Output Feedback Guaranteeing Cost Control by Matrix Inequalities for Discrete-Time Delay Systems

ÉVA GYURKOVICS
Budapest University of Technology
and Economics
Institute of Mathematics
Műegyetem rkp. 3, H-1521, Budapest
HUNGARY
gye@math.bme.hu

TIBOR TAKÁCS
ECOSTAT Institute for
Strategic Research
of Economy and Society
Margit krt. 85, H-1024, Budapest
HUNGARY
tibor.takacs@ecostat.hu

Abstract: This paper investigates the construction of guaranteeing cost dynamic output feedback controller for linear discrete-time delay systems with time-varying parameter uncertainties and with exogenous disturbances. A necessary and sufficient condition for a controller to have the guaranteeing cost property is given by a nonlinear matrix inequality. As a sufficient condition, a linear matrix inequality is derived, the solution of which can be used for constructing a control of the desired type. It is also shown that the resulted trajectory is input-to-state stable. A numerical example illustrates the application of the results.

Key–Words: guaranteeing cost control, delay systems, discrete-time systems, matrix inequalities

1 Introduction

Systems with delay occur in several areas of engineering, therefore they have always been one of the focuses of control theory. Especially in the past decade a huge amount of papers has been published on the stabilization, guaranteeing cost and H_∞ control problems for uncertain time-delay systems primarily in the continuous-time case, but in the discrete-time case as well. (To mention but a few of them see e.g. [3], [5], [11], [13], [18], [19], [20] and the references therein, while a sampled tracking of delay system is presented by [16]). The relatively modest amount of work devoted to discrete-time delay systems can be explained by the fact that such systems can be transformed into augmented systems without delay. However, this approach suffers from the "curse dimension", if the delay is large and inappropriate for systems with unknown or time-varying delays. The present paper deals with the determination of guaranteeing cost output feedback control for uncertain discrete-time delay systems with given constant delay, though a part of the results remains valid also in the case of unknown but bounded delay. We note that to the best of our knowledge, time-varying delays are considered in the literature under the assumption that at least the current state is available for measurement, and memoryless state-feedback is to be constructed (e.g. in [2], [4], [5], [15], [19], [21]). Most papers consider only system uncertainty of either norm-bounded or polytopic type. Similarly to paper [19], we consider both system uncertainty and exogenous disturbance, but the class of system uncertainty considered here is more general. This is the uncertainty of linear fractional form. Authors are not aware of results on uncertain delay systems with this type of uncertainty.

Firstly, we formulate a necessary and a sufficient condition for a dynamic feedback to be a guaranteeing cost robust controller. This condition can be transformed into a bilinear matrix inequality (BMI). A linear matrix inequality (LMI) will be shown, the solution of which is also a solution of this BMI. In order to set up the LMI, the method of the seminal work [6] will be applied. A further novelty of the present paper is that, unlike [13], [19] (see remark 2 in [19]), the coefficient matrix of the delayed initial states in the cost function bound should not be fixed, but it is computed parallel to the weighting matrix of the undelayed initial state and the parameters of the dynamic output feedback controller. It is also shown that the resulted trajectory is input-to-state stable. We note that, under a special choice of the weighting matrices of the cost function, robust H_∞ results can be derived from the results of the present paper.

The organization of this paper is as follows: After fixing the problem statement, we provide some definitions and a preliminary lemma in Section 2. Our main results are stated and proved in Section 3. A numerical example is given in Section 4 to illustrate the application of the results, and finally the conclusions are drawn.

Standard notation is applied. The transpose of matrix A is denoted by A^T, and $P > 0$ (≥ 0) denotes the positive (semi-) definiteness of P. The minimum and maximum eigenvalues of the symmetric matrix P are respectively denoted by $\lambda_m(P)$ and $\lambda_M(P)$. Notation $\mathbf{w}_\infty$ (or simply $\mathbf{w}$) is used for the infinite vector series $\{w_j\}_{j=0}^{\infty}$, while $\mathbf{w}_k$ denotes its truncation to

$$\{w_{k-\tau}, w_{k-\tau+1}, ..., w_k\},$$

where τ is a given positive integer. $\|w\|$ denotes the Euclidean norm of the vector w, while $\|\mathbf{w}\|_\infty$ and $\|\mathbf{w}\|_2$ are defined by

$$\|\mathbf{w}\|_\infty = \sup_{k\in\mathbf{N}} \|w_k\|,$$

and

$$\|\mathbf{w}\|_2 = \left(\sum_{k=0}^{\infty} \|w_k\|^2\right)^{1/2}.$$

Notations l_2 and l_∞ are used for the linear space of infinite vector series with finite norms. Symbol I denotes the identity matrix of appropriate dimension. The notation of time-dependence is omitted, if it does not cause any confusion. For the sake of brevity, asterisk replaces blocks in hypermatrices, and matrices in expressions that are inferred readily by symmetry.

2 Problem statement and preliminaries

2.1 Discrete uncertain time-delay systems

Consider the following discrete-time state-delayed uncertain system:

$$\begin{aligned} x_{k+1} &= \overline{A}x_k + \overline{A}_d x_{k-\tau} + \overline{B}u_k \\ &\quad + \overline{E}w_k + H_x p_k^{(x)}, \qquad k \in \mathbf{Z}^+ \\ x_k &= \phi_k,\ k = 0, -1, ..., -\tau, \\ y_k &= \overline{C}x_k + \overline{C}_d x_{k-\tau} + H_y p_k^{(y)}, \\ q_k^{(x)} &= \overline{A}_q x_k + \overline{A}_{d,q} x_{k-\tau} + \overline{B}_q u_k + G_x p_k^{(x)}, \\ q_k^{(y)} &= \overline{C}_q x_k + \overline{C}_{d,q} x_{k-\tau} + G_y p_k^{(y)}, \end{aligned}$$

where $x \in \mathbf{R}^n$ is the state, $u \in \mathbf{R}^m$ is the control, $w \in \mathbf{R}^s$ is the exogenous disturbance and $y \in \mathbf{R}^p$ is the measured output. All the matrices are of appropriate dimensions. The time delay τ is assumed to be a known constant. The uncertainty appears in the system as

