Investigation on spectrum of the adjacency matrix and Laplacian matrix of graph $\mathcal{G}_l$

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Abstract: Let $\mathcal{G}_l$ be the graph obtained from K_l by adhering the root of isomorphic trees $\mathcal{T}$ to every vertex of K_l, and d_{k-j+1} be the degree of vertices in the level j. In this paper we study the spectrum of the adjacency matrix $A(\mathcal{G}_l)$ and the Laplacian matrix $L(\mathcal{G}_l)$ for all positive integer l, and give some results about the spectrum of the adjacency matrix $A(\mathcal{G}_l)$ and the Laplacian matrix $L(\mathcal{G}_l)$. By using these results, an upper bound for the largest eigenvalue of the adjacency matrix $A(\mathcal{G}_l)$ is obtained:

$$\lambda_1(A(\mathcal{G}_l)) < \max\{\max_{2\le j\le k-2}\{\sqrt{d_j-1}+\sqrt{d_{j+1}-1}\}, \sqrt{d_{k-1}-1}+\sqrt{d_k-l+1}, \sqrt{d_k-l+1}+l-1\},$$

and an upper bound for the largest eigenvalue of the Laplacian matrix $L(\mathcal{G}_l)$ is also obtained:

$$\mu_1(L(\mathcal{G}_l)) < \max\left\{\max_{2\le j\le k-2}\{\sqrt{d_j-1}+d_j+\sqrt{d_{j+1}-1}\}, \sqrt{d_{k-1}-1}+d_{k-1}+\sqrt{d_k-l+1}, \sqrt{d_k-l+1}+d_k+1\right\}.$$

Key–Words: Adjacency matrix, Laplacian matrix, complete graph, spectrum

1 Introduction

Let G be a simple undirected graph with vertex set $V = \{v_1, v_2, ...v_n\}$, which $n = |V|$. Let $A(G)$ be a $(0,1)$-adjacency matrix of G. Since $A(G)$ is a real symmetric matrix, all of its eigenvalues are real. Without loss of generality, that they are ordered in non-increasing order, i.e.,

$$\lambda_1(G) \ge \lambda_2(G) \ge ... \ge \lambda_n(G), \quad (1)$$

and call them the spectrum of G, The largest eigenvalue $\lambda_1(G)$ is called the spectral radius of G.

If the destine eigenvalues of $A(G)$ are

$$\lambda_1(G) > \lambda_2(G) > ... > \lambda_s(G),$$

and their muliplicities are

$$m(\lambda_1), m(\lambda_2), ..., m(\lambda_s),$$

then we shall write

$$SpecA(G) = \begin{pmatrix} \lambda_1(G) & \lambda_2(G) & ... & \lambda_s(G) \\ m(\lambda_1) & m(\lambda_2) & ... & m(\lambda_s) \end{pmatrix}.$$

For example, the complete graph K_n has n vertices, and each distinct pair are adjacent. Thus, the graph K_4 has adjacency matrix

$$A(K_4) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

and an easy calculation shows that that spectrum of K_4 is:

$$SpecA(K_4) = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$$

About the spectrum and the spectral radius of graphs, a great deal of investigation is carried out [1][2][3]. Specially, to the special graphs, for example [4] studied the spectral radius of bicyclic graphs with n vertices and diameter d, [5] studied the spectral radius of trees with fixed diameter. In [6] V. Nikiforov proved that if G is a graph of order $n \ge 2$, maximum degree Δ, and girth at least 5, then

$$\lambda_1(G) \le \min\{\Delta, \sqrt{n-1}\},$$

where $\lambda_1(G)$ is the largest eigenvalue of the adjacency matrix of G.

In [7] X.D.Zhang proved

$$\lambda_1(G) < \Delta - \frac{2\Delta - 1 - 2\sqrt{\Delta(\Delta-1)}}{n(n-1)\Delta},$$

where G be a simple connected non-regular graph of order n and Δ be the maximum degree of G.

Let $d(v_i)$ denote the degree of $v_i \in V$, $i = 1, 2, ..., n$, and let

$$D(G) = diag(d(v_1), d(v_2), ..., d(v_n))$$

be the diagonal matrix of vertex degrees. The Laplacian matrix of G is $L(G) = D(G) - A(G)$. Clearly, $L(G)$ is a real symmetric matrix. From this fact and Geršgorin's Theorem, it follows that its eigenvalues are nonnegative real numbers. Therefore, the eigenvalues of $L(G)$, which are call the Laplacian eigenvalue of G , can be denote by

$$\mu_1(G) \geq \mu_2(G) \geq ... \geq \mu_n(G) = 0.$$

We call $\mu_1(G)$ the Laplacian spectral radius of G.

If the destine eigenvalues of $L(G)$ are

$$\mu_1(G) > \mu_2(G) > ... > \mu_s(G),$$

and their muliplicities are

$$m(\mu_1), m(\mu_2), ..., m(\mu_s),$$

then we shall write

$$SpecL(G) = \begin{pmatrix} \mu_1(G) & \mu_2(G) & ... & \mu_s(G) \\ m(\mu_1) & m(\mu_2) & ... & m(\mu_s) \end{pmatrix}.$$

For the Laplacian eigenvalues and the Laplacian spectral radius of simple graphs, there are many good results. In [8], some of the many results known for Laplacian matrices are given. Fiedler [9] proved that G is a connected graph if and only if the second smallest eigenvalue of $L(G)$ is positive. This eigenvalue is call the algebraic connectivity of G, denoted by $\alpha(G)$. In [10] Li and Pan proved the following result:

Let G be a simple connected graph with n vertices and m edges. Denote by Δ, δ the largest and smallest degrees of vertices in G. Then

$$\mu_1(G) \leq \sqrt{2\Delta^2 + 4m - 2\delta(n-1) + 2\Delta(\delta-1)}$$

In [11], Shu,Hong and Wenren proved a sharp upper bound as follows:

$$\mu_1(G) \leq d_n + \frac{1}{2} + \sqrt{(d_n - \frac{1}{2})^2 + \sum_{i=1}^{n} d_i(d_i - d_n)}.$$

where $d_1 \geq d_2 \geq ... \geq d_n$ is the degree sequence of G.

2 Preliminaries

Let $\mathcal{T}$ be an unweighted rooted tree of k levels such that in each level the vertices have equal degree. K_l be a complete graph on l vertices. Let $\mathcal{G}_l$ be the graph obtained from K_l by adhering the root of isomorphic trees $\mathcal{T}$ to every vertex of K_l. Similar to the definition of tree's level, we agree that the complete graph K_l is at level 1, and that $\mathcal{G}_l$ has k levels. Thus the vertices in the level k have degree 1.

