Linear Time-Varying Systems: Theory and Identification of Model Parameters

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Abstract: A strategy is proposed to model the complex industrial systems using linear time-varying system ($LTVS$). The proposed methodology is independent of model structure and the model may take any classic linear structure such as finite impulse response, input-output relation structures etc. To take into account the error between system and model due to model order reduction, variation of system behavior in time and perturbations, model's parameters are considered varying but bounded variables characterized by intervals. The output of this model is characterized by a function of the piecewise linear parameters which contains all possible system's responses taking into account modeling error as well as the perturbations.

Key–Words: Time-Varying System, Identification of Model Parameters, Interval Analysis.

1 Introduction

Description of complex industrial processes generally leads to mathematical models of very large orders. Examples of these processes are mobile arc welding robot ($MAWR$) or wood cutting system [7, 24, 25, 26]. These models are very time-consuming from processing point of view. Moreover, from an engineering point of view, one is more interested in treating a simpler and consequently less accurate mathematical model rather than a complex and accurate one. In this objective, mathematical model simplification is usually performed using model reduction methods [6, 12, 16, 22]. Having been simplified, model describes system's behavior in a less accurate manner and hence, there is generally a difference between observed and estimated values which is called *modeling error*. This error can also be due to improper model structure, inadequate parameter identification, the variation of system's behavior in time, etc.

Handling modeling error is among the most challenging problems in almost all identification procedures. This matter is more important when the model is developed for critical applications in which modeling error should be reduced as much as possible. Classical method to handle modeling error is the probabilistic approach in which model's parameters are constant scalars and modeling error is characterized by means of a certain *Probability Density Function* (PDF). However, it is not always possible to characterize modeling error by a certain PDF. Moreover, the properties of PDF may change in different calculation steps specially if one uses iterative algorithms.

Another alternative is set-membership approach in which model's parameters are supposed to be unwell-known (uncertain) or time-variant but bounded. Uncertain bounded parameters are then characterized by intervals [14, 18, 18, 5]. If the parameters are correctly characterized, it is then guaranteed that the model is able to determine all possible system's responses. This fundamental property of the set-membership approach is the main motivation to explore it to describe dynamic systems in critical industrial applications in which one needs guaranteed results.

A methodological approach independent of model structure is proposed in this paper to characterize the parameters of linear time-varying model ($LTVM$) which is then implemented to $MAWR$ system. After explaining the system under study, proposed methodology is explained in sections 3, 4 and 5. In section 6, numerical results of parameter characterization of the $MAWR$ system are given.

2 Mobile Arc Welding Robot System

This system is a relatively new application of robotics, even though robots were first introduced during the 1960s. Growth is primarily limited by high equipment costs, and the resulting restriction to high-production applications. Arc welding robot has begun growing quickly just recently, and already it commands about 20% of industrial robot applications. The major components of arc welding robots are the manipulator or the mechanical unit and the controller which are shown in Figs 1, 2 and 3.

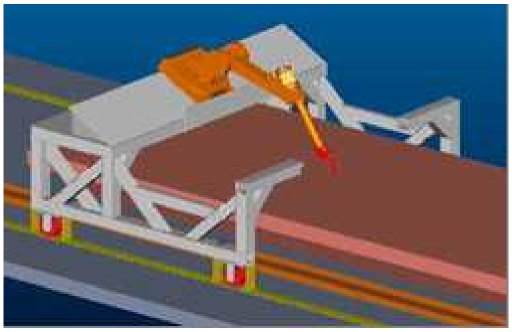

Figure 1: Mobile welding robot arm

Figure 2: Mobile platform equipments

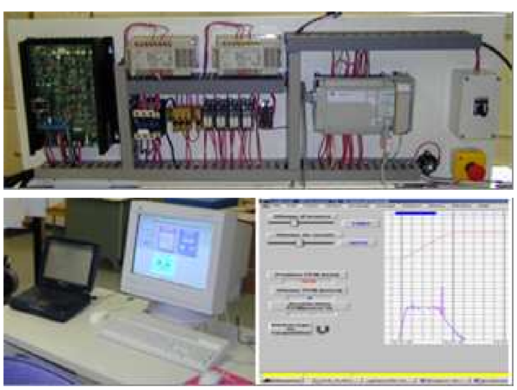

Figure 3: Data acquisition equipments

Representing this system mathematically leads to a high order model [24, 26]. Reducing the model to a second or to a first order increases modeling error. This error which is shown in Fig. 4, can also be due to wide variation of mechanical and physical properties of the system such as changing the static friction between the wheels and the rail, the unbalance platform gravity affected by changing the robot configuration and the variable platform mass due to the electrode systems.

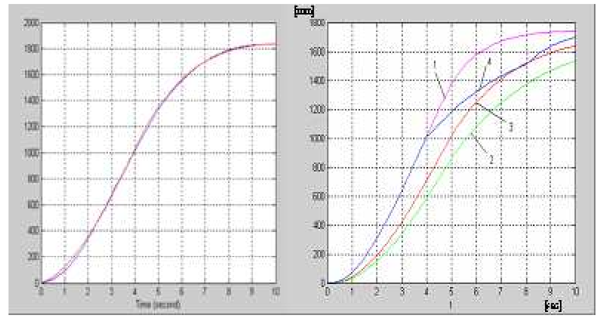

Figure 4: Measured system response without (Left and curve 2) and with perturbations (curves 1,3,4)

To collect data for parameter identification, a microcomputer-based data acquisition equipment shown in Fig. 3 is used which provides to us set $DS_o = \{u_{o,k}, y_{o,k}\}$ in which $u_{o,k}$ and $y_{o,k}$ represent respectively system's input and response.

