An Algorithm for Improving the Accuracy of Z-Coordinate Determination for USBL Systems

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Abstract: - This paper presents an approach to resolve the problem of low Z-coordinate determination accuracy for ultra-short baseline (USBL) acoustic systems in which the location measurement of the underwater objects is based on the determination of the distance to the object and the angle object position. The antenna with five receiving elements and different spatial orientation of receiving bases is proposed. It is assumed that the receiving antenna may have the significant inclinations and the orientation of the USBL antenna which is controlled by measuring of the pitch and roll angles. The problem of the low precision Z-coordinate determination is resolved by means of utilizing various spatial orientations of the receiving bases in a modernized five element USBL system. The computer simulation results of the proposed algorithm are presented to demonstrate reliable operation of the algorithm for different angular antenna positions and various locations of the object. The results of the calculation of the errors of the determination of the object coordinates and accuracy evaluation of modernized USBL system are presented.

Key-Words: - Ultra-short baseline (USBL) system, underwater object, transponder, carrier coordinate system, local coordinate systems, pitch and roll angles.

1 Introduction

The determination of the position of an underwater object with respect to a given reference point is required for the wide range of tasks in ocean engineering. In some cases the accuracy in determining the position of the object may be critical. Within the scope of this task, the design of the methods of adaptation of underwater systems to the variations of operational conditions has a special significance. For some classes of systems, in particular for the underwater navigation and positioning systems, this implies taking into account the conditions of sea surface and inhomogeneity of water. The methods and the systems for determining the position of underwater objects are being constantly developed with a view to improving the reliability and accuracy of object coordinate measurements.

During the past few decades the special attention in this class of systems was paid to application of new signal processing techniques and signal filtering. In this paper, we examine a coordinate determination algorithm for the systems designed to measure the location of underwater objects equipped with transponders. The proposed algorithm is focused on resolving the problem of low precision Z-coordinate determination for ultra-short baseline (USBL) systems.

2 Problem Formulation

2.1 USBL measurement principle

To determine the position of the dynamic underwater objects, as a rule, three basic methods are used [1][2]. The first method utilizes the measurement of the distances between the object and the seabed reference points with the known coordinates. The distances are obtained through the measurement of the propagation times of the acoustic pulses emitted by an emitter placed on the object and by the transponders placed on the sea floor. The values of coordinates of transponders and the measured distances from the object to transponders enable us to calculate the coordinates of the object. This method requires at least three bottom transponders and special polygon calibration to determine transponder coordinates. Systems utilizing this method are called long baseline (LBL) systems. The second method to determine the position of underwater objects uses the measurement of the distance to the object and its angular location. The measuring system is located in

a reference point, while the underwater object is equipped with the transponder. The distance being determined by the measurement of the values of propagation times of the requested impulse of the system and the response transponder pulse. The angular location of the object is determined by the measurement of the delays of the envelope of the transponder pulse on the elements of the receiving antenna. The minimum number of elements of the receiving antenna is three. As a rule, these three elements are placed in a horizontal plane and form two receiving bases (one of the elements is utilized for both bases), two receiving bases form a right angle. These types of systems are called short baseline (SBL) systems.
In this paper we describe coordinate determination algorithm for ultra-short baseline (USBL) acoustic systems, which employs the same principle as SBL systems: the distance to the object and its angular location are measured. Unlike the SBL method (where the delays of the front of the transponder impulse are measured), the phase difference of the pulse carrier frequency on the elements of the receiving antenna is measured. For an unambiguous coordinate determination, the size of the receiving bases must be less than half of the acoustic wavelength of the transponder pulse carrier frequency.
Before representing the designed coordinate determination algorithm, we will consider the technique of the coordinate determination for the case of the three element USBL system. Let system *Σ=(0,x,y,z)* be the left hand Cartesian coordinate system. Let the plane of the USBL antenna be aligned with the *(x,y)* coordinate plane and the z-axis has the downwards direction. We align the origin *O* of coordinate system with the receiving element 2. We align the x-axis with the receiving

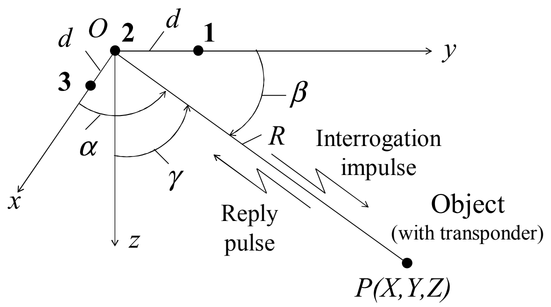


Fig.1 Three element USBL antenna and the point *P* in introduced coordinate system

base 3-2 and the y-axis with the receiving base 1-2. Three element USBL antenna and the point *P* in the introduced coordinate system *Σ=(0,x,y,z)* are represented in Fig.1. Let the size of the receiving bases be *d*. Now we can define the coordinates of the point *P(X,Y,Z)*. The time delays τ_{12} and τ_{32} of the signal on the outputs of the elements of the base 1-2 and the base 3-2 (it is supposed that *R>>d*) can be expressed as:

$$\tau_{12} = \frac{d\cos\beta}{c}, \quad \tau_{32} = \frac{d\cos\alpha}{c}, \tag{1}$$

where *c* is the speed of the sound in the water.
With *d/c* defined as τ_d we can write the direction cosines $cos\alpha$ and $cos\beta$:

$$\cos\alpha = \tau_{32}/\tau_d, \quad \cos\beta = \tau_{12}/\tau_d \,. \tag{2}$$