For $j = 1, 2, 3, ..., k$, let n_{k-j+1} and d_{k-j+1} be the number of vertices and the degree of them in the level j. Observe that $n_k = l$ is the number of vertices in level 1 and n_1 the number of vertices in level k(the number of pendant vertices). Then,

$$n_{k-1} = (d_k - l + 1)n_k,$$

$$n_{k-j} = (d_{k-j+1} - 1)n_{k-j+1}, j = 2, 3, ..., k-1.$$

Observe that d_k is the degree of vertices of the complete graph K_l in $\mathcal{G}_l$, d_1 is the degree of the vertices in the level k, $n_k = l$. The total number of vertices in the graph $\mathcal{G}_l$ is

$$n = \sum_{j=1}^{k-1} n_j + l.$$

Example 2.1 Follow (Fig.1) is an example of a such graph $\mathcal{G}_4$ for $k = 3, n_1 = 24, n_2 = 8, n_3 = 4, d_1 = 1, d_2 = 4, d_3 = 5$.

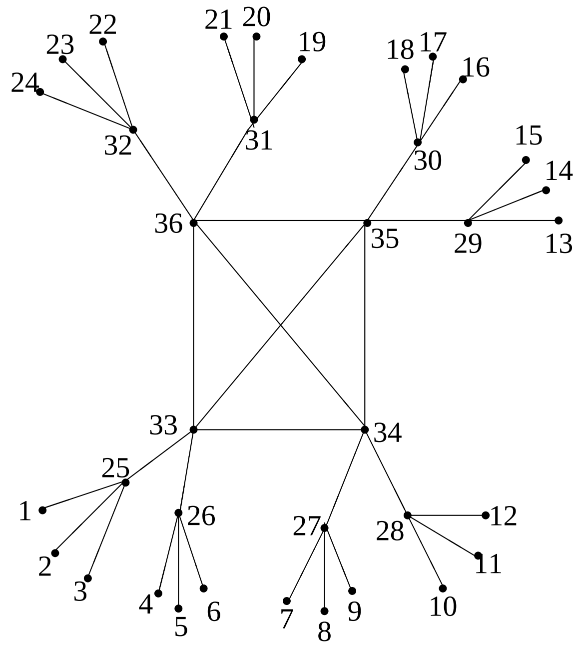

Fig.1 graph $\mathcal{G}_4$

In general, using the labels $n, n-1..., 1$, in this order, our labeling for the vertices of $\mathcal{G}_l$ is:

(1) First, we label the vertices of K_l with clockwise direction.

(2) For one of vertices of level $j(j = 1, 2, ..., k-1)$, the bigger its labeling is , then the vertex of level $j+1$ adjacent to it should be labeled first.

(3) Label from level 1 to level k in turn.

[12], [13] studied the spectrum of the adjacency matrix $A(\mathcal{G}_l)$ and the eigenvalues of Laplacian matrix $L(\mathcal{G}_l)$ for case $l = 1$ and $l = 2$ respectively. In this paper we will study the spectrum of the adjacency matrix $A(\mathcal{G}_l)$ and the eigenvalues of Laplacian matrix $L(\mathcal{G}_l)$ for all positive integer l.

We introduce the following notations:

(1) $\mathbf{0}$ is the all zeros matrix, the order of $\mathbf{0}$ will be clear from the context in which it is used.

(2)$\mathbf{I_m}$ is the identity matrix of order $m \times m$.

(3) $m_j = \frac{n_j}{n_{j+1}}$, for $j = 1, 2, ..., k-1$.

(4) $\mathbf{e_m}$ is the all ones column vetor of dimension m.

For $j = 1, 2, ..., k-1$, C_j is the block diagonal matrix

$$C_j = \begin{pmatrix} \mathbf{e_{m_j}} & & & \\ & \mathbf{e_{m_j}} & & \\ & & \ddots & \\ & & & \mathbf{e_{m_j}} \end{pmatrix}$$

with n_{j+1} diagonal blocks. Thus, the order of C_j is $n_j \times n_{j+1}$.

For example we use these notation with the graph $\mathcal{G}_4$ in Fig.1 $m_1 = \frac{n_1}{n_2} = 3, m_2 = \frac{n_2}{n_3} = 2$, then

$$C_1 = diag\{\mathbf{e_3}, \mathbf{e_3}, \mathbf{e_3}, \mathbf{e_3}, \mathbf{e_3}, \mathbf{e_3}, \mathbf{e_3}, \mathbf{e_3}\},$$

$$C_2 = diag\{\mathbf{e_2}, \mathbf{e_2}, \mathbf{e_2}, \mathbf{e_2}\},$$

The adjacency matrix $A(\mathcal{G}_4)$ in Fig.1 become

$$A(\mathcal{G}_4) = \begin{pmatrix} \mathbf{0} & C_1 & \mathbf{0} \\ C_1^T & \mathbf{0} & C_2 \\ \mathbf{0} & C_2^T & B_4 \end{pmatrix},$$

where $B_4 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$ and

$$L(\mathcal{G}_4) = \begin{pmatrix} d_1\mathbf{I_{24}} & -C_1 & \mathbf{0} \\ -C_1^T & d_2\mathbf{I_8} & -C_2 \\ \mathbf{0} & -C_2^T & U_4 \end{pmatrix},$$

where

$$U_4 = d_3\mathbf{I_4} - B_4 = \begin{pmatrix} 5 & -1 & -1 & -1 \\ -1 & 5 & -1 & -1 \\ -1 & -1 & 5 & -1 \\ -1 & -1 & -1 & 5 \end{pmatrix}.$$

In general, our labeling yields to

$$A(\mathcal{G}_l) = \begin{pmatrix} \mathbf{0} & C_1 & & & & \\ C_1^T & \mathbf{0} & C_2 & & & \\ & C_2^T & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & C_{k-1} \\ & & & & C_{k-1}^T & B_l \end{pmatrix},$$

where $B_l = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix}$.

$L(\mathcal{G}_l) =$

$$\begin{pmatrix} d_1\mathbf{I_{n_1}} & -C_1 & & & & \\ -C_1^T & d_2\mathbf{I_{n_2}} & -C_2 & & & \\ & -C_2^T & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & d_{k-1}\mathbf{I_{n_{k-1}}} & -C_{k-1} \\ & & & & -C_{k-1}^T & U_l \end{pmatrix},$$

where

$$U_l = d_k\mathbf{I_{n_k}} - B_l = \begin{pmatrix} d_k & -1 & -1 & \cdots & -1 \\ -1 & d_k & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & d_k \end{pmatrix}.$$

Apply the Gaussian elimination procedure we obtained the following lemma:

Lemma 2.1 Let $M =$

$$\begin{pmatrix} \alpha_1\mathbf{I_{n_1}} & C_1 & & & & \\ C_1^T & \alpha_2\mathbf{I_{n_2}} & C_2 & & & \\ & C_2^T & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \alpha_{k-1}\mathbf{I_{n_{k-1}}} & C_{k-1} \\ & & & & C_{k-1}^T & \alpha_k\mathbf{I_{n_k}} + B_l \end{pmatrix},$$

let

$$\beta_1 = \alpha_1$$

and

$$\beta_j = \alpha_j - \frac{n_{j-1}}{n_j}\frac{1}{\beta_{j-1}}, j = 2, 3, ..., k, \beta_{j-1} \neq 0.$$