$$p_k^{(x)} = \Delta_k^{(x)} q_k^{(x)}$$

and

$$p_k^{(y)} = \Delta_k^{(y)} q_k^{(y)},$$

where the time varying unknown matrices representing the collection of all parametric uncertainties $\Delta_k^{(x)}$, $\Delta_k^{(y)}$ satisfy the constraint

$$(\Delta_k^{(.)})^T \Delta_k^{(.)} \leq I,$$

where dot replaces either x or y. It is assumed that the system is well posed, i.e. matrices $I - \Delta^{(x)}G_x$ and $I - \Delta^{(y)}G_y$ are invertible for all admissible realizations of $\Delta^{(x)}$ and $\Delta^{(y)}$. It can easily be shown by the matrix inversion lemma that the inverse of a matrix of type $I - \Delta G$ exists for all $\Delta^T\Delta \leq I$ if and only if

$$I - G^T G > 0.$$

This is also equivalent to

$$I - GG^T > 0.$$

By substitution we obtain that the uncertain dynamics is

$$\begin{aligned} x_{k+1} &= (\overline{A} + \delta\overline{A})x_k + (\overline{A}_d + \delta\overline{A}_d)x_{k-\tau} \\ &\quad + (\overline{B} + \delta\overline{B})u_k + \overline{E}w_k \qquad (1) \\ y_k &= (\overline{C} + \delta\overline{C})x_k + (\overline{C}_d + \delta\overline{C})x_{k-\tau}, \qquad (2) \end{aligned}$$

where

$$\begin{aligned} &\left(\delta\overline{A},\ \delta\overline{A}_d,\ \delta\overline{B}\right) = \\ &\quad = H_x(I - \Delta^{(x)}G_x)^{-1}\Delta^{(x)}\left(\overline{A}_q,\ \overline{A}_{d,q},\ \overline{B}_q\right), \\ &\left(\delta\overline{C},\ \delta\overline{C}_d\right) = \\ &\quad = H_y(I - \Delta^{(y)}G_y)^{-1}\Delta^{(y)}\left(\overline{C}_q,\ \overline{C}_{d,q}\right). \end{aligned}$$

Assign to system (1)-(2) the objective function

$$J(\mathbf{x}_0, \mathbf{u}_\infty, \mathbf{w}_\infty) = \sum_{k=0}^{\infty} L(x_k, u_k, w_k) \qquad (3)$$

with

$$L(x, u, w) = x^T\overline{Q}x + u^T Ru - w^T Sw,$$

where

$$\mathbf{x}_0 = \{\phi_{-\tau}, \phi_{-\tau+1}, ..., \phi_0\}$$

is the initial function, and matrices $\overline{Q}$, R and S are symmetric and positive definite. The purpose of this paper is to design a dynamic output feedback control guaranteeing a certain level of performance for system

(1)-(3). To this end, the output feedback controller is looked for in the form

$$\begin{aligned} \widehat{x}_{k+1} &= \widehat{A}\widehat{x}_k + \widehat{A}_d\widehat{x}_{k-\tau} + \widehat{L}y_k, && (4)\\ \widehat{x}_k &= 0, \qquad k = 0,-1,...,-\tau, && (5)\\ u_k &= \overline{K}\widehat{x}_k + \overline{K}_d\widehat{x}_{k-\tau}, && (6) \end{aligned}$$

where $\widehat{x} \in \mathbf{R}^n$ is the state of the controller, and the matrices $\widehat{A}$, $\widehat{A}_d$, $\widehat{L}$, $\overline{K}$ and $\overline{K}_d$ of appropriate dimensions should be determined. The application of the control (4)-(6) to system (1)-(2) results in the following closed loop system:

$$\begin{aligned} z_{k+1} &= (A_0 + \delta A_0)z_k + (A_{d,0} + \delta A_{d,0})z_{k-\tau} + \\ &\qquad + E_0 w_k, \qquad k \in \mathbf{Z}^+ \qquad (7)\\ z_k &= \zeta_k, \quad k = 0,-1,\ldots,-\tau, \end{aligned}$$

where

$$\begin{aligned} z_k &= \begin{pmatrix} x_k \\ \widehat{x}_k \end{pmatrix}, \qquad k \in \mathbf{Z}^+,\\ \zeta_k &= \begin{pmatrix} \phi_k \\ 0 \end{pmatrix}, \qquad k = 0,-1,\ldots,-\tau,\\ A_0 &= \begin{pmatrix} \overline{A} & \overline{B}\,\overline{K} \\ \widehat{L}\,\overline{C} & \widehat{A} \end{pmatrix}, \quad E_0 = \begin{pmatrix} \overline{E} \\ 0 \end{pmatrix},\\ A_{d,0} &= \begin{pmatrix} \overline{A}_d & \overline{B}\,\overline{K}_d \\ \widehat{L}\,\overline{C}_d & \widehat{A}_d \end{pmatrix},\\ \delta A_0 &= H\widetilde{\Delta}A_q, \ \ \delta A_{d,0} = H\widetilde{\Delta}A_{q,d}, \end{aligned}$$

with

$$\begin{aligned} H &= \begin{pmatrix} H_x & 0 \\ 0 & \widehat{L}H_y \end{pmatrix}, \ A_q = \begin{pmatrix} \overline{A}_q & \overline{B}_q\overline{K} \\ \overline{C}_q & 0 \end{pmatrix},\\ \widetilde{\Delta} &= (I - \Delta G)^{-1}\Delta,\\ G &= \begin{pmatrix} G_x & 0 \\ 0 & G_y \end{pmatrix}, \ \Delta = \begin{pmatrix} \Delta^{(x)} & 0 \\ 0 & \Delta^{(y)} \end{pmatrix},\\ A_{d,q} &= \begin{pmatrix} \overline{A}_{d,q} & \overline{B}_{d,q}\overline{K}_d \\ \overline{C}_{d,q} & 0 \end{pmatrix}. \end{aligned}$$

