overline{M}_r u_r(kT_0) + u_r^{\mathrm{T}}(kT_0)B_r^{\mathrm{T}}\overline{M}_r u_r(kT_0) \quad (29)$$

Using (28) and (29), the performance index J_M can be rewritten as follow

$$J_r = \sum_{kT_0=0}^{N_M}[x^{\mathrm{T}}(kT_0)Q_r\, x(kT_0) + 2x^{\mathrm{T}}(kT_0)M_r\, u_r(kT_0) + u_r^{\mathrm{T}}(kT_0)R_r\, u_r(kT_0)] \quad (30)$$

where

$Q_r = A_E^{\mathrm{T}} Q_e A_E$, $M_r = A_E^{\mathrm{T}}\overline{M}_r + A_E^{\mathrm{T}} Q_e B_r$,

$R_r = B_r^{\mathrm{T}} Q_e B_r + 2B_r^{\mathrm{T}}\overline{M}_r + \overline{R}_r$.

From the quadratic index given by (30), the plant described by (11) can be optimized. With linear quadratic performance index given by (30), utilizing current method, optimal control of the plant given by (11) can be solved, effectively.

6 Optimal state feedback

In this section, with linear quadratic performance index given by (30), we discuss optimal control problem of the plant described by (11). We have the below-mentioned theorem.

Theorem 4 finite-time optimal control problem

According to the quadratic index given by (30), the plant described by (11) has optimal control law as follow

$$u_r(kT_0) = -[K_r(k) + R_r^{-1}M_r^{\mathrm{T}}]x(kT_0) \quad (31)$$

where, $K_r(k)$ is solved by the following recursion

$$K_r(k) = [R_r + B_r^{\mathrm{T}} S_r(k+1)B_r]^{-1}B_r^{\mathrm{T}} S_r(k+1)\tilde{A}_r \quad (32)$$

and matrix $S_r(k)$ is solution of discrete time Riccati equation as follow

$$S_r(k) = \tilde{Q}_r + \tilde{A}_r^{\mathrm{T}} S_r(k+1)\tilde{A}_r - \tilde{A}_r^{\mathrm{T}} S_r(k+1)B_r[R_r + B_r^{\mathrm{T}} S_r(k+1)B_r]^{-1}B_r^{\mathrm{T}} S_r(k+1)\tilde{A}_r \quad (33)$$

with boundary condition

$$S_r(N_M) = \mathbf{0} \quad (34)$$

The optimal performance index function is

$$J_{min} = x^{\mathrm{T}}(0)S_r(0)x(0) \quad (35)$$

where

$$\tilde{Q}_r = Q_r - M_r R_r^{-1} M_r^{\mathrm{T}},\quad \tilde{A}_r = A_r - B_r R_r^{-1} M_r^{\mathrm{T}}.$$

Proof: Let $\overline{u}_r(kT_0) = u_r(kT_0) + R_r^{-1}M_r^{\mathrm{T}} x(kT_0)$, then, the state equation of (11) is equivalent to

$$x[(k+1)T_0] = \tilde{A}_r x(kT_0) + B_r\overline{u}_r(kT_0) \quad (36)$$

and the quadratic index given by (30) is equivalent to

$$\tilde{J}_r = \sum_{kT_0=0}^{\infty}[x^{\mathrm{T}}(kT_0)\tilde{Q}_r\, x(kT_0) + \overline{u}_r^{\mathrm{T}}(kT_0)R_r\,\overline{u}_r(kT_0)] \quad (37)$$

Because the performance index $\tilde{J}_r$ has no cross-item, according to the conclusion with respect to finite-time linear quadratic optimal control of discrete time system, we can obtain the optimal control law (31). □

Now, with the below-mentioned quadratic performance index

$$J = \sum_{kT_0=0}^{\infty}[x^{\mathrm{T}}(kT_0)Q_r\, x(kT_0) + 2x^{\mathrm{T}}(kT_0)M_r\, u_r(kT_0) + u_r^{\mathrm{T}}(kT_0)R_r\, u_r(kT_0)] \quad (38)$$

we discuss infinite-time linear quadratic optimal control for the plant given by (11). Utilizing the above-mentioned method and appropriately defining extended vectors and weighed matrices, the continuous time performance index

$$J_c = \int_0^{\infty}[x_c^{\mathrm{T}}(t)Q\, x_c(t) + u_c^{\mathrm{T}}(t)R\, u_c(t)]dt \quad (39)$$

can be discretely transformed into the performance index J given by (38). The below-mentioned theorem gives solution to the infinite-time linear quadratic optimal control (ITLQOC) problem.