The third direction cosine is:

$$\cos\gamma = \sqrt{1-\left(\tau_{32}/\tau_d\right)^2-\left(\tau_{12}/\tau_d\right)^2} \,. \tag{3}$$

Cartesian coordinates of the point *P* define as:

$$X = R\cos\alpha\,, \quad Y = R\cos\beta\,, \quad Z = R\cos\gamma\,. \tag{4}$$

Azimuth angle φ of point *P* can be obtained from expression:

$$\varphi = arctg(\cos\alpha/\cos\beta)\,. \tag{5}$$

2.2 Accuracy problem

The object position accuracy in real marine conditions is the key to the development of the USBL systems. There are four main factors that can cause errors in the measurement of the position of underwater objects by the USBL systems. These factors are as follows: the inhomogeneity of the propagation medium and in particularly the phenomenon of the acoustic refraction in the water; the phenomenon of multipath interference due to multiple reflections of acoustic rays from the sea surface, the sea floor and the carrier hull; the phenomenon of diffraction of the acoustic waves in the body of the receiving antenna; the instability of the position of the receiving antenna [3][4].
The errors caused by the first factor can be considered as a systematic component of the errors of the coordinate measurements. These errors can be taken into account by means of the calculation of the real trajectories of acoustic rays with the

subsequent determination of the real position of the object [5]. The second factor can be resolved by suppressing the signals coming from unexpected directions and by taking into account that these signals have a time lag relative to the first incoming impulse [6]. To improve the accuracy of USBL systems various special signal processing techniques were designed. In particular the chirp signals and greater inter-element array separation [7] were used. Also the acoustic digital spread spectrum [8] and modulated Barker-coded signals [9] were applied. In [10] the USBL system with frequency-hopped pulses was investigated.

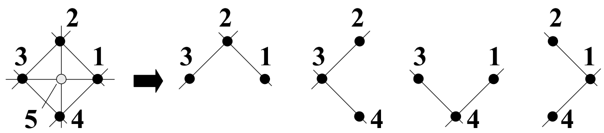
Reduction of impact of the third factor can be achieved by the improvement of antenna body construction and by optimization of the choice of size of sensors and operating frequency [11]. One of the methods of reducing the influence of antenna body diffraction errors is to use the redundant receiving elements by means of the increase of the number of elements of the receiving antenna. The fourth factor can be considered as a cause of the systematic errors as in the first one. This error component can be eliminated by measuring the inclination of the USBL antenna in the moment of measuring the antenna time delays and the following recalculation of object position [12].

In this paper we will concentrate on the fourth aspect of the accuracy problem. We also solve the problem of low precision in coordinate determination for the case the object found in the plane of receiving bases. The method to resolve specified problems includes the increase of the number of receiving bases of the USBL antenna and the design of an algorithm that enables the system to take into account the occasional inclination of the USBL antenna. It is assumed that the space orientation of the antenna is controlled by the measurement of the pitch and roll angles. Furthermore, in this paper we also assume that the propagation medium is homogeneous and multipath interference is absent.

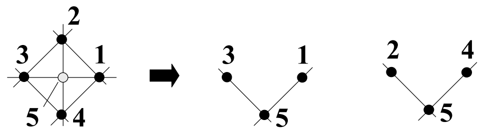
3 Problem Solution

3.1 Antenna and coordinate systems

Before introducing an explanation of the algorithm, some changes in the antenna construction must be made. At first, we add one more element to the antenna array in the horizontal plane. This element is added in such a way that a square is formed (the sides of the square form the receiving bases of the antenna). As result we have four basic three element USBL arrays (see Fig.2, a)).



a) decomposition of the receiving antenna on four basic USBL antennas in horozontal plane



b) decomposition of the receiving antenna on two basic USBL antennas in two ortogonal vertical planes

Fig.2. The five element receiving antenna and its decomposition on the basic USBL arrays.

The second modification of the antenna construction is the aggregation of the fifth element in another horizontal plane. The fifth element is placed underneath the four element USBL antenna plane exactly underneath its geometric center, so that the four obtained inclined bases would be the same size as the size of the horizontal receiving bases. In this case, the USBL system obtains two additional three element basic USBL arrays placed in two orthogonal vertical planes. Finally, we have the five element USBL array. The decomposition of the five element USBL antenna on the six basic USBL antennas is presented in Fig.2.