The objective function for the closed-loop system (7), equivalent to the original one under the application of (4)-(6), can be expressed as

$$J_z(\mathbf{z}_0, \mathbf{w}_\infty) = \sum_{k=0}^{\infty} L_z(\mathbf{z}_k, w_k), \qquad (8)$$

where

$$\begin{aligned} L_z(\mathbf{z}_k, w_k) &= \begin{pmatrix} z_k^T, & z_{k-\tau}^T \end{pmatrix} Q \begin{pmatrix} z_k \\ z_{k-\tau} \end{pmatrix} - \\ &\qquad - w_k^T S w_k \end{aligned}$$

with

$$\begin{aligned} Q &= \begin{pmatrix} \Gamma \\ 0 \end{pmatrix} \overline{Q} \begin{pmatrix} \Gamma^T, & 0 \end{pmatrix} + \\ &\quad + \begin{pmatrix} K^T \\ K_d^T \end{pmatrix} R \begin{pmatrix} K, & K_d \end{pmatrix},\\ \Gamma^T &= \begin{pmatrix} I, & 0 \end{pmatrix},\\ K &= \begin{pmatrix} 0, & \overline{K} \end{pmatrix}, \ K_d = \begin{pmatrix} 0, & \overline{K}_d \end{pmatrix}. \end{aligned}$$

Definition 1 *Consider the uncertain system (1)-(2) with the cost function (3). A controller of the form (4)-(6) is said to be a guaranteeing cost output feedback controller for (1)-(2) with (3), if there exist positive definite symmetric matrices P_0 and P_d such that for the function*

$$V(\mathbf{z}_k) = z_k^T P_0 z_k + \sum_{i=1}^{\tau} z_{k-i}^T P_d\, z_{k-i} \qquad (9)$$

inequality

$$V(\mathbf{z}_{k+1}) - V(\mathbf{z}_k) + L_z(\mathbf{z}_k, w_k) < 0 \qquad (10)$$

holds true for all $k \in \mathbf{N}$, for any disturbance sequence $\mathbf{w}_\infty$ and any realization of the uncertainty, where $\mathbf{z}_k = \left(z_{k-\tau}^T, ..., z_k^T\right)^T$ and $\mathbf{z}_\infty$ is the solution of (7).

Remark 2 *Definition 1 is the generalization of that given in [8] inasmuch as it allows the appearance of delayed states in (7) and (8). It is well-known that, by augmentation, system (7) is equivalent to an undelayed discrete-time system with a state space of dimension $(\tau+1)n$. Thus the above mentioned definition could directly be used to the augmented system. However, to avoid the application of a $(\tau+1)n \times (\tau+1)n$ matrix as a quadratic cost matrix, function V is defined like a Lyapunov-Krasovskii function. This definition is analogous of those of [10], [14], [17] and [22]. Other papers as e.g. [1] and [7] accept a definition formulated directly by the objective function. The connections of these two approaches will be discussed below in corollary 6 and in remark 7.*

2.2 Input-to-state stability of discrete-time time-delay systems

Consider the general discrete-time time-delay system

$$\begin{aligned} x(k+1) &= f(x(k), x(k-\tau), w(k)), \qquad (11)\\ &\qquad k \in \mathbf{Z}^+,\\ x(k) &= \phi(k), \quad k = 0,-1,...,-\tau. \end{aligned}$$

Definition 3 *System (11) is (globally) input-to-state stable (ISS), if there exist a $\mathcal{KL}$-function $\beta : \mathbf{R}_{\geq 0} \times \mathbf{R}_{\geq 0} \to \mathbf{R}_{\geq 0}$ and a $\mathcal{K}$-function γ such that, for each $\mathbf{w} \in l_\infty$ and each $\underline{\phi} = \{\phi(-\tau), ..., \phi(0)\}$ it holds that*

$$\left\| x(k; \underline{\phi}, \mathbf{w}) \right\| \leq \beta(\left\| \underline{\phi} \right\|, k) + \gamma(\|\mathbf{w}\|_\infty) \quad (12)$$

for each $k \in \mathbf{Z}^+$, where $x(.; \underline{\phi}, \mathbf{w})$ denotes the solution of (11).

As it has already been mentioned, system (11) is equivalent to an augmented undelayed discrete-time system. This gives the possibility of a straightforward re-formulation of results in [9] on the ISS property of discrete-time systems for delayed discrete-time systems, if an ISS-Lyapunov function can be shown for the augmented system. However, sometimes this may be difficult. The problem comes from the fact that it is not easy to give an upper estimation for the forward difference of a Lyapunov function candidate along the solution, which contains a strictly negative definite term with respect to the *whole augmented state*. It can be seen that this is the case in the problem under consideration. The difficulty can be overcome, if the required estimation holds true with a certain part of the augmented state as it is given in the following lemma.

Lemma 4 *If there exist a continuous function $V : \mathbf{R}^{n(\tau+1)} \to \mathbf{R}_{\geq 0}$, two $\mathcal{K}_\infty$- functions α_1, α_2 and two positive constants α_3 and σ such that*

(i) $$\alpha_1 \left(\left\| \underline{\xi} \right\| \right) \leq V\left(\underline{\xi} \right) \leq \alpha_2 \left(\left\| \underline{\xi} \right\| \right) \quad \forall \underline{\xi} \in \mathbf{R}^{n(\tau+1)}$$

(ii) $$V(F(\underline{\xi}, w)) - V\left(\underline{\xi} \right) \leq -\alpha_3 \left\| \xi_0 \right\|^2 + \sigma \left\| w \right\|^2 \quad \forall \underline{\xi} \in \mathbf{R}^{n(\tau+1)},\ \forall w \in \mathbf{R}^s,$$

where

$$\underline{\xi}^T = \left(\xi_0^T, \xi_1^T, ..., \xi_\tau^T \right), \quad \xi_i \in \mathbf{R}^n,$$

$$F(\underline{\xi}, w)^T = \left(\xi_1^T, ..., \xi_\tau^T, f\left(\xi_\tau, \xi_0, w \right)^T \right),$$

then system (11) is ISS.