Theorem 5 infinite-time optimal control problem

According to the quadratic index given by (30), the plant described by (11) has infinite-time optimal control law as follow

$$u_r(kT_0) = -[K_r + R_r^{-1}M_r^{\mathrm{T}}]x(kT_0) \quad (40)$$

where

$$K_r = [R_r + B_r^{\mathrm{T}} S_r B_r]^{-1}B_r^{\mathrm{T}} S_r\tilde{A}_r \quad (41)$$

and matrix $S_r(k)$ is solution of algebra Riccati equation as follow

$$S_r = \tilde{Q}_r + \tilde{A}_r^{\mathrm{T}} S_r\tilde{A}_r - \tilde{A}_r^{\mathrm{T}} S_r B_r[R_r + B_r^{\mathrm{T}} S_r B_r]^{-1}B_r^{\mathrm{T}} S_r\tilde{A}_r \quad (42)$$

Proof: Let $\overline{u}_r(kT_0) = u_r(kT_0) + R_r^{-1}M_r^{\mathrm{T}} x(kT_0)$, then, the state equation of (11) is equivalent to

$$x[(k+1)T_0] = \tilde{A}_r x(kT_0) + B_r\overline{u}_r(kT_0)$$

and the quadratic index given by (37) is equivalent to

$$\tilde{J}_r = \sum_{kT_0=0}^{\infty}[\boldsymbol{x}^{\mathrm{T}}(kT_0)\tilde{Q}_r\ \boldsymbol{x}(kT_0) + \overline{\boldsymbol{u}}_r^{\mathrm{T}}(kT_0)R_r \overline{\boldsymbol{u}}_r(kT_0)] \quad (43)$$

Because the performance index $\tilde{J}_r$ has no cross-item, according to the conclusion with respect to infinite-time linear quadratic optimal control of discrete time system, we can obtain the optimal control law (40). □

Based on the above-mentioned theorem 4 and theorem 5, we can obtain design arithmetic of the quadratic optimal state feedback controller for the plant described by (11). For the infinite-time linear quadratic optimal control (ITLQOC) problem, although the regulator given by (40) is linear time-invariant, the state-feedback gain is time-varying at each sample time in frame period.

7 Optimal output feedback

We discuss the optimal output feedback control law which substitutes the optimal state feedback control law given by (31) and (40), in this section. For sampleness, we only discuss the optimal output feedback control law which substitutes the linear time-invariant optimal state feedback control law given by (40).

We introduce the external reference inputs $\boldsymbol{r}(t)$, where $\boldsymbol{r}(t)\in\mathbf{R}^v$ and the sample period of $\boldsymbol{r}(t)$ are as follow

$$T_{rj} = q_{rj}T \qquad (j = 1, 2, \ldots, v).$$

Thus, the extended reference inputs can be defined as follow

$$\boldsymbol{r}_e(kT_0) = \begin{bmatrix} \boldsymbol{u}(kT_0) \\ \boldsymbol{u}(kT_0+T) \\ \vdots \\ \boldsymbol{u}[kT_0+(q-1)T] \end{bmatrix}.$$

In $\boldsymbol{r}_e(kT_0)$, we delete the elements not corresponding to the sample time $(kT_0 + iT_{rj})$ of the reference inputs and obtain vector $\boldsymbol{r}_r(kT_0)$. With the reference inputs $\boldsymbol{r}_r(kT_0)$, the state feedback control law given by (40) can be rewritten as follow

$$\boldsymbol{u}_r(kT_0) = K_c\boldsymbol{r}_r(kT_0) - K\boldsymbol{x}(kT_0) \quad (44)$$

where

$$K = K_r + R_r^{-1}M_r^{\mathrm{T}}.$$

Corresponding to the state feedback control law given by (44), the output feedback control law is as follow

$$\boldsymbol{u}_r(kT_0) = L_r\boldsymbol{r}_r(kT_0) - L_y\boldsymbol{y}_r(kT_0) \quad (45)$$

Because of causality condition, L_y must be a block lower triangle matrix. We substitute the output equation of (10) into (44) and can derive

$$\boldsymbol{u}_r(kT_0) = L_r\boldsymbol{r}_r(kT_0) - L_y[C_r\boldsymbol{x}(kT_0) + D_r\boldsymbol{u}_r(kT_0)] \quad (46)$$

We define

$$W = (\mathrm{I} + L_yD_r)^{-1} \quad (47)$$

Then, equation (46) is equivalent to

$$\boldsymbol{u}_r(kT_0) = WL_r\boldsymbol{r}_r(kT_0) - WL_yC_r\boldsymbol{x}(kT_0).$$

Obviously, if L_r and L_y satisfy

$$WL_r = K_c \quad (48)$$

$$WL_yC_r = K \quad (49)$$

respectively, the output feedback control law given by (45) is equivalent to the state feedback control law given by (44). Equation (48) can be rewritten as follow

$$L_y(C_r - D_rK) = K \quad (50)$$

In (49), we solve L_y which satisfies causality condition. We substitute L_y into (48) and can derive

$$L_r = (\mathrm{I} + L_yD_r)K_c \quad (51)$$

Because of causality condition, L_r must be a block lower triangle matrix, too. Because L_y and D_r are block lower triangle matrix, L_r is a block lower triangle matrix while K_c is a block lower triangle matrix. If accurate solution of L_y can not be solve, linear least squares can be utilized to solve approximate solution of L_y.