If $\beta_j \neq 0$ for all $j = 1, 2, ..., k-1$, then

$$\begin{aligned} detM &= \beta_1^{n_1}\beta_2^{n_2}...\beta_{k-1}^{n_{k-1}} \\ &\quad \times(\beta_k + l - 1)(\beta_k - 1)^{l-1}. \end{aligned} \quad (2)$$

Proof. Apply the Gaussian elimination procedure, without row interchanges, to M to obtain the block upper triangular matrix

$$\begin{pmatrix} \beta_1\mathbf{I_{n_1}} & C_1 & & & & \\ & \beta_2\mathbf{I_{n_2}} & C_2 & & & \\ & & \beta_3\mathbf{I_{n_3}} & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & \beta_{k-1}\mathbf{I_{n_{k-1}}} & C_{k-1} \\ & & & & & \beta_k\mathbf{I_{n_k}}+B_l \end{pmatrix}.$$

Hence,

$$detM=\beta_1^{n_1}\beta_2^{n_2}...\beta_{k-1}^{n_{k-1}}det(\beta_k\mathbf{I_{n_k}}+B_l).$$

Since

$$det(\lambda\mathbf{I}-B_l)=(\lambda-l+1)(\lambda+1)^{l-1},$$

so

$$\begin{aligned} det(\beta_k\mathbf{I_{n_k}}+B_l) &= (-1)^l det(-\beta_k\mathbf{I_{n_k}}-B_l) \\ &= (\beta_k+l-1)(\beta_k-1)^{l-1}. \end{aligned}$$

Then

$$detM=\beta_1^{n_1}\beta_2^{n_2}...\beta_{k-1}^{n_{k-1}}(\beta_k+l-1)(\beta_k-1)^{l-1}.$$

Thus, (2) is proved. #

Lemma 2.2 Let $M_0=$

$$\begin{pmatrix} \alpha_1\mathbf{I_{n_1}} & -C_1 & & & & \\ -C_1^T & \alpha_2\mathbf{I_{n_2}} & -C_2 & & & \\ & -C_2^T & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \alpha_{k-1}\mathbf{I_{n_{k-1}}} & -C_{k-1} \\ & & & & -C_{k-1}^T & \alpha_k\mathbf{I_{n_k}}-B_l \end{pmatrix},$$

let

$$\beta_1=\alpha_1$$

and

$$\beta_j=\alpha_j-\frac{n_{j-1}}{n_j}\frac{1}{\beta_{j-1}}, j=2,3,...,k, \beta_{j-1}\neq 0.$$

If $\beta_j\neq 0$ for all $j=1,2,...,k-1$, then

$$\begin{aligned} detM_0 &= \beta_1^{n_1}\beta_2^{n_2}...\beta_{k-1}^{n_{k-1}} \\ &\quad \times(\beta_k-l+1)(\beta_k+1)^{l-1}. \end{aligned} \quad (3)$$

Proof. Apply the Gaussian elimination procedure, without row interchanges, to M_0 to obtain the block upper triangular matrix

$$\begin{pmatrix} \beta_1\mathbf{I_{n_1}} & -C_1 & & & & \\ & \beta_2\mathbf{I_{n_2}} & -C_2 & & & \\ & & \beta_3\mathbf{I_{n_3}} & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & \beta_{k-1}\mathbf{I_{n_{k-1}}} & -C_{k-1} \\ & & & & & \beta_k\mathbf{I_{n_k}}-B_l \end{pmatrix}.$$

Hence,

$$detM_0=\beta_1^{n_1}\beta_2^{n_2}...\beta_{k-1}^{n_{k-1}}det(\beta_k\mathbf{I_{n_k}}-B_l).$$

Since

$$det(\lambda\mathbf{I}-B_l)=(\lambda-l+1)(\lambda+1)^{l-1},$$

so

$$det(\beta_k\mathbf{I_{n_k}}-B_l)=(\beta_k-l+1)(\beta_k+1)^{l-1}.$$

Then

$$detM_0=\beta_1^{n_1}\beta_2^{n_2}...\beta_{k-1}^{n_{k-1}}(\beta_k-l+1)(\beta_k+1)^{l-1}.$$

Thus, (3) is proved. #

3 The spectrum of the adjacency matrix and the Laplacian matrix of $\mathcal{G}_l$

In this section we will apply Lemma3.1, Lemma 3.2 and Lemma 3.3 to study the spectrum of the adjacency matrix and the Laplacian matrix of $\mathcal{G}_l$

Lemma 3.1[14] Let H be $k\times k$ symmetric tridiagonal matrix:

$$H=\begin{pmatrix} a_1 & b_1 & & & & \\ b_1 & a_2 & b_2 & & & \\ & b_2 & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & a_{k-1} & b_{k-1} \\ & & & & b_{k-1} & a_k \end{pmatrix},$$

and $Q_j(\lambda)(j=0,1,2,...,k)$ be the characteristic polynomials of the $j\times j$ leading principal submatrix of matrix H. Then

$$\begin{aligned} Q_0(\lambda) &= 1, \\ Q_1(\lambda) &= \lambda-a_1, \\ Q_j(\lambda) &= (\lambda-a_j)Q_{j-1}(\lambda)-b_{j-1}^2Q_{j-2}(\lambda), \\ & \quad (j=2,3,...,k). \end{aligned} \quad (4)$$

Lemma 3.2[14] Let H and $Q_j(\lambda)(j=0,1,2,...,k)$ be matrix as in Lemma 3.1, then all roots $\lambda_i^{(j)}(i=1,2,...,j)$ of $Q_j(j=0,1,2,...,k)$ are real and simple:

$$\lambda_1^{(j)}>\lambda_2^{(j)}>...>\lambda_j^{(j)},$$

and the roots of Q_{j-1} and Q_j, respectively, separate each other strictly:

$$\lambda_1^{(j)}>\lambda_1^{(j-1)}>\lambda_2^{(j)}>\lambda_2^{(j-1)}>...>\lambda_{j-1}^{(j-1)}>\lambda_j^{(j)}.$$

Lemma 3.3[15] Let A, B be $n \times n$ Hermitian matrix. Assume that B is positive semidefite and that the eigenvalues of A and $A+B$ are arranged in decreasing order as in (1). Then

$$\lambda_i(A) \leq \lambda_i(A+B) \quad \text{for all} \quad i = 1, 2, ..., n.$$

3.1 The spectrum of the adjacency matrix of $\mathcal{G}_l$

Let

$$\phi = \{1, 2, ..., k-1\}$$

and

$$\Omega = \{j \in \phi : n_j > n_{j+1}\}.$$

Observe that $n_{k-j} = (d_{k-j+1} - 1)n_{k-j+1}, j = 2, 3, ..., k-1$ and $n_{k-1} = (d_k - l + 1)n_k$. Observe also that if $j \in \phi - \Omega$ then $n_j = n_{j+1}$ and C_j is the identity matrix of order n_j.