Proof. Let us consider the augmented system applied with an arbitrary initial state

$$\underline{\xi}^{(0)} \in \mathbf{R}^{n(\tau+1)}$$

and an arbitrary $(\tau + 1)$-tuple of disturbances $\{w(0), ..., w(\tau)\}$:

$$\begin{aligned} \underline{\xi}(k+1) &= F(\underline{\xi}(k), w(k)),\ k = 0, 1, ..., \tau. \\ \underline{\xi}(0) &= \underline{\xi}^{(0)} \end{aligned} \quad (13)$$

By property (ii) we have

$$\begin{aligned} &\sum_{k=0}^{\tau} [V(F(\underline{\xi}(k), w(k))) - V(\underline{\xi}(k))] \\ = &\ V(F(\underline{\xi}(\tau), w(\tau))) - V(\underline{\xi}(0)) \\ \leq &\ -\sum_{k=0}^{\tau} \alpha_3 \left\| \xi_0(k) \right\|^2 + \sum_{k=0}^{\tau} \sigma \left\| w(k) \right\|^2. \end{aligned} \quad (14)$$

Observe that, for $k = 0, ..., \tau$, the first n elements of $\underline{\xi}(k)$ denoted by $\xi_0(k)$ equals to $\xi_k^{(0)}$ and

$$\begin{aligned} &F(\underline{\xi}(\tau), w(\tau)) \\ = &\ (f(\xi_\tau(0), \xi_0(0), w(0))^T, ... \\ &\quad ..., f(\xi_\tau(\tau), \xi_0(\tau), w(\tau))^T)^T. \end{aligned}$$

In this way, a mapping

$$\mathcal{F} : \mathbf{R}^{n(\tau+1)} \times \mathbf{R}^{s(\tau+1)} \to \mathbf{R}^{n(\tau+1)}$$

can be defined so that for any $\zeta \in \mathbf{R}^{n(\tau+1)}$ and

$$\eta = (\eta_0^T, ..., \eta_\tau^T)^T \in \mathbf{R}^{s(\tau+1)},$$

the result of the recursion (13) for

$$\underline{\xi}^{(0)} = \zeta$$

and $w(k) = \eta_k$ is taken, and $\mathcal{F}(\zeta, \eta)$ is defined by

$$\mathcal{F}(\zeta, \eta) = F(\underline{\xi}(\tau), w(\tau)).$$

From (14) it follows that

$$V(\mathcal{F}(\zeta, \eta)) - V(\zeta) \leq -\alpha_3 \left\| \zeta \right\|^2 + \sigma \left\| \eta \right\|^2,$$

thus V is an ISS-Lyapunov function in the sense of Definition 3.2 in [9] for the discrete-time system

$$\zeta(k+1) = \mathcal{F}(\zeta(k), \eta(k)),\ \zeta(0) = \zeta^0. \quad (15)$$

Therefore, Lemma 3.5 of [9] gives that (15) is ISS. Since

$$\zeta_j(k) = x(k(\tau+1) + j - \tau)$$

and

$$\left\| \eta(.) \right\|_\infty \leq \sqrt{\tau+1} \left\| w(.) \right\|_\infty,$$

the required inequality (12) immediately follows.

3 Main results

In this section we propose a guaranteeing cost robust minimax strategy for system (1)-(2). Firstly, a necessary and a sufficient condition will be established for the controller (4)-(6) to be a guaranteeing cost robust minimax strategy for system (1)-(2).

3.1 A necessary and sufficient condition

Set

$$\widehat{G} = (I - G^T G)^{-1},\ \widehat{\widehat{G}} = (I - GG^T)^{-1}.$$

Introduce the following notations:

$$\begin{aligned} A_1 &= A_0 + H\widehat{G}G^T A_q, \\ A_{d,1} &= A_{d,0} + H\widehat{\widehat{G}}G^T A_{d,q}. \end{aligned}$$

Theorem 5 *The controller (4)-(6) is a guaranteeing cost output feedback for (1)-(2) with (3), if and only if there exist positive definite matrices P_0 and P_d and a positive constant ε such that*

$$\Psi = \begin{pmatrix} \Psi_{11} & * \\ \Psi_{21} & \Psi_{22} \end{pmatrix} < 0. \qquad (16)$$

where

$$\begin{aligned} \Psi_{11} &= \begin{pmatrix} P_d - P_0 & * & * & * \\ 0 & -P_d & * & * \\ 0 & 0 & -S & * \\ A_1 & A_{d,1} & E_0 & -P_0^{-1} \end{pmatrix} \\ \Psi_{21} &= \begin{pmatrix} A_q & A_{d,q} & 0 & 0 \\ 0 & 0 & 0 & H^T \\ \Gamma^T & 0 & 0 & 0 \\ K & K_d & 0 & 0 \end{pmatrix} \\ \Psi_{22} &= diag\left\{ -\frac{1}{\varepsilon^2}(\widehat{\widehat{G}})^{-1}, \right. \\ & \left. -\varepsilon^2(\widehat{G})^{-1},\ -\overline{Q}^{-1},\ -R^{-1} \right\}. \end{aligned}$$

Proof. Set

$$\begin{aligned} \widehat{P} &= P_d - P_0 + \Gamma\overline{Q}\Gamma^T + KRK^T, \\ \widehat{K}_d &= K_d R K_d^T - P_d. \end{aligned}$$

By simple substitution from (7), we obtain that (10) can equivalently be written as

$$\begin{aligned} 0 &> V(\mathbf{z}_{k+1}) - V(\mathbf{z}_k) + L(\mathbf{z}_k, w_k) = \\ & z_{k+1}^T P_0 z_{k+1} + z_k^T(P_d - P_0)z_k + \\ & + \begin{pmatrix} z_k^T, & z_{k-\tau}^T \end{pmatrix} Q \begin{pmatrix} z_k \\ z_{k-\tau} \end{pmatrix} - \\ & z_{k-\tau}^T P_d z_{k-\tau} - w_k^T S w_k \\ &= \left(z_k^T,\ z_{k-\tau}^T,\ w_k^T\right) \left[\begin{pmatrix} A_0^T + \delta A_0^T \\ A_d^T + \delta A_d^T \\ E_0^T \end{pmatrix} P_0 \right. \\ & \times (A_0 + \delta A_0,\ A_d + \delta A_d,\ E_0) + \\ & + \begin{pmatrix} P_d - P_0 & 0 & 0 \\ 0 & -P_d & 0 \\ 0 & 0 & -S \end{pmatrix} + \\ & \left. \begin{pmatrix} \Gamma\overline{Q}\Gamma^T + KRK^T & KRK_d^T & 0 \\ K_d R K^T & K_d R K_d^T & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \\ & \begin{pmatrix} z_k \\ z_{k-\tau} \\ w_k \end{pmatrix}, \end{aligned} \qquad (17)$$

i.e. the matrix in the square brackets is negative definite. By Schur complement this holds true if and only if

$$\begin{pmatrix} \widehat{P} & * & * & * \\ K_d R K^T & \widehat{K}_d & * & * \\ 0 & 0 & -S & * \\ A_0 + \delta A_0 & A_d + \delta A_d & E_0 & -P_0^{-1} \end{pmatrix} < 0. \qquad (18)$$