3 Interval Analysis

As much as we know, Archimedes was of the early pioneers who used bounded numbers in his work to calculate π [8]. In new age and in the beginning of the 20^{th} century, the concept of the bounded value functions was discussed in [30] and a formal algebra of multi-value numbers and interval analysis were developed in [29, 28, 23] and [14] respectively. Interval analysis has also found its place in engineering [10] and especially in control engineering [9, 13, 27].

Definition 1 *Interval* $[x] = [\underline{x}, \overline{x}]$ *is a closed set of convex and continuous real numbers defined by lower bound* $\underline{x} \in \Re$ *and upper bound* $\overline{x} \in \Re$.

Any uncertain variable $x \in \Re$ whose true value is not known can be characterized by interval $[\underline{x}, \overline{x}]$ such that $\underline{x} \leq x \leq \overline{x}$.[1] Operations on intervals are also defined in such way that the resulting interval always contains the true result that would be obtained by using exact inputs and exact calculations.

Theorem 1 (Interval arithmetic operations) [15] *In interval arithmetic*

1. *For all intervals,*

$$-[x] = [-\overline{x}, -\underline{x}]$$

[1] During this work, x (respect. X) is a real-valued variable (respect. a real-valued vector) and $[x]$ (respect. $[X]$) is an interval (respect. interval vector).

2. *For* $\diamond \in \{+, -, \times, \div\}$, *if* $(x \diamond y)$ *is defined for all* $x \in [x]$ *and* $y \in [y]$, *we have:*

$$[x] \diamond [y] = [\min(\underline{x} \diamond \underline{y}, \overline{x} \diamond \overline{y}, \underline{x} \diamond \overline{y}, \overline{x} \diamond \underline{y}),\\ \max((\underline{x} \diamond \underline{y}, \overline{x} \diamond \overline{y}, \underline{x} \diamond \overline{y}, \overline{x} \diamond \underline{y}))]$$

3. *For monotone function* ξ,

$$\xi([x]) = [\min(\xi(\underline{x}), \xi(\overline{x})), \max(\xi(\underline{x}), \xi(\overline{x}))]$$

where

$$\xi([x]) = \{\xi(x) | \forall x \in [x]\}$$

In arithmetic expressions and real functions, one can replace the variables with intervals and evaluate the resulting expressions using interval arithmetics.

Definition 2 $[f]([x])$ *is defined as interval extension of real function* $f(x)$ *by replacing real argument* x *by interval* $[x]$ *and real arithmetic operations by their interval counterparts.*

Theorem 2 (Inclusion property) [15] *Suppose that the arithmetic expression* $f(z_1, ..., z_n)$ *can be evaluated at* $z_1, ..., z_n \in \Re$, *and let*

$$[x_1] \subseteq [z_1], ..., [x_n] \subseteq [z_n]$$

Then:

1. f *can be evaluated at* $[x_1], ..., [x_n]$ *and*

$$[f]([x_1], ..., [x_n]) \subseteq [f]([z_1], ..., [z_n])$$

2. $f([z_1], ..., [z_n]) \subseteq [f]([z_1], ..., [z_n])$

The former is called the *inclusion isotonicity* property and the latter, the *range inclusion* property.

Any interval can also be described in the normalized form. Mathematically stated:

$$[x] = [\overline{x}, \underline{x}] = x_c + \lambda_x \times [-1, 1]$$
$$mid([x]) = x_c = \tfrac{\overline{x}+\underline{x}}{2},\; rad([x]) = \lambda_x = \tfrac{\overline{x}-\underline{x}}{2} \geq 0$$

where x_c is called the *midpoint* and λ_x is called the *radius* of interval $[x]$. Normalized interval form substitutes original one to simplify interval operations by eliminating $\min(.)$ and $\max(.)$ functions from calculations [19].

Definition 3 *Interval vector* $[X]$ *is the counterpart of vector* X *whose entries are intervals; that is to say:*

$$[X]^T = [[x_1], [x_2], ..., [x_n]]$$

An interval vector can also be described in the normalized form. In this case, X_c is the vector of midpoints and λ_X is the vector of radiuses of the entries of interval vector $[X]$:

$$[X] = X_c + \lambda_X . * [\upsilon]$$

where

$$X_c = \begin{bmatrix} x_{c,1} \\ \vdots \\ x_{c,n} \end{bmatrix}, \lambda_X = \begin{bmatrix} \lambda_{x_1} \\ \vdots \\ \lambda_{x_n} \end{bmatrix}, [\upsilon] = \begin{bmatrix} [-1, 1] \\ \vdots \\ [-1, 1] \end{bmatrix}$$

Symbol $.*$ represents entry-by-entry product of two vectors.