Theorem 3.1.1 Let

$$S_0(\lambda) = 1, S_1(\lambda) = \lambda,$$

$S_j(\lambda) = \lambda S_{j-1}(\lambda) - \frac{n_{j-1}}{n_j} S_{j-2}(\lambda)$, for $j = 2, 3, ..., k$,

$$S_k^-(\lambda) = (\lambda+1)S_{k-1}(\lambda) - \frac{n_{k-1}}{l} S_{k-2}(\lambda)$$

and

$$S_k^+(\lambda) = (\lambda - l + 1)S_{k-1}(\lambda) - \frac{n_{k-1}}{l} S_{k-2}(\lambda).$$

Then

(i) If $S_j(\lambda) \neq 0$, for $j = 1, 2, ..., k-1$, then

$$\begin{aligned} det(\lambda \mathbf{I} - A(\mathcal{G}_l)) &= (S_k^-(\lambda))^{l-1} S_k^+(\lambda) \\ &\quad \times \prod_{j \in \Omega} S_j^{n_j - n_{j+1}}(\lambda). \end{aligned} \tag{5}$$

(ii)The spectrum of $A(\mathcal{G}_l)$ is $\sigma(A(\mathcal{G}_l)) = (\cup_{j \in \Omega}\{\lambda : S_j(\lambda) = 0\}) \cup \{\lambda : S_k^-(\lambda) = 0\} \cup \{\lambda : S_k^+(\lambda) = 0\}$.

Proof. Suppose $S_j(\lambda) \neq 0$ for all $j = 1, 2, ..., k-1$. We apply Lemma 2.2 to $M_0 = \lambda \mathbf{I} - A(\mathcal{G}_l)$.

$M_0 = \lambda \mathbf{I} - A(\mathcal{G}_l) =$

$$\begin{pmatrix} \lambda \mathbf{I_{n_1}} & -C_1 & & & & \\ -C_1^T & \lambda \mathbf{I_{n_2}} & -C_2 & & & \\ & -C_2^T & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \lambda \mathbf{I_{n_{k-1}}} & -C_{k-1} \\ & & & & -C_{k-1}^T & \lambda \mathbf{I_{n_k}} - B_l \end{pmatrix}.$$

We have

$$\beta_1 = \lambda = S_1(\lambda) \neq 0,$$

$$\begin{aligned} \beta_2 &= \lambda - \frac{n_1}{n_2}\frac{1}{\beta_1} \\ &= \lambda - \frac{n_1}{n_2}\frac{1}{S_1(\lambda)} \\ &= \frac{\lambda S_1(\lambda) - \frac{n_1}{n_2} S_0(\lambda)}{S_1(\lambda)} \\ &= \frac{S_2(\lambda)}{S_1(\lambda)} \neq 0. \end{aligned}$$

Similarly, for $j = 3, 4, ..., k-1, k$

$$\begin{aligned} \beta_j &= \lambda - \frac{n_{j-1}}{n_j}\frac{1}{\beta_{j-1}} \\ &= \lambda - \frac{n_{j-1}}{n_j}\frac{S_{j-2}(\lambda)}{S_{j-1}(\lambda)} \\ &= \frac{\lambda S_{j-1}(\lambda) - \frac{n_{j-1}}{n_j} S_{j-2}(\lambda)}{S_{j-1}(\lambda)} \\ &= \frac{S_j(\lambda)}{S_{j-1}(\lambda)} \neq 0. \end{aligned}$$

Thus

$$\begin{aligned} \beta_k + 1 &= \frac{S_k(\lambda)}{S_{k-1}(\lambda)} + 1 \\ &= \frac{S_k(\lambda) + S_{k-1}(\lambda)}{S_{k-1}(\lambda)} \\ &= \frac{(\lambda+1)S_{k-1}(\lambda) - \frac{n_{k-1}}{l} S_{k-2}(\lambda)}{S_{k-1}(\lambda)} \\ &= \frac{S_k^-(\lambda)}{S_{k-1}(\lambda)}, \end{aligned}$$

$$\begin{aligned} \beta_k - l + 1 &= \frac{S_k(\lambda)}{S_{k-1}(\lambda)} - l + 1 \\ &= \frac{S_k(\lambda) - (l-1)S_{k-1}(\lambda)}{S_{k-1}(\lambda)} \\ &= \frac{(\lambda - l + 1)S_{k-1}(\lambda) - \frac{n_{k-1}}{l} S_{k-2}(\lambda)}{S_{k-1}(\lambda)} \\ &= \frac{S_k^+(\lambda)}{S_{k-1}(\lambda)}. \end{aligned}$$

Therefore, from Lemma 2.2,

$$\begin{aligned} & det(\lambda I - A(\mathcal{G}_l)) \\ = \ & S_1^{n_1}(\lambda)\frac{S_2^{n_2}(\lambda)}{S_1^{n_2}(\lambda)} \cdots \frac{S_{k-1}^{n_{k-1}}(\lambda)}{S_{k-2}^{n_{k-1}}(\lambda)} \frac{S_k^+(\lambda)}{S_{k-1}(\lambda)} \frac{(S_k^-(\lambda))^{l-1}}{S_{k-1}^{l-1}(\lambda)} \\ = \ & S_1^{n_1-n_2}(\lambda) S_2^{n_2-n_3}(\lambda) ... S_{k-1}^{n_{k-1}-n_k}(\lambda) \\ & \times S_k^+(\lambda)(S_k^-(\lambda))^{l-1} \\ = \ & (S_k^-(\lambda))^{l-1} S_k^+(\lambda) \prod_{j \in \Omega} S_j^{n_j - n_{j+1}}(\lambda). \end{aligned}$$

Thus (i) is proved. Similar to the proof in [13], we can get (ii) by (i) . #

Let R_k^+ and R_k^- be the $k \times k$ symmetric tridiagonal matrices

$R_k^+ =$

$$\begin{pmatrix} 0 & \sqrt{d_2-1} & & & \\ \sqrt{d_2-1} & 0 & \sqrt{d_3-1} & & \\ & \sqrt{d_3-1} & \ddots & \ddots & \\ & \sqrt{d_{k-1}-1} & 0 & \sqrt{d_k-l+1} & \\ & & \sqrt{d_k-l+1} & l-1 & \end{pmatrix}$$

and $R_k^- =$

$$\begin{pmatrix} 0 & \sqrt{d_2-1} & & & \\ \sqrt{d_2-1} & 0 & \sqrt{d_3-1} & & \\ & \sqrt{d_3-1} & \ddots & \ddots & \\ & \sqrt{d_{k-1}-1} & 0 & \sqrt{d_k-l+1} & \\ & & \sqrt{d_k-l+1} & -1 & \end{pmatrix}.$$

Observe that

$$R_k^+ = R_k^- + diag\{0, 0, ..., 0, l\}.$$

Theorem 3.1.2 For $j = 1, 2, 3, ..., k-1$, let R_j be the $j \times j$ leading principal submatrix R_k^+. Then

$$\begin{aligned} det(\lambda \mathbf{I} - R_j) &= S_j(\lambda), j = 1, 2, ..., k-1, \\ det(\lambda \mathbf{I} - R_k^-) &= S_k^-(\lambda), \\ det(\lambda \mathbf{I} - R_k^+) &= S_k^+(\lambda). \end{aligned}$$

Proof. We apply Lemma 3.1, in our case, $a_1 = a_2 = ... = a_{k-1} = 0, a_k = l-1$(or $\quad a_k = -1$) and

$$b_{k-1} = \sqrt{\frac{n_{k-1}}{n_k}} = \sqrt{d_k - l + 1},$$

$$b_j = \sqrt{\frac{n_j}{n_{j+1}}} = \sqrt{d_{j+1} - 1}$$
$$\text{for} \quad j = 1, 2, 3, ..., k-2.$$

For these values, the recursion formula (4) gives the polynomials $S_j(\lambda), j = 0, 1, 2, ..., k-1, S_k^+(\lambda)$ and $S_k^-(\lambda)$.