The proposed antenna construction allows the elimination of the known drawback of the USBL systems with only horizontal plane array arrangement: the low accuracy of measuring the vertical coordinate when the object depth is close to the operating depth of the receiving antenna.

The receiving antenna orientation and the axis of the USBL system carrier are shown in Fig.3. From hereon in this paper, the carrier is represented by means of its longitudinal l-l' and lateral b-b' axes (the straight arrowed line shows the direction of movement of the carrier). The coordinates of an object are determined in the carrier coordinate system.

Let $\Sigma=(0,x,y,z)$ be the carrier coordinate system with the origin in the point O in such a way that the x-coordinate goes throughout the axis l-l' (the positive direction coincides with the direction of the straight arrowed line) and the y-coordinate goes throughout the axis b-b', and z-axis goes downwards. We will also continue to utilize the earlier introduced basic USBL coordinate system (see Fig.1). We now define it as a local coordinate system.

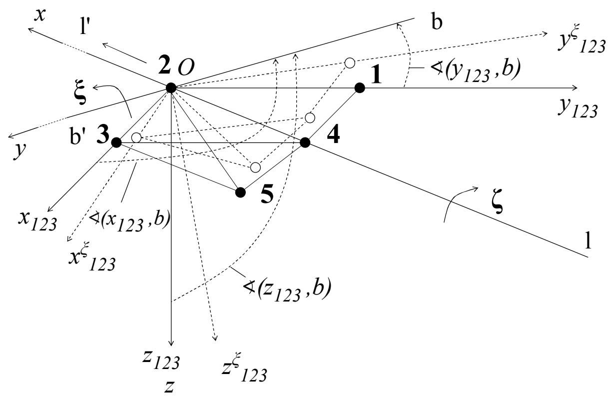


Fig.3. The receiving antenna orientation and the carrier longitudinal and lateral axes.

Let $\Sigma_{123}=(0,x_{123},y_{123},z_{123})$ be the local coordinate system for the basic USBL system with antenna elements 1,2,3. We will later introduce the coordinate systems for the other basic three-element USBL system.

3.2 Calculation expressions

In this section, we examine this newly-introduced five element USBL system when the receiving antenna may be in an unstable position. We assume that the USBL system is equipped with the special block to measure the pitch and roll angles of the receiving antenna. Let angles ξ and ζ be pitch and roll angles of the receiving USBL antenna. The case of the pitch inclination of the receiving antenna (rotation of the antenna about the b-b' axis by the angle ξ) is shown in Fig.3.

Let $USBL_{123}$ system be the basic USBL system with receiving elements 1,2,3. If the antenna of $USBL_{123}$ system has a pitch angle, this signifies the rotation of the coordinate system $\Sigma_{123}=(0,x_{123},y_{123},z_{123})$ around the axis b-b', with the transformation of the $\Sigma_{123}=(0,x_{123},y_{123},z_{123})$ in a new coordinate system $\Sigma^{\xi}_{123}=(0,x^{\xi}_{123},y^{\xi}_{123},z^{\xi}_{123})$. For the coordinate system Σ_{123}, we have the following direction cosines for the axis b-b' (see the Fig.3):

$$\cos\sphericalangle(x_{123},b)=-\sqrt{2}/2\,;$$
$$\cos\sphericalangle(y_{123},b)=\sqrt{2}/2\,;$$
$$\cos\sphericalangle(z_{123},b)=0\,. \tag{6}$$

To obtain the compact expressions for the coordinates, we introduce the following designations for the direction cosines:

$$\cos\sphericalangle(x_{123},b)=\eta_{123}\,;$$
$$\cos\sphericalangle(y_{123},b)=\chi_{123}\,;$$
$$\cos\sphericalangle(z_{123},b)=\nu_{123}\,. \tag{7}$$

Let X^{ξ}_{123}, Y^{ξ}_{123}, Z^{ξ}_{123} be the coordinates of the object in the coordinate system Σ^{ξ}_{123}. Knowing the direction cosines for the coordinate system Σ_{123} and the axis b-b' we can find the coordinates of the object in the new coordinate system $\Sigma^{\xi}_{123}=(0,x^{\xi}_{123},y^{\xi}_{123},z^{\xi}_{123})$. The coordinates X^{ξ}_{123}, Y^{ξ}_{123}, Z^{ξ}_{123} in the coordinate system Σ^{ξ}_{123} are obtained with the formulas [13]:

$$\begin{aligned}
X^{\xi}_{123}&=X_{123}(\cos\xi+\eta^{2}_{123}(1-\cos\xi))+Y_{123}(\nu_{123}\sin\xi+\eta_{123}\chi_{123}(1-\cos\xi))+Z_{123}(-\chi_{123}\sin\xi+\eta_{123}\nu_{123}(1-\cos\xi))\\
Y^{\xi}_{123}&=X_{123}(-\nu_{123}\sin\xi+\chi_{123}\eta_{123}(1-\cos\xi))+Y_{123}(\cos\xi+\chi^{2}_{123}(1-\cos\xi))+Z_{123}(\eta_{123}\sin\xi+\chi_{123}\nu_{123}(1-\cos\xi))\\
Z^{\xi}_{123}&=X_{123}(\chi_{123}\sin\xi+\eta_{123}\nu_{123}(1-\cos\xi))+Y_{123}(-\eta_{123}\sin\xi+\nu_{123}\chi_{123}(1-\cos\xi))+Z_{123}(\cos\xi+\nu^{2}_{123}(1-\cos\xi))
\end{aligned} \tag{8}$$