Remark 1 *A vector with scalar entries determines a point in space* $\Re^n$ *whereas a vector of intervals represents a hypercube in this space.*

As mentioned before, normalized form facilitates arithmetic operations. In which follows, one will need to calculate the resulting interval of multiplying two interval vectors. Using this form, we obtain [19]:

$$\begin{aligned}[z] &= [X]^T \times [Y] \\ &= (X_c + \lambda_X . * [\upsilon])^T \times (Y_c + \lambda_Y . * [\upsilon])\end{aligned}$$

where

$$\begin{aligned}\overline{z} &= X_c^T Y_c + |X_c^T| \lambda_Y + \lambda_X^T |Y_c| + \lambda_X^T \lambda_Y \\ \underline{z} &= X_c^T Y_c - |X_c^T| \lambda_Y - \lambda_X^T |Y_c| - \lambda_X^T \lambda_Y\end{aligned} \tag{1}$$

4 Linear Time-Varying Model

Parameter characterization using intervals has already been studied for input-output and state-space models in [13, 3, 17] and [2]. Methods which have already been proposed treat the case in which only system's response in Dataset DS is interval. Moreover, depending on model structure and identification semantic, parameter characterization procedure differs [4, 3, 1]. In this section, a generic approach is proposed for parameter characterization of any linear structure while both observed input and output are characterized by intervals.

During the data acquisition experience and because of diverse reason like measurement error, true system's input u_k and response y_k may not be equal to observed values $u_{o,k}$ and $y_{o,k}$. If maximum values of e_u and e_y are known:

$$e_u = \max_k(|u_k - u_{o,k}|)\ ,\ e_y = \max_k(|y_k - y_{o,k}|)$$

then the true values can be characterized by intervals:

$$[u]_k = [u_{o,k} - e_u, u_{o,k} + e_u]$$
$$[y]_k = [y_{o,k} - e_y, y_{o,k} + e_y]$$

which guarantee that:

$$\forall k,\ u_k \in [u]_k \wedge\ y_k \in [y]_k$$

Dataset $DS = \{[u]_k, [y]_k\}$ is then used in parameter characterization procedure.

4.1 Semantic of identification

In the case that system's responses are characterized by intervals, the identification semantic defines the properties of model's parameters and consequently, the specifications of the model's output [4]. It is described in the form of a mathematical (logical) expression made up of quantifiers $\forall$, $\exists$ and $\neg$, parameters and system's inputs and responses. For instance, if parameters are identified using the following semantic:

$$\forall k,\ \exists y_k \in [y]_k, \exists \theta_k \in [\theta] \mid y_k = \theta_k^T \times \varphi_k \qquad (2)$$

at every instant, system's response $[y]_k$ and model's output $[env]_k$ have at least one common point; that is:

$$\forall k,\ [y]_k \bigcap [env]_k \neq \phi$$

whereas if the semantic is defined as follows:

$$\forall k,\ \forall y_k \in [y]_k, \exists \theta_k \in [\theta] \mid y_k = \theta_k^T \times \varphi_k \qquad (3)$$

the model's output includes the system's response at any instance; that is:

$$\forall k,\ [y]_k \subseteq [env]_k$$

or in other words:

$$\forall k, \begin{cases} \overline{y}_k \leq \overline{env}_k \\ \underline{env}_k \leq \underline{y}_k \end{cases} \qquad (4)$$

The system's response and the model's output for identification semantics 2 and 3 are shown in the left and in the right hand-side of Fig. 5 respectively. The second one is chosen in this work to characterize model's parameters.

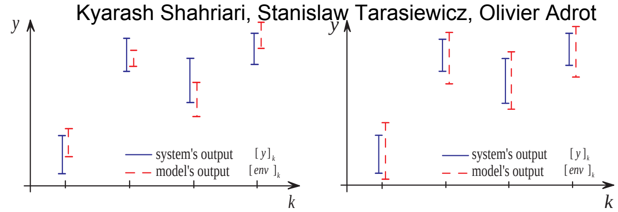

Figure 5: Different identification semantics.

4.2 Optimization criterion

The set of model's outputs at different instances determines the wrapping envelope of system's response:

$$\{[env]\} = \{[env]_1, [env]_2, ...\}$$

Smaller the radius of the wrapping envelope, more precisely the possible system's responses are characterized. Therefore, the radius of wrapping envelope is defined as the optimization criterion of model's parameters.

Definition 4 *The radius of wrapping envelope in time interval from $k = i$ until $k = j$ is the mean-value of its radius at different instants. Mathematically stated:*

$$OC_{i,j} = \frac{1}{j-i+1} \sum_{k=i}^{j} \frac{\overline{env}_k - \underline{env}_k}{2} \qquad (5)$$

Definition 5 *The precision of the wrapping envelope in time interval from $k = i$ until $k = j$ is the exponential function-value of $(-OC_{i,j})$; that is:*

$$P_{i,j} = \exp(-OC_{i,j})$$

For more detalis and demonstrations see [21, 19].

4.3 Model Structure

Probably, the simplest mathematical relationship between inputs and outputs of a linear time-invariant system in discrete time is represented by its transfer function:

$$G_{(z)} = \frac{y_k}{u_k} = \frac{b_1 z^{-1} + b_2 z^{-2} + ... + b_{n_b} z^{-n_b}}{1 + a_1 z^{-1} + a_2 z^{-2} + ... + a_{n_a} z^{-n_a}}$$

By developing and reformulating it in vectoriel form, one obtains [11]:

$$y_k = \theta^T \times \varphi_k + e_k \qquad (6)$$

where

$$\theta = [a_1, ..., \ a_{n_a}, \ b_1, ..., \ b_{n_b},]^T$$
$$\varphi_k = [-y_{k-1}, ..., \ -y_{k-n_a}, \ u_{k-1}, ..., \ u_{k-n_b}]^T$$

θ is the *parameters* and φ_k the *regression* vector of the model. Additive term e_k is usually added to compensate modeling error. If one fixes $n_a = n_b = N$, the model is an *input-output* model whereas if he/she fixes $n_a = 0$ and $n_b = N$, it is the *finite impulse response* of the system. In both cases, N is called the *moving horizon* of the model.