Substituting the definition of δA_0, $\delta A_{d,0}$, we obtain that

$$\begin{aligned} 0 > & \begin{pmatrix} \widehat{P} & * & * & * \\ K_d R K^T & \widehat{K}_d & * & * \\ 0 & 0 & -S & * \\ A_0 & A_d & E_0 & -P_0^{-1} \end{pmatrix} + \\ & \begin{pmatrix} 0 \\ 0 \\ 0 \\ H \end{pmatrix} \widetilde{\Delta} \begin{pmatrix} A_q, & A_{d,q}, & 0, & 0 \end{pmatrix} + \\ & \begin{pmatrix} A_q^T \\ A_{d,q}^T \\ 0 \\ 0 \end{pmatrix} \widetilde{\Delta} \begin{pmatrix} 0, & 0, & 0, & H^T \end{pmatrix}. \end{aligned} \qquad (19)$$

Applying Lemma 2.6. of [17], (19) holds if and only if there is a positive constant ε such that

$$0 > \begin{pmatrix} \widehat{P} & * & * & * \\ K_d R K^T & \widehat{K}_d & * & * \\ 0 & 0 & -S & * \\ A_0 & A_d & E_0 & -P_0^{-1} \end{pmatrix} +$$
$$\begin{pmatrix} \frac{1}{\varepsilon^2}\begin{pmatrix} A_q^T \\ A_{d,q}^T \\ 0 \end{pmatrix} \widehat{\widehat{G}} \begin{pmatrix} A_q, & A_{d,q}, & 0 \end{pmatrix} & * \\ H\widehat{G}G^T \begin{pmatrix} A_q, & A_{d,q}, & 0 \end{pmatrix} & \varepsilon^2 H\widehat{G}H^T \end{pmatrix}. \tag{20}$$

By taking into account the definition of A_1 and $A_{d,1}$, inequality (20) is obviously equivalent to

$$0 > \begin{pmatrix} P_d - P_0 & * & * & * \\ 0 & -P_d & * & * \\ 0 & 0 & -S & * \\ A_1 & A_{d,1} & E_0 & -P_0^{-1} \end{pmatrix} +$$
$$\begin{pmatrix} A_q^T & 0 & \Gamma & K^T \\ A_{d,q}^T & 0 & 0 & K_d^T \\ 0 & 0 & 0 & 0 \\ 0 & H & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\varepsilon^2}\widehat{\widehat{G}} & 0 & 0 & 0 \\ 0 & \varepsilon^2 \widehat{G} & 0 & 0 \\ 0 & 0 & \overline{Q} & 0 \\ 0 & 0 & 0 & R \end{pmatrix} (*),$$

from which (16) can be obtained applying the Schur complement again.

Remark 6 *If the delay τ is unknown but constant, then (4)-(6) is not well-defined, and V depends also on the unknown parameter τ. However, if a memoryless dynamic feedback is applied, i.e. $\widehat{A}_d = 0$ and $\overline{K}_d = 0$ are chosen, then the proposed approach is applicable, and theorem 5 remains valid, since (16) doesn't depend on the delay, though the above restriction may give more conservative results.*

Corollary 7 *If (4)-(6) is a guaranteeing cost output feedback controller for (1)-(2) with (3), then (7) is ISS and*

$$\sup_{\mathbf{w}_\infty \in l_2} J_z(\mathbf{z}_0, \mathbf{w}_\infty) \le V(\mathbf{z}_0). \tag{21}$$

Proof. Under the condition of the corollary, the strict inequality (16) holds true. But then there exists a $\mu > 0$ so that (16) remains valid if $\mu\widetilde{\Gamma}\widetilde{\Gamma}^T$ is added to the right hand side of the inequality, where $\widetilde{\Gamma}^T = (0, I, 0, 0, 0, 0, 0, 0)$. Let us denote a corresponding modification of L_z by

$$\widetilde{L}_z(\mathbf{z}_k, w_k) =$$
$$= \begin{pmatrix} z_k^T, & z_{k-\tau}^T \end{pmatrix} \begin{pmatrix} \overline{Q} & 0 \\ 0 & \mu I \end{pmatrix} \begin{pmatrix} z_k \\ z_{k-\tau} \end{pmatrix} -$$
$$- w_k^T S w_k.$$

Going backward along the equivalent relations, from the modified inequality it follows that

$$V(\mathbf{z}_{k+1}) - V(\mathbf{z}_k) + \widetilde{L}_z(\mathbf{z}_k, w_k) < 0 \tag{22}$$

holds true for all $k \in \mathbf{N}$, for all disturbance sequences $\mathbf{w}_\infty$ and for any realization of the uncertainty. Then (22) involves that the conditions of Lemma 4 are satisfied with

$$\alpha_1(s) = \min\{\lambda_m(P_0), \lambda_m(P_d)\}s^2,$$
$$\alpha_2(s) = \max\{\lambda_M(P_0), \lambda_M(P_d)\}s^2,$$

$\alpha_3 = \mu$ and $\sigma = \lambda_M(S)$, thus the ISS property follows from Lemma 4.

On the other hand, summing up (10) for $k = 0, ..., N$ we have that

$$\sum_{k=0}^{N} L_z(z_k, w_k) \le V(\mathbf{z}_0).$$

Since $\sum_{k=0}^{\infty} w_k^T S w_k$ is finite for any $\mathbf{w}_\infty \in l_2$, $J_z(\mathbf{z}_0, \mathbf{w}_\infty)$ is finite as well, and (21) holds true.