This system can be rewritten in the matrix form:

$$\boldsymbol{p}^{\xi}_{123}=\boldsymbol{A}\boldsymbol{p}_{123}\,, \tag{9}$$

where:
$\boldsymbol{p}^{\xi}_{123}=[X^{\xi}_{123},Y^{\xi}_{123},Z^{\xi}_{123}]^{T}$; $\boldsymbol{p}_{123}=[X_{123},Y_{123},Z_{123}]^{T}$ and $\boldsymbol{A}$ is matrix :

$$\boldsymbol{A}=\begin{bmatrix}
\cos\xi+\eta^{2}_{123}(1-\cos\xi) & \nu_{123}\sin\xi+\eta_{123}\chi_{123}(1-\cos\xi) & -\chi_{123}\sin\xi+\eta_{123}\nu_{123}(1-\cos\xi)\\
-\nu_{123}\sin\xi+\chi_{123}\eta_{123}(1-\cos\xi) & \cos\xi+\chi^{2}_{123}(1-\cos\xi) & \eta_{123}\sin\xi+\chi_{123}\nu_{123}(1-\cos\xi)\\
\chi_{123}\sin\xi+\eta_{123}\nu_{123}(1-\cos\xi) & -\eta_{123}\sin\xi+\nu_{123}\chi_{123}(1-\cos\xi) & \cos\xi+\nu^{2}_{123}(1-\cos\xi)
\end{bmatrix} \tag{10}$$

Solving the matrix equation (9) we obtain:

$$\boldsymbol{p}_{123}=\boldsymbol{A}^{-1}\boldsymbol{p}^{\xi}_{123}\,. \tag{11}$$

If the antenna of the USBL system, in addition to the pitch angle, has a roll angle ζ, this will transform the coordinate system Σ^{ξ}_{123} in a new coordinate system $\Sigma^{\xi,\zeta}_{123}$ by the rotation of the coordinate system Σ^{ξ}_{123} around the axis l-l'.

In the case of the rotation of the coordinate system Σ^{ξ}_{123} around the axis l-l' in the roll angle, the direction cosines for the x^{ξ}_{123}-axis (base 2-3) and for the y^{ξ}_{123}-axis (base 1-2) will be equal. We find the expression of the direction cosine for the x^{ξ}_{123}-axis (base 2-3). To obtain this expression we consider the rotation of the base 2-3 around the b-b' axis on the pitch angle ξ ($0\le\xi\le90°$). The rotation geometry

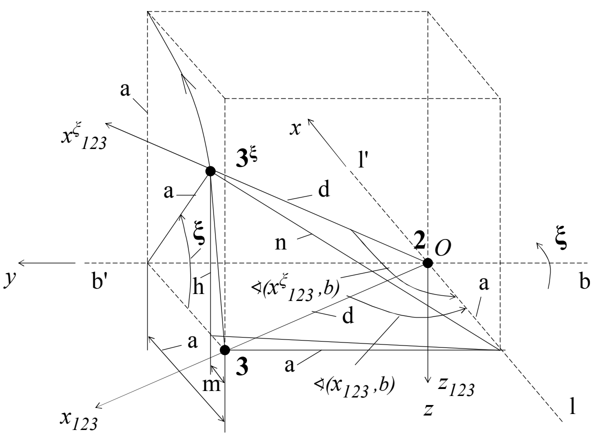


Fig.4. The rotation geometry of the base 2-3 around the b-b' axis on the pitch angle ξ.

of the base 2-3 is shown in Fig.4. In Fig.4 the direction cosine to find is designated as $\cos\sphericalangle(x^{\xi}_{123},l)$.

By introducing in the figure the set of right-angled triangles with the sides d, a, m, n y h we can write the set of equations:

$$h = a(\sin\xi);$$
$$m = a(1-\cos\xi);$$
$$n^2 = m^2 + a^2 + h^2.$$

From the set of equations we can express n^2 through the value a and the angle ξ. Also we can write the n^2 through the direction cosine $\cos\sphericalangle(x^{\xi}_{123},l)$ by applying the cosine law for the triangle with the sides d, a ($a = d\sqrt{2}/2$) and n.

$$n^2 = a^2 + d^2 - 2ad\cos\sphericalangle(x^{\xi}_{123},l).$$

The direction cosine for the z^{ξ}_{123}-axis is determined by pitch angle. Finally, the direction cosines can be expressed as follows:

$$\cos\sphericalangle(x^{\xi}_{123},l) = \cos\xi/\sqrt{2},$$
$$\cos\sphericalangle(y^{\xi}_{123},l) = \cos\xi/\sqrt{2},$$
$$\cos\sphericalangle(z^{\xi}_{123},l) = \cos(\pi/2-\xi). \quad (12)$$