We have already argued that modeling error is due to many facts among which model order reduction and variation of physical and mechanical properties of system can be cited. If model's parameters are considered time-varying, one can project modeling error on its parameters and eliminate e_k from 6; that is to say:

$$y_k = \theta_k^T \times \varphi_k$$

where

$$\theta_k = [a_{1,k}, ..., \ a_{n_a,k}, \ b_{1,k}, ..., \ b_{n_b,k}]^T$$

Parameter vector θ_k carries index k to demonstrate that its entries are not constant scalars anymore and vary in time.

System analysis using a model with time-varying parameters is complicated and time-consuming. To obtain a time-invariant model which is true at any instant, one can characterized time-variant parameters vector θ_k^T by interval vector $[\theta]$ such that:

$$\forall k, \ \theta_k \in [\theta]$$

In this case, model's output is obtain from:

$$[env]_k = [\theta]^T \times \varphi_k$$

If vector φ_k contains also uncertain entries, it is replaced by interval vector $[\varphi]_k$ for which it is guaranteed that $\varphi_k \in [\varphi]_k$. One consequently obtains a more general vectoriel form of the model:

$$[env]_k = [\theta]^T \times [\varphi]_k$$

Considering 1, the upper and the lower bounds of output interval $[env]_k$ are:

$$\overline{env}_k = \theta_c^T \varphi_{k,c} + |\theta_c^T| \lambda_{\varphi_k} + \lambda_\theta^T |\varphi_{k,c}| + \lambda_\theta^T \lambda_{\varphi_k}$$
$$\underline{env}_k = \theta_c^T \varphi_{k,c} - |\theta_c^T| \lambda_{\varphi_k} - \lambda_\theta^T |\varphi_{k,c}| - \lambda_\theta^T \lambda_{\varphi_k} \quad (7)$$

where

$$\theta_c = mid([\theta]) \in \Re^{N+1} \varphi_{k,c} = mid([\varphi]_k) \in \Re^{N+1}$$
$$\lambda_\theta = rad([\theta]) \in \Re^{N+1} \quad \lambda_{\varphi_k} = rad([\varphi]_k) \in \Re^{N+1}$$

In 7, θ_c and λ_θ are model's parameters which should be identified.

There exist different numerical methods to characterize parameters of $LTVM$ [3, 2]. Parameters characterization using the semantic of 3 can easily be reformulated in the form of an optimization problem subject to a set of constraints. Considering optimization criterion in 5 and the pair of inequalities in 4 we obtain:

$$\min_{\theta_c, \lambda_\theta} (OC_{i,j}) = \min_{\theta_c, \lambda_\theta} \left(\frac{1}{j-i+1} \sum_{k=i}^{j} \frac{\overline{env}_k - \underline{env}_k}{2} \right) \quad (8)$$

subject to

$$\forall k, \begin{cases} \overline{y}_k \le \overline{env}_k \\ \underline{env}_k \le \underline{y}_k \end{cases}$$

in which $\overline{env}_k$ and $\underline{env}_k$ are substituted from 7. By finding the minimizer[2] of objective function $OC_{i,j}$, one can obtain model's parameter vector $[\theta]$. Because of non-linear term $|\theta_c^T|$ in the objective function and in the constraints, this optimization problem is a non-linear optimization problem subject to non-linear constraints with respect to the parameters.

In any dynamic model, parameters can not be characterized before instant $k < N$ as the entries of regressor vector are not available. This problem also appears in simulation where the model's output can not be calculated before instant $k \le N$. In other words, the model is not valid in its moving horizon. A parameter identification procedure based on model order reduction/increase respectively during/after model's moving horizon has been proposed in [20]. Using this procedure, a set of parameters is assigned to the model and consequently, model's output can be calculated at any instant.

5 Piecewise Parameters of LTVM

In the previous sections, a method has been proposed to identify the parameters of complex systems. To obtain a time-invariant model, variant parameters have

[2] A local (respect. global) minimizer is a set of parameters that minimizes locally (respect. globally) the value of the objective function.

been characterized by intervals. However, if the variations of parameters are considerable, interval parameters will be large and consequently, $LTVM$ produces a wide wrapping envelope which may not be of use from academical or technical point of view.

To improve the precision, we suggest a novel strategy to split system's operating regime (SOR) into several segments and to represent every segment by a set of parameters. In which follows, every segment of SOR is called an *operating mode* (OM) and its corresponding parameters set is called *local parameters set* (LPS). At any instant, model uses the parameters set which matches the best to current SOP. For a non-linear system, this approach is very likely to linearizing the system around different operating points and for a hybrid system, it means to represent every system's operating mode by a set of LPS.

It is supposed that every SOR can be determined by system's state variables in observable state space. For piecewise linear modeling of such a system, following steps are performed.