Remark 8 *Since $J_z(\mathbf{z}_0, \mathbf{w}_\infty)$ and $J(\mathbf{x}_0, \mathbf{u}_\infty, \mathbf{w}_\infty)$ are identical, if $\mathbf{u}_\infty$ is generated by (4)-(6), corollary 6 shows that the guaranteeing cost output feedback controller yields a cost bounded by*

$$V(\mathbf{z}_0) = \widetilde{V}(\mathbf{x}_0)$$

independently from the disturbance $\mathbf{w}_\infty \in l_2$ and the uncertainty. We note that one can allow any bounded disturbance sequence to have the ISS property. At the same time, in the investigation of the cost function over an infinite horizon, one has to restrict the attention to output feedback controllers (4)-(6), which ensure that the objective functional is well defined for a class of admissible disturbances in the sense that the cost value belongs to $\mathbf{R} \cup \{+\infty\} \cup \{-\infty\}$. For any $\mathbf{w}_\infty \in l_2$ the corresponding cost value

$$J_z(\mathbf{z}_0, \mathbf{w}_\infty) \in \mathbf{R} \cup \{+\infty\} \cup \{-\infty\},$$

thus l_2 is a suitable class of admissible disturbances.

Remark 9 *If τ is unknown, then $V(\mathbf{z}_0)$ in (21) cannot be computed. Nevertheless, if $0 < \tau \le \overline{\tau}$ with given $\overline{\tau}$, we have that*

$$V(\mathbf{z}_0) \le \overline{V}(\mathbf{z}_0) := z_0^T P_0 z_0 + \sum_{i=1}^{\overline{\tau}} z_{-i}^T P_d z_{-i}$$

and $\overline{V}(\mathbf{z}_0)$ is a computable upper bound for the cost function.

3.2 Derivation of an LMI

Matrix inequality (16) is clearly nonlinear. On the basis of the approach of [6], an LMI will be shown, the solution of which is also a solution of (16). This means that an LMI can be given for the unknowns $\overline{K}$, $\overline{K}_d$, $\widehat{L}$, $\widehat{A}$, $\widehat{A}_d$, P_0 and P_d, which yields a sufficient condition for (4)-(6) to be a guaranteeing cost output feedback controller. This is established by the next theorem. To formulate it, introduce further notation as follows. Set

$$\Pi = P_0^{-1} P_d P_0^{-1},$$

and partition matrices P_0 and P_0^{-1} into $n \times n$ blocks as

$$P_0 = \begin{pmatrix} X & M \\ M^T & Z \end{pmatrix}, \quad P_0^{-1} = \begin{pmatrix} Y & N \\ N^T & W \end{pmatrix}.$$

Introduce matrices

$$F_1 = \begin{pmatrix} X & I \\ M^T & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} I & Y \\ 0 & N^T \end{pmatrix},$$

and

$$\widetilde{\Pi} = F_1^T \Pi F_1.$$

Set furthermore

$$\begin{aligned}
\overline{A}_1 &= \overline{A} + H_x(I - G_x^T G_x)^{-1} G_x^T \overline{A}_q, \\
\overline{A}_{d,1} &= \overline{A}_d + H_x(I - G_x^T G_x)^{-1} G_x^T \overline{A}_{d,q}, \\
\overline{B}_1 &= B + H_x(I - G_x^T G_x)^{-1} G_x^T \overline{B}_q, \\
\overline{C}_1 &= \overline{C} + H_y(I - G_y^T G_y)^{-1} G_y^T \overline{C}_q, \\
\overline{C}_{d,1} &= \overline{C}_d + H_y(I - G_y^T G_y)^{-1} G_y^T \overline{C}_{d,q},
\end{aligned}$$

and

$$\widetilde{K} = \overline{K} N^T, \quad \widetilde{K}_d = \overline{K}_d N^T, \quad \widetilde{L} = M \widehat{L}, \tag{23}$$

$$\widetilde{A} = X \overline{A}_1 Y + X \overline{B}_1 \widetilde{K} + \widetilde{L} \overline{C}_1 Y + M \widehat{A} N^T, \tag{24}$$

$$\widetilde{A}_d = X \overline{A}_{d,1} Y + X \overline{B}_1 \widetilde{K}_d + \widetilde{L} \overline{C}_{d,1} Y + M \widehat{A}_d N^T. \tag{25}$$

Theorem 10 *Let $\varepsilon > 0$ be a fixed positive constant. If the LMI*

$$\Omega = \begin{pmatrix} \Omega_{11} & * & * & * \\ \Omega_{21} & \Omega_{22} & * & * \\ 0 & \Omega_{32} & \Omega_{33} & * \\ \Omega_{41} & 0 & 0 & \Omega_{44} \end{pmatrix} < 0, \tag{26}$$

holds for X, Y, $\widetilde{\Pi}$, $\widetilde{K}$, $\widetilde{K}_d$, $\widetilde{L}$, $\widetilde{A}$ and $\widetilde{A}_d$, where

$$\Omega_{11} = \begin{pmatrix} \widetilde{\Pi} - \begin{pmatrix} X & I \\ I & Y \end{pmatrix} & 0 \\ 0 & -\widetilde{\Pi} \end{pmatrix},$$

$$\Omega_{22} = \begin{pmatrix} -S & \overline{E}^T X & \overline{E}^T & 0 \\ X\overline{E} & -X & -I & 0 \\ \overline{E} & -I & -Y & 0 \\ 0 & 0 & 0 & -\varepsilon^{-2}(\widehat{\widehat{G}})^{-1} \end{pmatrix},$$

$$\Omega_{33} = -\varepsilon^2 (\widehat{G})^{-1},$$

$$\Omega_{44} = \begin{pmatrix} -\overline{Q}^{-1} & 0 \\ 0 & -R^{-1} \end{pmatrix},$$

$$\Omega_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \psi_{21} & \widetilde{A} & \psi_{23} & \widetilde{A}_d \\ \overline{A}_1 & \psi_{32} & \overline{A}_{d,1} & \psi_{34} \\ \overline{A}_q & \psi_{42} & \overline{A}_{d,q} & \psi_{44} \\ \overline{C}_q & \overline{C}_q Y & \overline{C}_{d,q} & \overline{C}_{d,q} Y \end{pmatrix},$$

$$\begin{aligned}
&\psi_{21} = X\overline{A}_1 + \widetilde{L}\overline{C}_1, \ \psi_{23} = X\overline{A}_{d,1} + \widetilde{L}\overline{C}_{d,1}, \\
&\psi_{32} = \overline{A}_1 Y + \overline{B}_1 \widetilde{K}, \ \psi_{34} = \overline{A}_{d,1} Y + \overline{B}_1 \widetilde{K}_d, \\
&\psi_{42} = \overline{A}_q Y + \overline{B}_q \widetilde{K}, \ \psi_{44} = \overline{A}_{d,q} Y + \overline{B}_q \widetilde{K}_d,
\end{aligned}$$

$$\Omega_{32} = \begin{pmatrix} 0 & H_x^T X & H_x^T & 0 \\ 0 & H_y^T \widetilde{L} & 0 & 0 \end{pmatrix},$$

$$\Omega_{41} = \begin{pmatrix} I & Y & 0 & 0 \\ 0 & K & 0 & K_d \end{pmatrix},$$

then a guaranteeing cost output feedback controller for (1)-(2) with (3) can be expressed in the form of (4)-(6) by solving (23)-(25).