To obtain compact expressions, we introduce the formulas:

$$\cos\sphericalangle(x^{\xi}_{123},l) = \eta^{\xi}_{123};$$
$$\cos\sphericalangle(y^{\xi}_{123},l) = \chi^{\xi}_{123};$$
$$\cos\sphericalangle(z^{\xi}_{123},l) = \nu^{\xi}_{123}. \quad (13)$$

Now we can find the coordinates $X^{\xi,\zeta}_{123}$, $Y^{\xi,\zeta}_{123}$, $Z^{\xi,\zeta}_{123}$ in the coordinate system $\Sigma^{\xi,\zeta}_{123}$ that were obtained by rotation of the coordinate system Σ^{ξ}_{123} around the axis l-l' in roll angle. In matrix form, it will be:

$$\boldsymbol{p}^{\xi,\varsigma}_{123} = \boldsymbol{B}\boldsymbol{p}^{\xi}_{123}, \quad (14)$$

where: $\boldsymbol{p}^{\xi,\zeta}_{123}=[X^{\xi,\zeta}_{123}, Y^{\xi,\zeta}_{123}, Z^{\xi,\zeta}_{123}]^T$; $\boldsymbol{p}^{\xi}_{123}=[X^{\xi}_{123},Y^{\xi}_{123},Z^{\xi}_{123}]^T$; and $\boldsymbol{B}$ is matrix:

$$\boldsymbol{B} = \begin{bmatrix} \cos\zeta+\left(\eta^{\xi}_{123}\right)^2(1-\cos\zeta) & \nu^{\xi}_{123}\sin\zeta+\eta^{\xi}_{123}\chi^{\xi}_{123}(1-\cos\zeta) & -\chi^{\xi}_{123}\sin\zeta+\eta^{\xi}_{123}\nu^{\xi}_{123}(1-\cos\zeta) \\ -\nu^{\xi}_{123}\sin\zeta+\chi^{\xi}_{123}\eta^{\xi}_{123}(1-\cos\zeta) & \cos\zeta+\left(\chi^{\xi}_{123}\right)^2(1-\cos\zeta) & \eta^{\xi}_{123}\sin\zeta+\chi^{\xi}_{123}\nu^{\xi}_{123}(1-\cos\zeta) \\ \chi^{\xi}_{123}\sin\zeta+\eta^{\xi}_{123}\nu^{\xi}_{123}(1-\cos\zeta) & -\eta^{\xi}_{123}\sin\zeta+\nu^{\xi}_{123}\chi^{\xi}_{123}(1-\cos\zeta) & \cos\zeta+\left(\nu^{\xi}_{123}\right)^2(1-\cos\zeta) \end{bmatrix} \quad (15)$$

Solving the matrix equation (14), we obtain:

$$\boldsymbol{p}^{\xi}_{123} = \boldsymbol{B}^{-1}\boldsymbol{p}^{,\varsigma}_{123}. \quad (16)$$

Sustaining (16) in (11), we obtain the coordinates of the object in the basic coordinate system Σ_{123}:

$$\boldsymbol{p}_{123} = \boldsymbol{A}^{-1}\boldsymbol{B}^{-1}\boldsymbol{p}^{\xi,\varsigma}_{123}. \quad (17)$$

In order to obtain the coordinates of the object in a carrier coordinates system $\Sigma=(0,x,y,z)$ it is necessary to make one more rotation of the coordinate system Σ_{123} around the axis z on the angle of 135° (see Fig.3). For the Σ coordinate system, we have the following direction cosines for the z-axis:

$$\cos\sphericalangle(x,z) = 0;$$
$$\cos\sphericalangle(y,z) = 0;$$
$$\cos\sphericalangle(z,z) = 1. \quad (18)$$

To find the coordinates in the carrier coordinate system, we write the matrix equation:

$$\boldsymbol{p}_{123} = \boldsymbol{C}\boldsymbol{p}, \quad (19)$$

where: $\boldsymbol{p}_{123}=[X_{123},Y_{123},Z_{123}]^T$; $\boldsymbol{p}=[X,Y,Z]^T$ and $\boldsymbol{C}$ is the matrix that realizes rotation of the coordinate system Σ_{123} around the axis z:

$$\boldsymbol{C} = \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (20)$$

Solving the matrix equation (19), we obtain the coordinates of object in carrier coordinate system:

$$\boldsymbol{p} = \boldsymbol{C}^{-1}\boldsymbol{A}^{-1}\boldsymbol{B}^{-1}\boldsymbol{p}_{123}^{\xi,\varsigma} \;. \qquad (21)$$