This completes the proof. #

Theorem 3.1.3 Let $R_j, j = 1, 2, ..., k-1, R_k^+$ and R_k^- as above. then

(i)$\sigma(A(\mathcal{G}_l)) = (\cup_{j\in\Omega}\sigma(R_j)) \cup \sigma(R_k^+) \cup \sigma(R_k^-)$.

(ii) The multiplicity of each eigenvalue of the matrix R_j, as an eigenvalue of $A(\mathcal{G}_l)$, is at least $n_j - n_{j+1}$ for $j \in \Omega$, 1 for the eigenvalues of R_k^+ and $l-1$ for the eigenvalues of R_k^-.

Proof. (i) is an immediate consequence of Theorem 3.1.1 and Theorem 3.1.2. From Lemma 3.2 that the eigenvalues of $R_j, j = 1, 2, ..., k-1, R_k^+$ and R_k^- are simply. Finally, we use (5) and Theorem 3.1.2 to obtain(ii). #

Theorem 3.1.4 Let $A(\mathcal{G}_l)$ be the adjacency matrix of $\mathcal{G}_l$. Then

(a$_1$) $\sigma(R_{j-1}) \cap \sigma(R_j) = \phi$ for $j = 2, 3, ..., k-1$.

(a$_2$) $\sigma(R_{k-1}) \cap \sigma(R_k^+) = \phi$ and $\sigma(R_{k-1}) \cap \sigma(R_k^-) = \phi$.

(a$_3$) The largest eigenvalue of R_k^+ is the largest eigenvalue of $A(\mathcal{G}_l)$ and the largest eigenvalue of R_k^- is the second largest eigenvalue of $A(\mathcal{G}_l)$.

Proof. (a$_1$) and (a$_2$) follow from Lemma 3.2.

By Lemma 3.3 and

$$R_k^+ = R_k^- + diag\{0, 0, ..., 0, l\},$$

we can get the eigenvalues of R_k^+ are greater or equal to the eigenvalues of R_k^-. Now (a$_3$) follow from this fact and Lemma3.2. #

Example 3.1.1

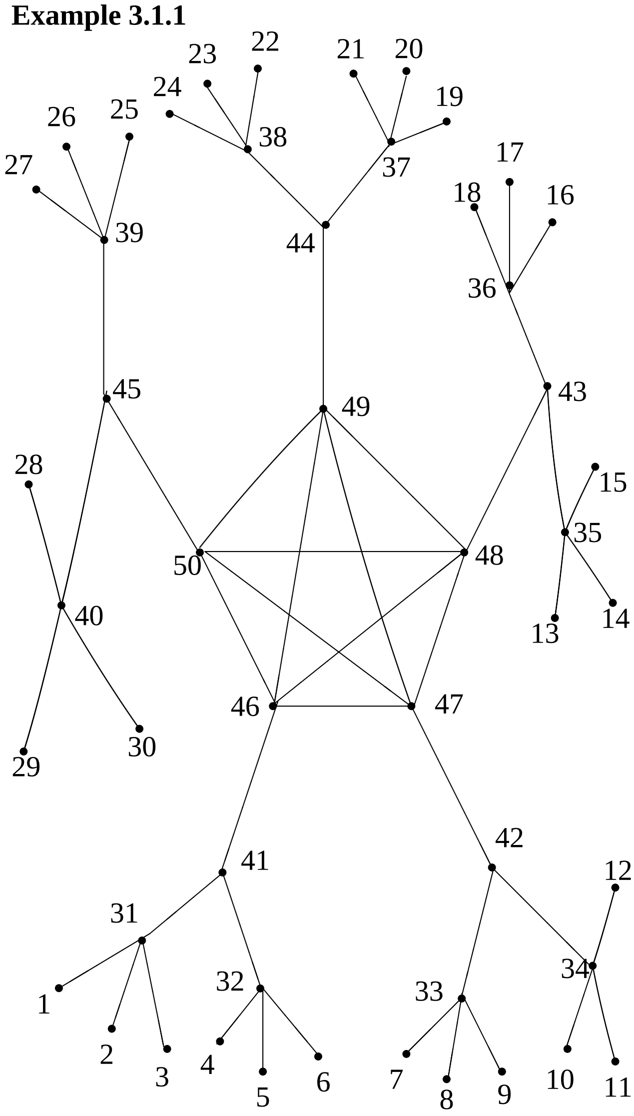

Fig.2 graph $\mathcal{G}_5$

For the graph $\mathcal{G}_5$ in Fig.2 for $l = 5, k = 4, n_1 = 30, n_2 = 10, n_3 = n_4 = 5, d_1 = 1, d_2 = 4, d_3 = 3, d_4 = 5$.

$$R_1 = 0, R_2 = \begin{pmatrix} 0 & \sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}$$

$$R_3 = \begin{pmatrix} 0 & \sqrt{3} & \\ \sqrt{3} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$R_4^+ = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

$$R_4^- = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

and $\Omega = \{1, 2\}$. By Theorem 3.1.3, the eigenvalues of $A(\mathcal{G}_5)$ in Fig.2 are the eigenvalues of R_1, R_2, R_4^+ and R_4^-, they are

$$\begin{array}{lcccc} R_1: & 0 & & & \\ R_2: & -1.7320 & 1.7320 & & \\ R_4^-: & -2.4142 & -1.3028 & 0.4142 & 2.3028 \\ R_4^+: & -2.2696 & -0.1444 & 2.1444 & 4.2696 \end{array}$$

The spectral radius of $\mathcal{G}_5$ in Fig.2 is $\lambda_1(A(\mathcal{G}_5)) = 4.2696$.