Proof. Fix ε in (16). Apply the congruence transformation

$$diag\{P_0^{-1}, \, P_0^{-1}, \, I, I, \, I, \, I, \, I, \, I\}$$

to (16), then multiply the obtained inequality by

$$diag\{F_1, \, F_1, \, I \; F_1, \, I, \, I, \, I, \, I\}$$

from the right and by its transpose from the left. Let us compute the blocks of Ω. Block Ω_{11} can immediately be obtained by observing that

$$F_1^T P_0^{-1} F_1 = \begin{pmatrix} X & I \\ I & Y \end{pmatrix}. \tag{27}$$

Block Ω_{22} is received from the third to the fifth rows and columns of Ψ in (16):

$$\Omega_{22} = \begin{pmatrix} -S & * & * \\ F_1^T E_0 & -F_1^T P_0^{-1} F_1 & * \\ 0 & 0 & -\varepsilon^{-2}(I - GG^T) \end{pmatrix}.$$

Here

$$F_1^T E_0 = \begin{pmatrix} X\overline{E} \\ \overline{E} \end{pmatrix},$$

thus by (27), block Ω_{22} is verified. Since the last three block rows and columns are multiplied by $diag\{I, I, I\}$, Ω_{33} and Ω_{44} are obviously valid. To compute Ω_{21}, we observe that $P_0^{-1} F_1 = F_2$. Thus

$$\Omega_{21} = \begin{pmatrix} 0 & 0 \\ F_1^T A_1 F_2 & F_1^T A_{d,1} F_2 \\ A_q F_2 & A_{d,q} F_2 \end{pmatrix}.$$

By substitution we obtain that

$$\begin{aligned} F_1^T A_1 F_2 = {} & \begin{pmatrix} X & M \\ I & 0 \end{pmatrix} \times \\ & \times \begin{pmatrix} \overline{A}_1 & \overline{B}_1 \overline{K} \\ \widehat{L}\overline{C}_1 & \widehat{A} \end{pmatrix} \begin{pmatrix} I & Y \\ 0 & N^T \end{pmatrix} \\ = {} & \begin{pmatrix} \psi_{21} & \widetilde{A} \\ \overline{A}_1 & \psi_{32} \end{pmatrix}, \end{aligned}$$

and analogously

$$\begin{aligned} F_1^T A_{d,1} F_2 &= \begin{pmatrix} \psi_{23} & \widetilde{A}_d \\ \overline{A}_{d,1} & \psi_{34} \end{pmatrix}, \\ A_q F_2 &= \begin{pmatrix} \overline{A}_q & \psi_{42} \\ \overline{C}_q & \overline{C}_q Y \end{pmatrix}, \\ A_{d,q} F_2 &= \begin{pmatrix} \overline{A}_{d,q} & \psi_{44} \\ \overline{C}_{d,q} & \overline{C}_{d,q} Y \end{pmatrix}, \end{aligned}$$

which gives the required expression of Ω_{21}. Furthermore,

$$\Omega_{32} = (0, H^T F_1, 0),$$

and

$$H^T F_1 = \begin{pmatrix} H_x^T X & H_x^T \\ H_y^T \widetilde{L}^T & 0 \end{pmatrix},$$

thus Ω_{32} is also verified. Finally,

$$\begin{aligned} \Omega_{41} &= \begin{pmatrix} \Gamma^T & 0 \\ K & K_d \end{pmatrix} \begin{pmatrix} F_2 & 0 \\ 0 & F_2 \end{pmatrix} \\ &= \begin{pmatrix} I & Y & 0 & 0 \\ 0 & \overline{K} & 0 & \overline{K}_d \end{pmatrix}, \end{aligned}$$

because

$$\begin{pmatrix} I, & 0 \end{pmatrix} F_2 = \begin{pmatrix} I, & Y \end{pmatrix},$$
$$\begin{pmatrix} 0, & K \end{pmatrix} F_2 = \begin{pmatrix} 0, & \overline{K} \end{pmatrix}.$$

Thus for a fixed $\varepsilon > 0$, (16) follows from the LMI (26). As far as the solvability of (23)-(25) is concerned, we observe that from (26) it follows that

$$\begin{pmatrix} X & I \\ I & Y \end{pmatrix} > 0,$$

thus $I - XY$ is invertible. Because of the definition,

$$MN^T = I - XY,$$

therefore M and N are also invertible, and they can be determined e.g. by singular value factorization of $I - XY$. Thus the system of matrix equations (23)-(25) can subsequently be solved for the required matrices $\overline{K}$, $\overline{K}_d$, $\widehat{L}$, $\widehat{A}$ and $\widehat{A}_d$.

Remark 11 *Observe that matrices P_0 and P_d are simultaneously determined here by the LMI (26) fixing ε. Several papers (see e.g. [11]) proposed an LMI method for systems without exogenous disturbances, where ε was set to 1 and P_d was also fixed.*

Remark 12 *We note that in the case of unknown delay, the dynamic output feedback (4)-(6) had to be considered with $\widehat{A}_d = 0$ and $\overline{K}_d = 0$. However in the derivation of (26), it was crucial to introduce $\widetilde{A}_d$ defined by equation (25) as a new unknown. This equation may not hold, if $\widehat{A}_d = 0$ is fixed. Therefore, the approach of this paper is not suitable to reduce the construction of a guaranteeing cost dynamic output feedback controller to the solution of an LMI, if the delay is not given.*

Remark 13 *From the proof of Theorem 10 one can see that inequality (16) can be transformed into a bilinear one in the variables $\overline{K}$, $\overline{K}_d$, $\widehat{L}, P_0$, P_d and ε. However, the solution of BMIs requires more sophisticated tools (see e.g. [12]).*