In case of the USBL system with the antenna of five elements, the system consists of six basic USBL systems with six different orientations: $USBL_{123}$, $USBL_{234}$, $USBL_{341}$, $USBL_{412}$, $USBL_{153}$ and $USBL_{452}$. The coordinates of the object are determined individually in each basic USBL system. The $USBL_{123}$ system differs from the $USBL_{234}$, $USBL_{341}$ and $USBL_{412}$ systems in its own values of the pitch and roll angles and the angle of rotation of each basic three element USBL antenna around the z-axis. The algorithm of coordinate determination for the $USBL_{153}$ and $USBL_{452}$ systems is practically the same as for the $USBL_{123}$ system. It is necessary to make all algorithm steps represented for the $USBL_{123}$ system, taking into account the corresponding values of the pitch and roll angles. Before the final transformation (to obtain the coordinates in the carrier coordinate system) it is necessary to make one additional rotation of the local coordinate systems to reduce $USBL_{153}$ and $USBL_{452}$ systems to the horizontal plane. In order to distinguish the results of the measured coordinates by different basic USBL systems we introduce new designations for the vectors and transformation matrixes:

$$\begin{aligned}
\boldsymbol{p}_{\mathrm{USBL}123} &= \boldsymbol{C}^{-1}_{123}\boldsymbol{A}^{-1}_{123}\boldsymbol{B}^{-1}_{123}\boldsymbol{p}^{\xi,\varsigma}_{123}\,; \\
\boldsymbol{p}_{\mathrm{USBL}234} &= \boldsymbol{C}^{-1}_{234}\boldsymbol{A}^{-1}_{234}\boldsymbol{B}^{-1}_{234}\boldsymbol{p}^{\xi,\varsigma}_{234}\,; \\
\boldsymbol{p}_{\mathrm{USBL}341} &= \boldsymbol{C}^{-1}_{341}\boldsymbol{A}^{-1}_{341}\boldsymbol{B}^{-1}_{341}\boldsymbol{p}^{\xi,\varsigma}_{341}\,; \\
\boldsymbol{p}_{\mathrm{USBL}412} &= \boldsymbol{C}^{-1}_{412}\boldsymbol{A}^{-1}_{412}\boldsymbol{B}^{-1}_{412}\boldsymbol{p}^{\xi,\varsigma}_{412}\,; \\
\boldsymbol{p}_{\mathrm{USBL}153} &= \boldsymbol{C}^{-1}_{153}\boldsymbol{D}^{-1}_{153}\boldsymbol{A}^{-1}_{153}\boldsymbol{B}^{-1}_{153}\boldsymbol{p}^{\xi,\varsigma}_{153}\,; \\
\boldsymbol{p}_{\mathrm{USBL}452} &= \boldsymbol{C}^{-1}_{452}\boldsymbol{D}^{-1}_{452}\boldsymbol{A}^{-1}_{452}\boldsymbol{B}^{-1}_{452}\boldsymbol{p}^{\xi,\varsigma}_{452}\,. \qquad (22)
\end{aligned}$$

3.3 Problem of Z-coordinate measuring

The important part of the algorithm is solving the problem of the low precision of coordinate determination in cases when an object is found in the plane of receiving bases. We will demonstrate this problem for the case of three element $USBL_{123}$ system and when the receiving does not have inclination (ξ=0, ζ=0). We will estimate the coordinate determination accuracy of the USBL system. Let X, Y, Z be the true values of the coordinates of the object in the carrier coordinate system $\Sigma=(0,x,y,z)$. Let R be the true incline distance to the object. Let X_{USBL}, Y_{USBL}, Z_{USBL} be the values of the coordinates obtained by means of the developed algorithm (the coordinates of the object are calculated with utilizing of the expression (22) for $USBL_{123}$ system). The values of the true errors of determination of the coordinates can be written as $\Delta X=X_{USBL}-X$; $\Delta Y= Y_{USBL}-Y$, $\Delta Z= Z_{USBL}-Z$. The values of the relative true errors of the coordinates we can write as follows: $\Delta X/R$; $\Delta Y/R$; $\Delta Z/R$. Simulating the situation when the object is located close to the horizontal plane of the receiving USBL antenna we assume the distance to the object R=100m and relative depth Z=5m. In process of the simulation the azimuth angle φ is changing clockwise (if look down on horizontal plane, see Fig.3) from 0° to 360° with 1° step in the (x,y) coordinate plane (zero reading is coincided with x-axis of the $\Sigma=(0,x,y,z)$ coordinate system, see Fig.3). We can also calculate the spherical coordinates of the object or coordinate vector in spherical coordinate system. This vector can be expressed as follows:

$$\boldsymbol{q}_{123}^{\xi,\varsigma} = (R_{123}, \psi_{123}^{\xi,\varsigma}, \varphi_{123}^{\xi,\varsigma})^T \qquad (23)$$

The colatitude angle $\psi^{\xi,\zeta}_{123}$ in the spherical coordinate system measures the declination from vertical (from z-axis). In our case it is the more practical to use the altitude angle which will characterize the degree of closeness of the object to the plane of the measuring system. The modulus of the altitude angle to the object relatively the $USBL_{123}$ system plane can be expressed as $|\psi^{\xi,\zeta}_{123} - 90°|$. The results of the calculation of relative errors and the modulus of attitude angle $|\psi^{\xi,\zeta}_{123} - 90°|$ are shown in Fig.5.