$SpecA(\mathcal{G}_5) =$

$$\begin{pmatrix} 4.2696 & 2.3028 & 2.1444 & 1.7320 & 0.4142 & 0 & -0.1444 & -1.3028 & -1.7320 & -2.2696 & -2.4142 \\ 1 & 4 & 1 & 5 & 4 & 20 & 1 & 4 & 5 & 1 & 4 \end{pmatrix}.$$

3.2 The spectrum of the Laplacian matrix of $\mathcal{G}_l$

Theorem 3.2.1 Let

$$P_0(\mu) = 1, P_1(\mu) = \mu - 1,$$

$P_j(\mu) = (\mu - d_j)P_{j-1}(\mu) - \frac{n_{j-1}}{n_j}P_{j-2}(\mu)$, for $j = 2, 3, ..., k$,

$$P_k^+(\mu) = (\mu - (d_k + 1))P_{k-1}(\mu) - \frac{n_{k-1}}{l}P_{k-2}(\mu)$$

and

$$P_k^-(\mu) = (\mu - (d_k - l + 1))P_{k-1}(\mu) - \frac{n_{k-1}}{l}P_{k-2}(\mu).$$

Then

(i) If $P_j(\mu) \neq 0$, for all $j = 1, 2, ..., k - 1$, then

$$\begin{aligned} det(\mu I - L(\mathcal{G}_l)) &= (P_k^+(\mu))^{l-1}P_k^-(\mu) \\ &\quad \times \textstyle\prod_{j\in\Omega} P_j^{n_j - n_{j+1}}(\mu). \end{aligned} \tag{6}$$

(ii)The spectrum of $L(\mathcal{G}_l)$ is $\sigma(L(\mathcal{G}_l)) = (\cup_{j\in\Omega}\{\mu : P_j(\mu) = 0\}) \cup \{\mu : P_k^-(\mu) = 0\} \cup \{\mu : P_k^+(\mu) = 0\}$.

Proof. Suppose $P_j(\mu) \neq 0$ for all $j = 1, 2, ..., k - 1$. We apply Lemma 2.1 to $M = \mu I - L(\mathcal{G}_l)$, we denote $\mu - d_j = x_j, (j = 1, 2, ..., k)$, then

$M = \mu\mathbf{I} - L(\mathcal{G}_l) =$

$$\begin{pmatrix} x_1\mathbf{I_{n_1}} & C_1 & & & & \\ C_1^T & x_2\mathbf{I_{n_2}} & C_2 & & & \\ & C_2^T & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & x_{k-1}\mathbf{I_{n_{k-1}}} & C_{k-1} \\ & & & & C_{k-1}^T & x_k\mathbf{I_{n_k}} + B_l \end{pmatrix}.$$

We have

$$\beta_1 = \mu - d_1 = \mu - 1 = P_1(\mu) \neq 0,$$

$$\begin{aligned} \beta_2 &= (\mu - d_2) - \frac{n_1}{n_2}\frac{1}{\beta_1} \\ &= \mu - d_2 - \frac{n_1}{n_2}\frac{1}{P_1(\mu)} \\ &= \frac{(\mu - d_2)P_1(\mu) - \frac{n_1}{n_2}P_0(\mu)}{P_1(\mu)} \\ &= \frac{P_2(\mu)}{P_1(\mu)} \neq 0. \end{aligned}$$

Similarly, for $j = 3, 4, ..., k - 1, k$

$$\begin{aligned} \beta_j &= (\mu - d_j) - \frac{n_{j-1}}{n_j}\frac{1}{\beta_{j-1}} \\ &= \mu - d_j - \frac{n_{j-1}}{n_j}\frac{P_{j-2}(\mu)}{P_{j-1}(\mu)} \\ &= \frac{(\mu - d_j)P_{j-1}(\mu) - \frac{n_{j-1}}{n_j}P_{j-2}(\mu)}{P_{j-1}(\mu)} \\ &= \frac{P_j(\mu)}{P_{j-1}(\mu)} \neq 0. \end{aligned}$$

Thus

$$\begin{aligned}\beta_k - 1 &= \frac{P_k(\mu)}{P_{k-1}(\mu)} - 1\\ &= \frac{P_k(\mu) - P_{k-1}(\mu)}{P_{k-1}(\mu)}\\ &= \frac{(\mu - d_k - 1)P_{k-1}(\mu) - \frac{n_{k-1}}{l}P_{k-2}(\mu)}{P_{k-1}(\mu)}\\ &= \frac{P_k^+(\mu)}{P_{k-1}(\mu)},\end{aligned}$$

$$\begin{aligned}\beta_k + l - 1 &= \frac{P_k(\mu)}{P_{k-1}(\mu)} + l - 1\\ &= \frac{P_k(\mu) + (l-1)P_{k-1}(\mu)}{P_{k-1}(\mu)}\\ &= \frac{(\mu - d_k + l - 1)P_{k-1}(\mu) - \frac{n_{k-1}}{l}P_{k-2}(\mu)}{P_{k-1}(\mu)}\\ &= \frac{P_k^-(\mu)}{P_{k-1}(\mu)}.\end{aligned}$$

Therefore, from Lemma 2.1,

$$\begin{aligned}&det(\mu\mathbf{I} - L(\mathcal{G}_l))\\ =\ & P_1^{n_1}(\mu)\frac{P_2^{n_2}(\mu)}{P_1^{n_2}(\mu)}\cdots\frac{P_{k-1}^{n_{k-1}}(\mu)}{P_{k-2}^{n_{k-1}}(\mu)}\frac{P_k^-(\mu)}{P_{k-1}(\mu)}\frac{(P_k^+(\mu))^{l-1}}{P_{k-1}^{l-1}(\mu)}\\ =\ & P_1^{n_1-n_2}(\mu)P_2^{n_2-n_3}(\mu)\ldots P_{k-1}^{n_{k-1}-n_k}(\mu)\\ & \times P_k^-(\mu)(P_k^+(\mu))^{l-1}\\ =\ & (P_k^+(\mu))^{l-1}P_k^-(\mu)\prod_{j\in\Omega}P_j^{n_j-n_{j+1}}(\mu).\end{aligned}$$

Thus (i) is proved. Similar to the proof Theorem 3.1.1, we can get (ii) by (i) . #

Let W_k^+ and W_k^- be the $k\times k$ symmetric tridiagonal matrices

$W_k^+ =$

$$\begin{pmatrix} 1 & \sqrt{d_2-1} & & & \\ \sqrt{d_2-1} & d_2 & \sqrt{d_3-1} & & \\ & \sqrt{d_3-1} & \ddots & \ddots & \\ & & \sqrt{d_{k-1}-1} & d_{k-1} & \sqrt{d_k-l+1}\\ & & & \sqrt{d_k-l+1} & d_k+1 \end{pmatrix}$$

and $W_k^- =$

$$\begin{pmatrix} 1 & \sqrt{d_2-1} & & & \\ \sqrt{d_2-1} & d_2 & \sqrt{d_3-1} & & \\ & \sqrt{d_3-1} & \ddots & \ddots & \\ & & \sqrt{d_{k-1}-1} & d_{k-1} & \sqrt{d_k-l+1}\\ & & & \sqrt{d_k-l+1} & d_k-l+1 \end{pmatrix}.$$