Theorem 10 provides a constructive method to find an appropriate control. There are efficient methods and software tools to find a feasible solution. Since the main purpose is to find a guaranteed cost, it is expedient to assign an objective function to the LMI, which assures as low guaranteed cost as possible. Partition $\widetilde{\Pi}$ into $n \times n$ blocks as follows:

$$\widetilde{\Pi} = \begin{pmatrix} \widetilde{\Pi}_{11} & \widetilde{\Pi}_{12} \\ \widetilde{\Pi}_{12}^T & \widetilde{\Pi}_{22} \end{pmatrix}.$$

The upper bound of the system's performance is given by (9), (21). Taking into account the initial value of $\widehat{\mathbf{x}}_0$, an upper bound U_b for the guaranteed cost value can be given by the upper left blocks of P_0 and P_d, which are X and $\widetilde{\Pi}_{11}$, respectively. Thus

$$U_b = \lambda_M(X)\,\|x_0\|^2 + \lambda_M(\widetilde{\Pi}_{11})\sum_{j=1}^{\tau}\|x_{-j}\|^2 .$$

To obtain a relatively low value of U_b, we consider two additional variables ω_1, and ω_2, add the LMIs

$$\omega_1 I > X, \qquad \omega_2 I > \widetilde{\Pi}_{11}$$

to LMI (26), and minimize the objective function

$$\theta\omega_1 + (1-\theta)\omega_2$$

with respect to the resulted in new LMI system, where $0 \le \theta \le 1$ is a given constant. To solve the problem with a fixed nonzero initial state, it can be chosen e.g. as

$$\theta = \|x_0\| \,/\, (\|x_0\| + ... + \|x_{-\tau}\|) .$$

4 A numerical example

Consider the dynamical system (1)-(3) with the following parameters: let $\tau = 5$,

$$\begin{aligned}
\overline{A} &= \begin{pmatrix} 1.2 & 0 \\ -1.2 & 0 \end{pmatrix}, \ \overline{A}_d = \begin{pmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{pmatrix}, \\
\overline{B} &= \begin{pmatrix} 1 \\ 0.01 \end{pmatrix}, \ \overline{E} = \begin{pmatrix} 0.01 \\ 0.01 \end{pmatrix}, \\
\overline{C} &= \begin{pmatrix} 1 & 0 \end{pmatrix}, \ \overline{C}_d = \begin{pmatrix} 0.05 & 0.05 \end{pmatrix}, \\
H_x &= \begin{pmatrix} 0.2 & 0.1 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}, \ H_y = 0.1, \\
\overline{A}_q &= \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \ \overline{A}_{d,q} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}, \\
\overline{B}_q &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \\
\phi_k &= \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad k = 0, -1, ..., -\tau, \\
\overline{C}_q &= \begin{pmatrix} 0.05 & 0.04 \end{pmatrix}, \ \overline{C}_{d,q} = \begin{pmatrix} 0 & 0 \end{pmatrix}, \\
G_x &= \begin{pmatrix} 0.01 & 0.02 & 0 \\ 0 & 0.04 & 0 \\ 0.01 & 0.02 & 0.05 \end{pmatrix}, \ G_y = 0.01,
\end{aligned}$$

and consider the following matrices in the performance index:

$$Q = 0.1 I_2, \quad R = 0.1, \quad S = 1.$$

Solving (26) for $\varepsilon = 1$ with the proposed objective function one obtains

$$\begin{aligned}
\widehat{A} &= \begin{pmatrix} -0.8105 & 0.1324 \\ 1.2772 & -0.2082 \end{pmatrix}, \\
\widehat{A}_d &= \begin{pmatrix} 0.0275 & 0.1060 \\ 0.0581 & 0.0534 \end{pmatrix}, \\
\widehat{L} &= \begin{pmatrix} 6.8764 \\ -5.2473 \end{pmatrix}, \\
K &= \begin{pmatrix} -0.1252 & 0.0221 \end{pmatrix}, \\
K_d &= \begin{pmatrix} -0.0019 & 0.0478 \end{pmatrix}, \\
\omega_1 &= 12.915, \ \ \omega_2 = 4.1596, \\
P_0 &= \begin{pmatrix} 12.2348 & * & * & * \\ 2.3827 & 4.3278 & * & * \\ -1.3848 & 0.6952 & 0.5861 & * \\ 0.3420 & 0.6813 & 0.1899 & 0.2496 \end{pmatrix}, \\
P_d &= \begin{pmatrix} 2.6126 & * & * & * \\ 1.8960 & 1.8271 & * & * \\ -0.0349 & 0.0295 & 0.0195 & * \\ 0.3338 & 0.2153 & 0.0090 & 0.0874 \end{pmatrix},
\end{aligned}$$

$\lambda_M(P_0) = 13.0230$, $\lambda_M(P_d) = 4.1941$ and the guaranteed cost with the given initial states is 15.0360. The state responses of above system with $\Delta^{(x)} = I, \quad \Delta^{(y)} = I$ are given in figure 1.

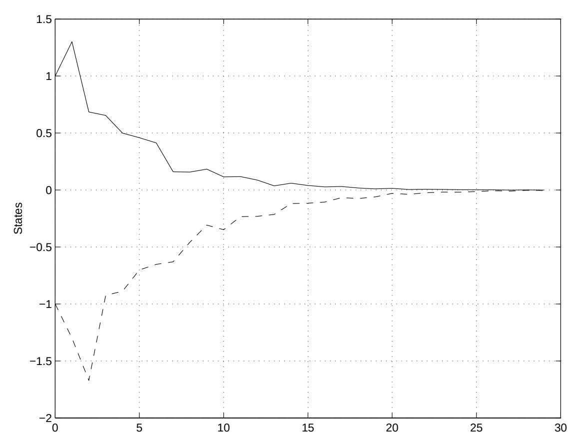

Figure 1: State responses of the closed-loop system.

5 Conclusions

This paper dealt with the construction of guaranteeing cost output feedback controller for linear uncertain discrete-time time-delay systems. The uncertainty involved time-varying parameter uncertainties of linear-fractional form and external disturbances. A necessary and sufficient condition for a controller to be a guaranteeing cost output feedback was formulated in terms of a (nonlinear) matrix inequality. A sufficient condition in terms of a linear matrix inequality was also given, which could effectively be solved. A numerical example illustrated the proposed method.

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