1) Determining characteristic variables: They are the variables by which every SOR can be determined in observable state space [25, 21]. Since they are not known a priori, the entries of the regressor vector are supposed to be. Hence, the space of the characteristic variables is regressor space χ_φ. At any instant k, φ_k (respect. $[\varphi_k]$) represents a point (respect. a hypercube) in this space.

2) Splitting system's operating regime: In this step, the objective is to split SOR into several segments SOR_i and to characterize LPS_i of $LTVM$ using subset $DS_i \subseteq DS$ which corresponds to SOR_i. $LTVM$ with LPS_i will then represent SOR_i. To do that, a strategy based on the precision of wrapping envelope is proposed in which follows.

Suppose that the desired precision is user-defined value $prcn$ and suppose that the last OM has already been finished at instant bgn. Before instant bgn, $i-1$ modes have been detected and therefore, at instant bgn, we enter in the i^{th} one. At the beginning, the initial data set for current mode is supposed to include only the minimum number of points to characterize LPS_i local parameters that is $DS_i = DS^{bgn,b+N} = \{([u]_k, [y]_k) | bgn \leq k \leq bgn + N\}$. Since this is a minimum number of necessary points, the wrapping envelope is the most possible precise one [3]. If $P_{bgn,bgn+N} < prcn$, then $prcn$ should be modified since even for the least number of points, it can not be achieved.

[3] if the true value of system inputs and outputs are known, then $P_{bgn,bgn+N} = 1$.

In the next step, point $bgn + N + 1$ is added to DS_i and local parameters are re-characterized using $DS_i = DS^{bgn,bgn+N+1}$. If $P_{bgn,bgn+N+1}$ is still superior than $prcn$, it demonstrates that this point can be included in OM_i. Following points are added one after the others until instant s at which if point $([u]_s, [y]_s)$ is added to DS_i, $P_{bgn,s}$ becomes less than $prcn$. This indicates that OM_i can not handle any other point. $DS_i = DS_o^{bgn,s-1}$, LPS_i are characterized using DS_i and new OM_{i+1} starts at instant $bgn + s$. This procedure is followed for all the points in DS. At the end of this procedure which is shown in Fig. 6 (see also appendix A for splitting algorithm), DS is divided into M disjoints subsets $DS_i,\ i = 1, ..., M$.

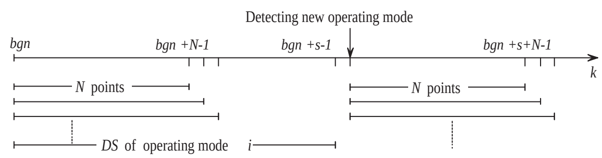

Figure 6: Splitting system's operating regime.

The value of $prcn$ plays an important role in producing proper SOR divisions. One may proceed a preliminary analysis to observe the evolution of precision $P_{i,j}$ with respect to i and j and to chose an adequate value for $prcn$. An alternative strategy for splitting SOR based on the consistency of system's response with estimated one has already been suggested in [21] [19].

3) Precision Improvement: As can be realized, the smaller the amount of precision is, the more accurate the model would be. Its cost is the complexity of the model that would appear in the number of LPS sets. So, a trade off should be done between the modeling precision and the number of its LPS sets. However, it happens that an slight increase in $prcn$ has no effect on the number of sets. Then, the most amount of precision is sought in interval $[prcn, 1]$ in such a way that the same number of modes is required to describe the system. It is accomplished by bisecting interval $[prcn, 1]$ and by observing the solutions, i.e. the number of modes, in the bisected sections. If it is found in the upper section $[\frac{1+prcn}{2}, 1]$, we continue bisecting the upper one otherwise $[prcn, \frac{1+prcn}{2}]$ would be the interval which would be bisected in the next step. The procedure is continued until the radius of obtained interval on $prcn$ becomes less than allowed tolerance $tlrn$ which is also a user defined value. The lower bound of the final interval would be the optimal value for $prcn$ (see appendix B for the algorithm).

4) Determining the validity domain of local parameters sets: To determine validity domain (VD) of any LPS, theorem 3 is presented (see [19] and appendix C for the proof):

Theorem 3 (The validity domain of LPS) *For a system of order n, suppose that the validity domain of any system's operating regime is a convex hull in observable state space. Then, it is also a convex hull in input-output space $S_{\chi^1} = \{u_{k-1}, \dots, u_{k-n}, y_{k-1}, \dots, y_{k-n}\}$ or reduced input-output space $S_{\chi^2} = \{u_{k-1}, \dots, u_{k-n-1}, y_k, \dots, y_{k-n-1}\}$.*

If the VD of any SOR_i is a convex hull in input-output space, then the VD of OM_i and consequently, the VD of LPS_i is also a convex hull in input-output space. Entries of any DS_i determines a set of points (or hypercubes) by vectors $\chi_k^1 = [u_{k-1}, \dots, u_{k-n}, y_{k-1}, \dots, y_{k-n}]$ or $\chi_k^2 = [u_{k-1}, \dots, u_{k-n-1}, y_k, \dots, y_{k-n-1}]$ which occupies a region in S_{χ_2} or S_{χ_2} respectively. This region is an approximation of LPS_i validity domain (see Fig. 7).