Observe that

$$W_k^+ = W_k^- + diag\{0, 0, ..., 0, l\}.$$

Theorem 3.2.2 For $j = 1, 2, 3, ..., k-1$, let W_j be the $j\times j$ leading principal submatrix W_k^+. Then

$$\begin{aligned}det(\mu\mathbf{I} - W_j) &= P_j(\mu), j = 1, 2, ..., k-1,\\ det(\mu\mathbf{I} - W_k^-) &= P_k^-(\mu),\\ det(\mu\mathbf{I} - W_k^+) &= P_k^+(\mu).\end{aligned}$$

Proof. Similar to the proof of Theorem 3.1.2, in this case, $a_1 = 1$, $a_j = d_j$for$j = 2, 3, ..., k-1$, $a_k = d_k + 1$ (or $a_k = d_k - l + 1$) and

$$b_{k-1} = \sqrt{\frac{n_{k-1}}{n_k}} = \sqrt{d_k - l + 1},$$

$$b_j = \sqrt{\frac{n_j}{n_{j+1}}} = \sqrt{d_{j+1} - 1}$$
$$\text{for}\quad j = 1, 2, 3, ..., k-2.$$

For these values, recursion formula (4) gives the polynomials $P_j(\mu), j = 0, 1, 2, ..., k-1, P_k^+(\mu)$ and $P_k^-(\mu)$.

This completes the proof. #

Similar to the proof of Theorem 3.1.3, we can get:

Theorem 3.2.3 Let $W_j, j = 1, 2, ..., k-1, W_k^+$ and W_k^- as above. then

(i)$\sigma(L(\mathcal{G}_l)) = (\cup_{j\in\Omega}\sigma(W_j))\cup\sigma(W_k^+)\cup\sigma(W_k^-)$.

(ii) The multiplicity of each eigenvalue of the matrix W_j, as an eigenvalue of $L(\mathcal{G}_l)$, is at least $n_j - n_{j+1}$ for $j\in\Omega$, 1 for the eigenvalues of W_k^- and $l-1$ for the eigenvalues of W_k^+.

Theorem 3.2.4 Let $L(\mathcal{G}_l)$ be the Laplacian matrix of $\mathcal{G}_l$. Then

(b_1) $\sigma(W_{j-1})\cap\sigma(W_j) = \phi$ for $j = 2, 3, ..., k-1$.

(b_2) $\sigma(W_{k-1})\cap\sigma(W_k^+) = \phi$ and $\sigma(W_{k-1})\cap\sigma(W_k^-) = \phi$.

(b_3) $\det W_j = 1$ for $j = 1, 2, ..., k-1$, $\det W_k^- = 0$ and $\det W_k^+ = l$.

(b_4) The largest eigenvalue of W_k^+ is the largest eigenvalue of $L(\mathcal{G}_l)$.

(b_5) The smallest eigenvalue of W_k^+ is the algebraic connectivity $\mathcal{G}_l$.

Proof. (b_1) and (b_2) follow from Lemma 3.2.

Now we apply the Gaussian elimination procedure, without row interchanges, to reduce W_j to the upper triangular matrix

$W_j \Rightarrow$

$$\begin{pmatrix} 1 & \sqrt{d_2-1} & & & \\ & 1 & \sqrt{d_3-1} & & \\ & & 1 & \ddots & \\ & & & \ddots & \sqrt{d_{j-1}-1} & \\ & & & & 1 & \sqrt{d_j-1}\\ & & & & & 1 \end{pmatrix}.$$

The same procedure applied W_k^+ and W_k^- gives the triangular matrices

$W_k^+ \Rightarrow$

$$\begin{pmatrix} 1 & \sqrt{d_2-1} & & & & \\ & 1 & \sqrt{d_3-1} & & & \\ & & \ddots & \ddots & & \\ & & & & \sqrt{d_{k-1}-1} & \\ & & & & 1 & \sqrt{d_k-l+1} \\ & & & & & l \end{pmatrix}$$

and $W_k^- \Rightarrow$

$$\begin{pmatrix} 1 & \sqrt{d_2-1} & & & & \\ & 1 & \sqrt{d_3-1} & & & \\ & & \ddots & \ddots & & \\ & & & & \sqrt{d_{k-1}-1} & \\ & & & & 1 & \sqrt{d_k-l+1} \\ & & & & & 0 \end{pmatrix}.$$

respectively. Thus (b_3) is proved and 0 is the smallest eigenvalue of W_k^-.

Since

$$W_k^+ = W_k^- + diag\{0, 0, ..., 0, l\},$$

by Lemma 3.3, the eigenvalues of W_k^+ are greater or equal to the eigenvalues of W_k^-. Now (b_4) and (b_5) follow from this fact and Lemma3.2. $\#$

Example 3.2.1 For the graph $\mathcal{G}_5$ in Fig.2

$$W_1 = 1, W_2 = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 4 \end{pmatrix}$$

$$W_3 = \begin{pmatrix} 1 & \sqrt{3} & 0 \\ \sqrt{3} & 4 & \sqrt{2} \\ 0 & \sqrt{2} & 3 \end{pmatrix}$$

$$W_4^+ = \begin{pmatrix} 1 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 4 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 3 & 1 \\ 0 & 0 & 1 & 6 \end{pmatrix}$$

$$W_4^- = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and $\Omega = \{1, 2\}$. By Theorem 3.2.2, the eigenvalues of $L(\mathcal{G}_5)$ in Fig.2 are the eigenvalues of W_1, W_2, W_4^+ and W_4^-, they are

W_1 :	1			
W_2 :	0.2088	4.7912		
W_4^- :	0	0.6385	2.8327	5.5287
W_4^+ :	0.0658	2.2511	5.2586	6.4244

The spectral radius of $\mathcal{G}_5$ in Fig.2 is $\mu_1(L(\mathcal{G}_5)) = 6.4244$.

$$SpecL(\mathcal{G}_5) = \begin{pmatrix} 6.4244 & 5.5287 & 5.2586 & 4.7912 & 2.8327 & 2.2511 & 1 & 0.6385 & 0.2088 & 0.0658 & 0 \\ 4 & 1 & 4 & 5 & 1 & 4 & 20 & 1 & 5 & 4 & 1 \end{pmatrix}.$$

4 An upper bound for the largest eigenvalue of the adjacency matrix $A(\mathcal{G}_l)$ and the Laplacian matrix $L(\mathcal{G}_l)$

In this section we will give an upper bound for the largest eigenvalue of the adjacency matrix and the Laplaican matrix of graph $\mathcal{G}_l$.

Lemma 4.1[3] Let $B = (b_{ij})$ be a nonnegative $n \times n$ matrix and λ is the largest eigenvalue of matrix B. Denote the ith row sum of B by $s_i(B)$. Then

$$\min_{1\le i\le n} s_i(B) \le \lambda \le \max_{1\le i\le n} s_i(B),$$

the left equality holds if and only if the right equality also holds.