From Fig.5 we can note that the accuracy of X- and Y-coordinates determining is good (0.025%) and for Z-coordinate is very poor (the relative errors $\Delta Z/R$ in some cases are more than 20 times greater than the $\Delta X/R$, $\Delta Y/R$, the values of relative errors $\Delta Z/R$ are changing in diapason from -0.4% to 0.4%). The increasing of the time delay counter frequency can improve the situation, but do not resolve the Z-coordinate accuracy requirement problem,

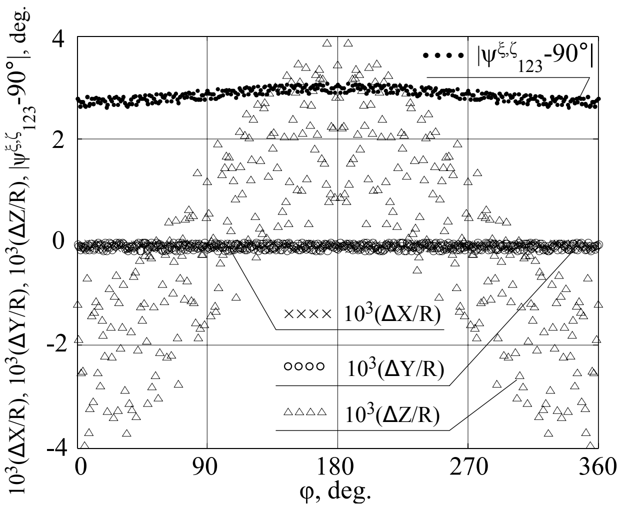


Fig.5. Variation of the relative errors of the object coordinates and modulus of the latitude angle to the transponder (object) relatively the $USBL_{123}$ plane; R=100m; Z=5m; ξ=0°; ζ=0°.

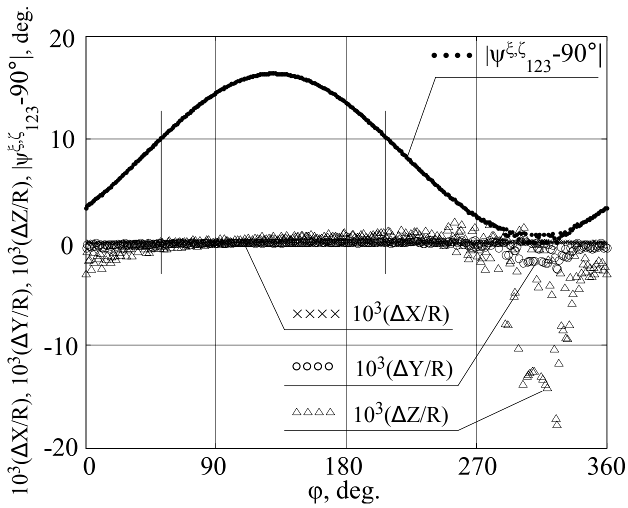


Fig.6. Variation of the relative errors of the object coordinates and modulus of the latitude angle to the transponder (object) relatively the $USBL_{123}$ plane; R=100m; Z=15 m; ξ=5°; ζ= –6°.

especially when the object is located closer to the plane of receiving USBL antenna (for example in the cases of inclination of receiving antenna). The next figure (Fig.6) shows the case when the object is located in the relative depth of 15m (with incline distance R=100m) and the receiving $USBL_{123}$ antenna has the inclination ((ξ=5°, ζ=-6°). As was expected the relative errors of determination of the Z-coordinate of the object are increased for the angle where the object is found even closer to the receiving antenna plane.

From Fig.6 we can note also that when the modulus of the latitude angle is more than some threshold value (we can designate this threshold as $\Delta_{lat.\ angle}$ and elect, for example, the threshold $\Delta_{lat.\ angle}$=10°) the values of the relative errors ΔZ/R become comparable with the values of X- and Y-coordinates relative errors. In the graph this diapason of azimuth angles (approximately from 50° to 210°) is marked with two vertical lines.

The graphs of relative coordinate errors on Fig.6, which were calculated with the first formula (22), show some increase of the Y-coordinates relative errors (in diapason from 300° to 330°), that can be explained by utilizing Z-coordinate (obtained in considering diapason of φ with certain error) in design formula (22). Thus, in fact, the problem of accurate determination of the object coordinates by means of USBL system is the problem of the increase of the Z-coordinate determination accuracy. The construction of antenna with five receiving elements proposed earlier allows the resolution of this problem by using the abundant data obtained with other elemental USBL systems that form the five element USBL system. In the considered case, the problem is resolved by means of the different orientations of the basic USBL antennas in the space. This allows at least one elemental USBL antenna which measures the Z-coordinate with high accuracy. As mentioned above the elemental USBL system also has the sign Z-coordinate problem. The possible inclinations of receiving antenna force to include this problem in the designed coordinate determination algorithm. The designed algorithm is presented in the next section. The algorithm has two parts. The first part supports the object coordinate calculation by using the data obtained on $USBL_{412}$ and $USBL_{153}$ systems. The second part of the algorithm utilizes the data obtained on $USBL_{123}$, $USBL_{234}$, $USBL_{341}$, $USBL_{412}$ systems and some results of the first part of the algorithm.