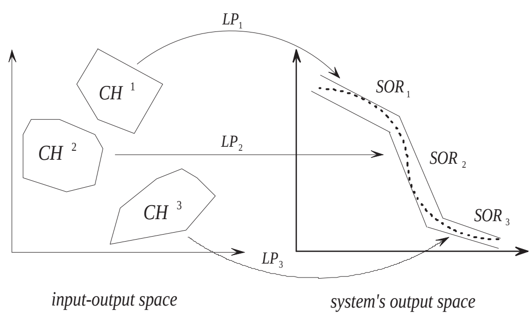

Figure 7: Validity of local parameters sets in S_χ.

Definition 6 *The validity domain of LPS_i is convex hull CH_χ^i of points/hypercubes determined by DS_i in space χ.*

Definition 7 *Consequently, LPS_i is valid at instant k iff vector χ_k is in convex hull CH_χ^i or has at least one common point with it. Mathematically stated:*

$$\chi_k \in CH_\chi^i \ \ or \ \ [\chi]_k \bigcap CH_\chi^i \neq \phi$$

As mentioned before, convex hull CH_χ^i is an approximation of LPS_i validity domain. More DS_i is informative, more CH_χ^i approaches the true domain. If calculating convex hull CH_χ^i is time-consuming (because its dimensions or the number of entries of DS_i) or its form is complex (because of the high number of vertexes and sides), for simplicity reasons, this convex hull can be approximated by its smallest outer hypercube noted as $\Diamond CH_\chi^i$. Therefore:

$$CH_\chi^i \subseteq \Diamond CH_\chi^i$$

In some cases, convex hulls i.e. validity domains of LPS intersect. The most important reason for this phenomenon is uncertainty on system's observations. For more details and discussions, see [19].

5) Aggregating similar local parameters sets: During data acquisition procedure, system may enter several times in an identical operating regime. In this case, the data corresponds to one OM may be found in disjoint time intervals in DS. As proposed splitting method dose not verify whether the following OM is a new or a mode which has already been identified, several LPS may be assigned to one SOR. Principally, this makes no problem. However, to diminish the number of LPS we try to aggregate PLs which represent an identical SOR. The following proposed solution is based on the validity domains of LPS. For two different convex hulls CH^i and CH^j, three different cases may happen.

The first case is that the former is the subset of the latter. Mathematically stated:

$$CH^i \subseteq CH^j$$

In this case regarding to definition 7, LPS_j is valid as soon as LPS_i. Therefore, the former is considered as subset of the latter. LPS_i is eliminated and TDS_i is added to DS_j. LPS_j are then re-characterized using $DS_i \cup DS_j$ and the validity domain of new LPS_j is convex hull CH^j.

The second case happens when they intersect, but neither does include the other one. i.e:

$$(CH^i \bigcap CH^j \neq \phi) \wedge$$
$$(CH^i \bigcap CH^j \neq CH^i \wedge CH^i \bigcap CH^j \neq CH^j)$$

In this case, only at some instances both LPS_i and LPS_j are valid simultaneously and non of them includes completely the other one. Therefore, no aggregation is performed since it may reduce modeling precision considerably.

In the third case, they are two disjoint convex hulls. In other words:

$$CH^i \bigcap CH^j = \phi$$

which means that model LPS_i and model LPS_j describe two different SOR. Therefore, no aggregation is made.

The above rules should be applied to every pair of validity domains (CH^i, CH^j) to eliminate as many repeated LPS as possible and to simplify the structure of $LTVM$. To make the aggregation procedure easier, one may also use outer hypercube approximations $(\Diamond CH^i, \Diamond CH^j)$ rather than convex hulls (CH^i, CH^i).

6 Numerical Example

The reduced order model of the system without any perturbation is described by:

$$G_{(S)} = \frac{Y_{(S)}}{E_{(S)}} = \frac{s + 2.3}{s^2 + 6.6s + 1.67} \tag{9}$$

If model's parameters are considered time-variant, the model follows system's response more precisely. This is shown in Fig. 8. Parameters variations of $LTVM$ are shown in Fig. 9.

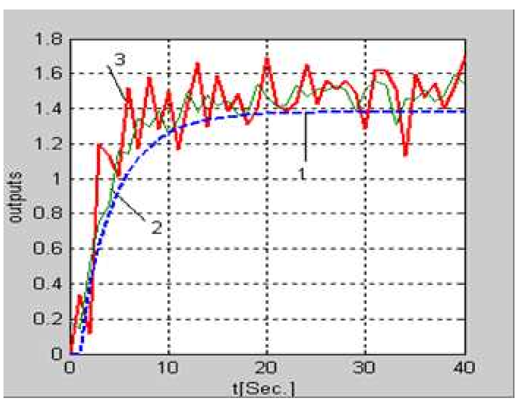

Figure 8: curve 1: System's response without perturbation. Curve 2: Tuning model response. Curve 3: System's response with perturbation.

This result illustrates that the piecewise parameters of $LTVM$ have a substantial effect on the system's responses. The precision of this model has been worked out using identification algorithms presented in appendixes A, B.