8 Simulation example

Suppose that plant is as follow

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$$

$$\boldsymbol{y}(t) = C\boldsymbol{x}(t)$$

where

$$A = \begin{bmatrix} 0 & 0 \\ 0.5 & -0.5 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Design the discrete time output feedback control law which demands the linear quadratic performance index

$$J = \int_0^{\infty}(x_1^2 + x_2^2 + 0.1u_1^2 + 0.1u_2^2)dt$$

to be optimal. Where, system sample period is T = 0.1s and the initial value of the states are given as $\boldsymbol{x}(0) = [1.0\ 1.0]^{\mathrm{T}}$. System samples $y_1(t)$, $y_2(t)$ and outputs $u_1(t)$ at the first sample instant. Also, samples $y_2(t)$ and outputs $u_2(t)$ at the second sample instant.

According to (12), the discrete time description of the plant is represented as follow

$$\boldsymbol{x}[(k+1)T] = F\boldsymbol{x}(kT) + G\boldsymbol{u}(kT)$$

$$\boldsymbol{y}(kT) = C_d\boldsymbol{x}(kT)$$

where

$$F = \begin{bmatrix} 1 & 0 \\ 0.04877 & 0.95123 \end{bmatrix}, G = \begin{bmatrix} 0.1 \\ 0.00246 \end{bmatrix},$$

$$C_d = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

As q = LCM($T_{\boldsymbol{u}}$, $T_{\boldsymbol{y}}$) = 2, frame period $T_0 = qT = 0.2s$. Suppose that sample period is T_0, the discrete time model of the plant is

$$\boldsymbol{x}[(k+1)T_0] = A_r\boldsymbol{x}_0(kT_0) + B_r\boldsymbol{u}_{T0}(kT_0)$$
$$\boldsymbol{y}(kT_0) = C_r\boldsymbol{x}(kT_0) + D_r\boldsymbol{u}_{T0}(kT_0)$$
$$\boldsymbol{u}_{T0}(kT_0) = \begin{bmatrix} u_1(kT_0) \\ u_2(kT_0+T) \end{bmatrix}$$

where

$$A_r = \begin{bmatrix} 1 & 0 \\ 0.09516 & 0.90484 \end{bmatrix},\ B_r = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.19754 \end{bmatrix},$$
$$C_r = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 2.04877 & 0.95123 \end{bmatrix},\ D_r = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.3 & 0 \end{bmatrix}.$$

From (12), we can obtain

$$Q_J = \begin{bmatrix} 0.10008 & 0.00238 \\ 0.00238 & 0.09516 \end{bmatrix},$$
$$M_J = \begin{bmatrix} 0.00516 & 0.00532 \\ 0.00484 & 0.00959 \end{bmatrix},$$
$$R_J = \begin{bmatrix} 0.01067 & 0.00099 \\ 0.00099 & 0.01164 \end{bmatrix}.$$

From (17), we can obtain

$$Q_r = \begin{bmatrix} 0.20062 & 0.00906 \\ 0.00906 & 0.18127 \end{bmatrix},$$
$$M_r = \begin{bmatrix} 0.01589 & 0.00579 \\ 0.01411 & 0.00913 \end{bmatrix},$$
$$R_r = \begin{bmatrix} 0.01267 & 0.00149 \\ 0.00149 & 0.01164 \end{bmatrix}.$$

From (26), (27) and (30), we solve the output control gain

$$L_y = \begin{bmatrix} -0.70120 & 1.86897 & 0 \\ 0 & -5.21083 & 9.08668 \end{bmatrix}.$$

The zero-input state response of the closed-loop control system is shown in Fig.2. As shown in Fig.2, by the control of the designed controller, the response of the closed-loop control system is asymptotically stable. We obtain a quite satisfactory control result.

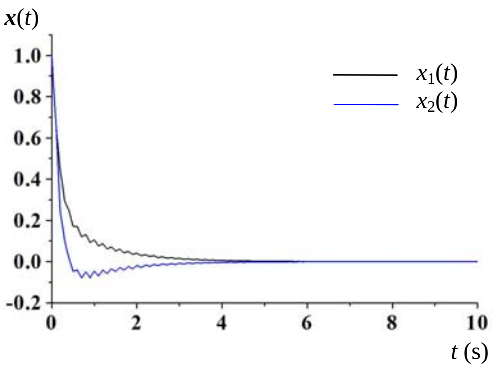

Fig. 2 Zero-input state response of the closed-loop control system

9 Conclusion

Optimal multirate control of the networked control system is mainly studied in this paper. Using “lifting” technology, a linear periodic time-varying system is transformed into a linear periodic time-invariant system, the linear quadratic performance index of a linear periodic time-varying system is transformed into that of a linear periodic time-invariant system. An optimal state feedback control and an optimal output feedback control are given. The simulation results show that the optimal output feedback control law proposed in this paper is effective.

Proposed multirate solutions in this paper have been extensively tested in simulation environment. In the future, a practical experiment will be need to test proposed multirate solutions in this paper. In addition, the selection of control and communication policies in robust will be investigated.

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