4.1 An upper bound for the largest eigenvalue of the adjacency matrix $A(\mathcal{G}_l)$

Theorem 4.1.1 Let $\mathcal{G}_l$ (l is a positive integer) be the graph as above and that $\mathcal{G}_l$ has k levels, d_{k-j+1} be the degree of vertices in the level j, then

$$\lambda_1(A(\mathcal{G}_l)) < \max \Big\{ \max_{2\le j\le k-2}\{\sqrt{d_j-1}+\sqrt{d_{j+1}-1}\}, \sqrt{d_{k-1}-1}+\sqrt{d_k-l+1}, \sqrt{d_k-l+1}+l-1 \Big\}.$$

Proof. Let $R_k^+ =$

$$\begin{pmatrix} 0 & \sqrt{d_2-1} & & & \\ \sqrt{d_2-1} & 0 & \sqrt{d_3-1} & & \\ & \sqrt{d_3-1} & \ddots & \ddots & \\ & \sqrt{d_{k-1}-1} & 0 & \sqrt{d_k-l+1} \\ & & \sqrt{d_k-l+1} & l-1 \end{pmatrix},$$

we now apply Lemma 4.1 to conclude

$$\lambda_1(R_k^+) < \max \Big\{ \max_{2\le j\le k-2}\{\sqrt{d_j-1}+\sqrt{d_{j+1}-1}\}, \sqrt{d_{k-1}-1}+\sqrt{d_k-l+1}, \sqrt{d_k-l+1}+l-1 \Big\}.$$

From Theorem 3.3.4, we easily have

$$\lambda_1(A(\mathcal{G}_l)) < \max \Big\{ \max_{2\le j\le k-2}\{\sqrt{d_j-1}+\sqrt{d_{j+1}-1}\}, \sqrt{d_{k-1}-1}+\sqrt{d_k-l+1}, \sqrt{d_k-l+1}+l-1 \Big\}.\#$$

4.2 An upper bound for the largest eigenvalue of the Laplacian matrix $L(\mathcal{G}_l)$

In this section we give an upper bound for the largest eigenvalue of the Laplacian matrix $L(\mathcal{G}_l)$.

Theorem 4.2.1 Let $\mathcal{G}_l$ (l is a positive integer) be the graph as above and that $\mathcal{G}_l$ has k levels, d_{k-j+1} be the degree of vertices in the level j, then

$$\mu_1(L(\mathcal{G}_l)) < \max \Big\{ \max_{2\le j\le k-2}\{\sqrt{d_j-1}+d_j+\sqrt{d_{j+1}-1}\}, \sqrt{d_{k-1}-1}+d_{k-1}+\sqrt{d_k-l+1}, \sqrt{d_k-l+1}+d_k+1 \Big\}.$$

Proof. Let $W_k^+ =$

$$\begin{pmatrix} 1 & \sqrt{d_2-1} & & & \\ \sqrt{d_2-1} & d_2 & \sqrt{d_3-1} & & \\ & \sqrt{d_3-1} & \ddots & \ddots & \\ & \sqrt{d_{k-1}-1} & d_{k-1} & \sqrt{d_k-l+1} \\ & & \sqrt{d_k-l+1} & d_k+1 \end{pmatrix}$$

we now apply Lemma 4.1 to conclude

$$\mu_1(W_k^+) < \max \Big\{ \max_{2\le j\le k-2}\{\sqrt{d_j-1}+d_j+\sqrt{d_{j+1}-1}\}, \sqrt{d_{k-1}-1}+d_{k-1}+\sqrt{d_k-l+1}, \sqrt{d_k-l+1}+d_k+1 \Big\}.$$

From Theorem 3.2.4, we easily have

$$\mu_1(L(\mathcal{G}_l)) < \max \Big\{ \max_{2\le j\le k-2}\{\sqrt{d_j-1}+d_j+\sqrt{d_{j+1}-1}\}, \sqrt{d_{k-1}-1}+d_{k-1}+\sqrt{d_k-l+1}, \sqrt{d_k-l+1}+d_k+1 \Big\}.\#$$

5 Conclusion

We studied the spectrum of the adjacency matrix $A(\mathcal{G}_l)$ and the spectrum of the Laplacian matrix $L(\mathcal{G}_l)$ for all positive integer l with an effective way.

(I). Let $R_j, j = 1,2,...,k-1, R_k^+$ and R_k^- as in section 3. We found that:

(I$_1$)$\sigma(A(\mathcal{G}_l)) = (\cup_{j\in\Omega}\sigma(R_j)) \cup \sigma(R_k^+) \cup \sigma(R_k^-)$.

(I$_2$) The multiplicity of each eigenvalue of the matrix R_j, as an eigenvalue of $A(\mathcal{G}_l)$, is at least $n_j - n_{j+1}$ for $j \in \Omega$, 1 for the eigenvalues of R_k^+ and $l-1$ for the eigenvalues of R_k^-.

(I$_3$)The largest eigenvalue of R_k^+ is the largest eigenvalue of $A(\mathcal{G}_l)$ and the largest eigenvalue of R_k^- is the second largest eigenvalue of $A(\mathcal{G}_l)$.

It is very convenient with conclusions (I$_1$)(I$_2$)(I$_3$) to calculate the spectrum of the adjacency matrix $A(\mathcal{G}_l)$.

In section 4.1, according to the results(I$_1$)(I$_2$)(I$_3$) and Lemma 4.1, an upper bound for the largest eigenvalue of the adjacency matrix $A(\mathcal{G}_l)$ is obtained:

$$\lambda_1(A(\mathcal{G}_l)) < \max \Big\{ \max_{2\le j\le k-2}\{\sqrt{d_j-1}+\sqrt{d_{j+1}-1}\}, \sqrt{d_{k-1}-1}+\sqrt{d_k-l+1}, \sqrt{d_k-l+1}+l-1 \Big\}.$$

(II). Let $W_j, j = 1,2,...,k-1, W_k^+$ and W_k^- as in section 3.2. We found that:

(II$_1$) $\sigma(L(\mathcal{G}_l)) = (\cup_{j\in\Omega}\sigma(W_j)) \cup \sigma(W_k^+) \cup \sigma(W_k^-)$.

(II$_2$) The multiplicity of each eigenvalue of the matrix W_j, as an eigenvalue of $L(\mathcal{G}_l)$, is at least $n_j - n_{j+1}$ for $j \in \Omega$, 1 for the eigenvalues of W_k^- and $l-1$ for the eigenvalues of W_k^+.

(II$_3$) The largest eigenvalue of W_k^+ is the largest eigenvalue of $L(\mathcal{G}_l)$.

(II$_4$) The smallest eigenvalue of W_k^+ is the algebraic connectivity $\mathcal{G}_l$.

In section 4.2, according to the results(II$_1$)(II$_2$)(II$_3$) and Lemma 4.1, an upper bound for the largest eigenvalue of the Laplacian matrix $L(\mathcal{G}_l)$ is obtained:

$$\mu_1(L(\mathcal{G}_l)) < \max \Big\{ \max_{2\le j\le k-2}\{\sqrt{d_j-1}+d_j+\sqrt{d_{j+1}-1}\}, \sqrt{d_{k-1}-1}+d_{k-1}+\sqrt{d_k-l+1}, \sqrt{d_k-l+1}+d_k+1 \Big\}.$$

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