7 Conclusion

In this paper, we demonstrated that in systems identification, how modeling error can be taken into account

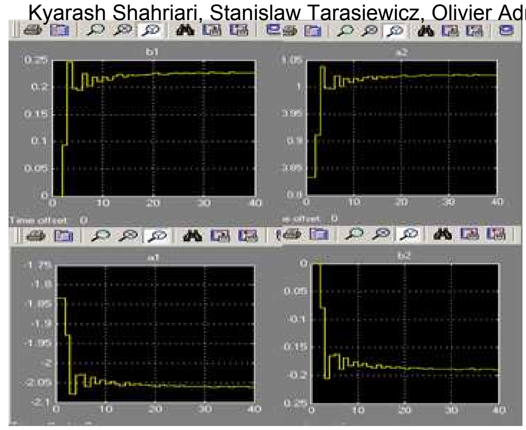

Figure 9: Time-Varying coefficients of $LTVM$ of Eq. 9.

using $LTVM$. Since manipulating such a model is time-consuming from processing point of view and complex to analysis, parameters have been charactarized by intervals. The model then can predict all possible system's responses encapsulated in a tube called wrapping envelope. This model can be used in system analysis and control in applications such as process safety in which ignorance of modeling error may cause catastrophic consequences. Moreover, the $LTVM$ model's moving horizon given in this paper is useful in determining the optimal parameters of controller or regulator.

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References:

[1] O. Adrot and J. Flaus. Trajectory computation of a dynamic uncertain system. *IEEE Conference on Decision and Control, CDC*, 2003.

[2] O. Adrot, K. Shahriari, and J. Flaus. Estimation of bounded model uncertainties. *11th IFAC Symposium on Automation in Mining and Metal Processing*, 2004.

[3] J. Armengol. *Application of Modal Interval Analysis to the simulation of the behaviour of dynamic systems with uncertain parameters*. PhD thesis, Universitat de Girona, 1999.

[4] R. Calm, M. Sainz, P. Herrero, J. Vehi, and J. Armengol. Parameter identification with quantifiers. *5th IFAC Symposium on Robust Control Design*, July 2006.

[5] C. Combastel. A state bounding observer for uncertain non-linear continuous-time systems based on zonotopes. *44th IEEE Conference on Decision and Control, European Control Conference. CDC-ECC.*, 2005.

[6] R. W. Freaud. Model reduction methods based on krylov subspaces. *Acta Numerica*, (3):267–319, 2003.

[7] J. C. Gille. *Systeme Lineaire: Equation d'etat*. Eyrolles, Paris, 1990.

[8] T. L. Heath, editor. *The Works of Archimedes*. Cambridge University Press, Cambridge, 1897. Dover edition, 1953.

[9] L. Jaulin, M. Kieffer, O. Didrit, and E. Walter. *Applied Interval Analysis*. Springer-Verlag, 2001.

[10] L. Kolev. *Interval Methods for Circuit Analysis*. World Scientific, 1993.

[11] L. Ljung. *System Identification*. Prentice Hall, New Jersey, 1999.

[12] T. N. Lucas. A useful algorithm in model reduction. *IEEE proceedings*, 130:362–366, 1984.

[13] M. Milanese, J. Norton, H. Piet-Lahanier, and E. Walter. *Bounding Approaches to System Identification*. Plenum Press, New York, London, 1996.

[14] R. Moore. *Interval Analysis*. Printice Hall, Englewood Cliffs, NJ, 1966.

[15] A. Neumaier. *Introduction to Numerical Analysis*. Cambridge Univ. Press, Cambridge, 2001.

[16] J. Philips, L. Daniel, and L. Silveria. Guaranteed passive balancing transformation for model order reduction. *IEEE transaction on CAD*, (8):1027–1041, 2003.

[17] S. Ploix, O. Adrot, and J. Ragot. Parameter uncertainty computation in static linear models. *Decision and Control, 1999. Proceedings of the 38th IEEE Conference on*, 2:1916–1921, 1999.

[18] H. Raynaud, L. Pronzato, and E. Walter. Robust identification and control based on ellipsoidal parametric uncertainty descriptions. *European Journal of Control*, 2000.

[19] K. Shahriari. *Analyse de Surete de Procedes Multi Mode par des Modeles a Base d'Intervalles*. PhD thesis, Joseph Fourier University, 2007.

[20] K. Shahriari, O. Adrot, and J. Flaus. Multi mode modeling with linear dynamic models using set-membership tools. *The 11th Information Processing and Management of Uncertainty in Knowledge-Based Systems International Conference*, July 2006.

[21] K. Shahriari, J. Flaus, and O. Adrot. Linear multi-mode modelling using set-membership approach. *5th IFAC Symposium on Robust Control Design*, July 2006.

[22] S. Skogestad. Simple analytic rules for model reduction and pid controller tuning. *Journal of Process Control*, (13):291–309, 2003.

[23] T. Sunaga. Theory of interval algebra and its application to numerical analysis. *RAAG Memoirs, Ggujutsu Bunken Fukuy-kai, Tokyo*, 2:29–46 (547–564), 1958.

[24] S. Tarasiewicz. Commande par mode de glissement pour robot mobile autonome. *Technique de l'Ingenieur*, (3):S7 853 – S7 853–8, 2004, Paris.

[25] S. Tarasiewicz. Theorie de systems lineaire a parametres variables. *Rapport technique, Universite Laval, Canada*, 2005.

[26] S. Tarasiewicz, M. Beaudoin, Y. Ding, and K. Belatra. Construction et realisation d'un robot mobile autonome: Synthese de recherche de 2001 a 2005. *Rapport technique, Universite Laval, Canada*, 2005.

[27] E. Walter and L. Pronzato. *Identification of Parametric Models: From Experimental Data*. Springer-Verlag Berlin and Heidelberg, GmbH & Co. K, 1997.

[28] M. Warmus. Calculus of approximations. *Bulletin de l'Academie Polonaise de Sciences*, 4(5):253–257, 1956.

[29] R. C. Young. The algebra of multi-valued quantities. *Mathematische Annalen*, 104:260–290, 1931.

[30] W. H. Young. Sull due funzioni a piu valori constituite dai limiti d'una funzione di variable reale a destra ed a sinistra di ciascun punto. *Rendiconti Academia di Lincei, Classes di Scienza Fiziche*, 17(5):582–587, 1908.

Appendix A: SOR Splitting Algorithm

prcn: user defined modeling precision
DS: System measurements
N: Moving horizon of $LTVM$

1. **Start**
2. $begin = 1, end = N, i = 1, DS_i = DS^{begin,end}$
3. **While** *end* $<$ (the number of DS's entries)
   (a) **Characterize** LPS_i using DS_i
   (b) **If** $P_{begin,end} < prcn$
      i. **Get** $DS_i = DS^{begin,end-1}$
      ii. **Characterize** LPS_i using DS_i
      iii. $i = i + 1$
      iv. $begin = end$
      v. $end = begin + N$
      vi. **Go to** 3
   (c) **Else**
      i. $end = end + 1$
      ii. $DS_i = DS^{begin,end}$
      iii. **Go to** 3
4. **End**

Appendix B: Precision Improvement Algorithm

$tlrn$: Allowed tolerance on $prcn$

1. **Start**
2. *upper=1* , *lower=prcn*
3. **While** *(upper-lower)* $>$ $tlrn$
   (a) $c = (upper + lower)/2$
   (b) **Repeat** algorithm 1 by taking $prcn = c$
   (c) **If** (number of operating modes) $> M$
      i. *upper=c*
   (d) **Else**
      i. *lower=c*
   (e) **End**
4. **End**

Appendix C: Proof of Theorem 3

The theorem is proven for a 1^{st} and a 2^{nd} order dynamic system. However, it can easily be proven in the same way for a system of any order.

Proof:
The dynamic of a 1^{st} order $SISO$ system in observable state space is described as follows:

$$x_{k+1} = a \times x_k + b \times u_k$$
$$y_k = x_k \tag{10}$$

where $a, b \in \Re$ are known, $x_k \in \Re$ is the state variable and $u_k, y_k \in \Re$ are the system's input and output respectively. According the hypothesis of the theorem, the validity domain of SOR_i is convex hull $CH^i_{\{x\}}$ in state space $S_{\{x\}}$ which can be determined by a set of linear inequality constraints as followings:

$$CH^i_{\{x\}} = \{x \mid \forall j,\ \alpha_{i,j} \times x + \gamma_{i,j} \leq 0\} \tag{11}$$

where j is the index of inequality constraint and $\alpha_{i,j}, \gamma_{i,j} \in \Re$ are known. From 10 one has:

$$x_k = y_k$$

Therefore, one obtains from 11:

$$CH^i_{\{y\}} = \{y \mid \forall j,\ \alpha_{i,j} \times y + \gamma_{i,j} \leq 0\} \tag{12}$$

Equations 11 and 12 show that if the validity domain of SOR_i is a convex hull in observable state space, it is also a convex hull in input-output space $S_{\{y\}}$.

Identical to the 1^{st} order system, the dynamic of a 2^{nd} order $SISO$ system can be described as follows:

$$X_{k+1} = \begin{bmatrix} 0 & a_2 \\ 1 & a_1 \end{bmatrix} \times X_k + B \times u_k$$
$$y_k = \begin{bmatrix} 0 & 1 \end{bmatrix} \times X_k \tag{13}$$

where $X_k = [x_{1,k}, x_{2,k}]^T$. According to the hypothesis of the theorem, the validity domain of SOR_i in the observable space space is convex hull $CH^i_{\{x_1,x_2\}}$:

$$CH^i_{\{x_1,x_2\}} = \left\{X \mid \begin{bmatrix} \alpha_{i,j} & \beta_{i,j} \end{bmatrix} \times X + \gamma_{i,j} \leq 0\right\} \tag{14}$$

where $\alpha_{i,j},\ \beta_{i,j},\ \delta_{i,j}\ \in \Re$ are known. Calculating $x_{1,k}$ and $x_{2,k}$ form 13 with respect to $a_1,\ a_2,\ b_1,\ b_2$ and y_k and inserting them into 14, then:

$$CH^i_{\chi^1} = \{\chi^1_k \mid \begin{bmatrix} \alpha_{i,j}\beta_{i,j} \end{bmatrix} \times \begin{bmatrix} a_2 & 0 & b_1 & 0 \\ a_1 & a_2 & b_2 & b_1 \end{bmatrix} \chi^1_k + \gamma_{i,j} \leq 0\} \tag{15}$$

and

$$CH^i_{\chi^2} = \{\chi^2_k \mid \begin{bmatrix} \alpha_{i,j}\beta_{i,j} \end{bmatrix} \times \begin{bmatrix} 0 & a_2 & b_1 \\ 1 & 0 & 0 \end{bmatrix} \chi^2_k + \gamma_{i,j} \leq 0\} \tag{16}$$

which show that if the hypothesis of the theorem is true, the validity domains of any SOR_i of a 2^{nd} order $SISO$ system is also a convex hull in input-output or reduced input-output spaces S_{χ^1} and S_{χ^2}.