3.4 Algorithm description

It is assumed that the measured values are: ξ – pitch angle of receiving antenna; ζ – roll angle of receiving antenna (see Fig.3); n_{12}, n_{32}, n_{23}, n_{43}, n_{34}, n_{14}, n_{41}, n_{24}, n_{15}, n_{35}, n_{45}, n_{25} - time delays for receiving bases of the corresponding $USBL_{123}$, $USBL_{234}$, $USBL_{341}$, $USBL_{412}$, $USBL_{153}$ and $USBL_{452}$ systems to secure the positive values of measured delays the output signals of each common receiving element of three element USBL system are inverting. The resulting time delays on each base can be expressed as follows: $n_{ij}=f_c(\tau_{ij}+T/2)$, where f_c is the frequency of the time delay counter, τ_{ij} is the time delay between the receiving elements of

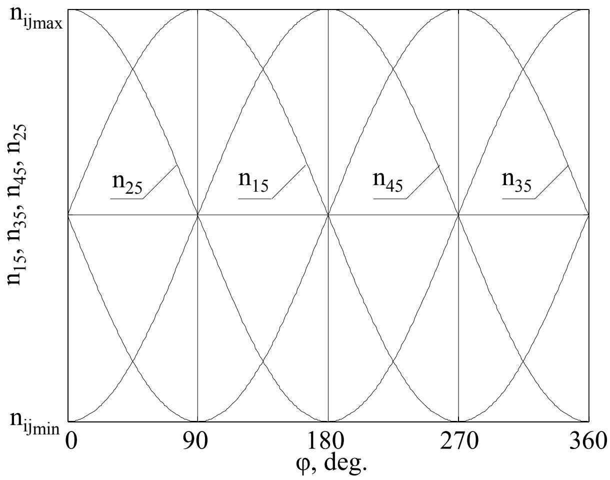


Fig.7. Time delays n_{15}, n_{35}, n_{45} and n_{25} measured in time delay counter pulses; case when the object is located in the plane of the $USBL_{123}$ ($USBL_{234}$, $USBL_{341}$, $USBL_{412}$) receiving bases.

the USBL system (j is the index of the common receiving element), *T* is the period of the transponder pulse carrier frequency.

First the vector $\boldsymbol{p}^{\xi,\zeta}_{153}=[X^{\xi,\zeta}_{153}, Y^{\xi,\zeta}_{153}, Z^{\xi,\zeta}_{153}]^T$ is calculated. The sign of the $Z^{\xi,\zeta}_{153}$ coordinate is defined utilizing time delay values obtained on the $USBL_{452}$ system (values n_{45}, n_{25}). The behavior of the time delays n_{15}, n_{35}, n_{45}, n_{25} ($n_{15}=n_{15}(\varphi)$, $n_{35}=n_{35}(\varphi)$, etc.) in the case of the non inclined receiving antenna ($\xi=0$, $\zeta=0$) and when the transponder is located in the plane of elemental horizontal USBL systems is shown in Fig.7.

The angle φ changes clockwise from 0° to 360° with 1° step in the *(x,y)* coordinate plane of the carrier coordinate system (as before, the zero reading is coincided with x-axis of the $\Sigma=(0,x,y,z)$ coordinate system, see Fig.3).

If $n_{45}>n_{25}$ φ lies in [270°-360°] or [0°-90] range the sign of the $Z^{\xi,\zeta}_{153}$ coordinate is negative; if $n_{45}\le n_{25}$ φ lies in [90°-270°] range the $Z^{\xi,\zeta}_{153}$ coordinate has a positive sign. With the obtained values of the vector $\boldsymbol{p}^{\xi,\zeta}_{153}=[X^{\xi,\zeta}_{153}, Y^{\xi,\zeta}_{153}, Z^{\xi,\zeta}_{153}]^T$ the values of n'_{45} y n'_{25} are computed for the $USBL_{452}$ system. The modules of the differences $\delta1=|n'_{45}-n_{45}|$ and $\delta2=|n'_{25}-n_{25}|$ are then calculated (values n_{45} y n_{25} are obtained through the measurement). If δ1 and δ2 are less than the threshold Δ_δ, it is assumed that the sign of the coordinate $Z^{\xi,\zeta}_{153}$ is defined correctly, if δ1 and δ2 are greater than the threshold Δ_δ, one should consider changing the sign of the coordinate $Z^{\xi,\zeta}_{153}$. The same procedure is applied to the $USBL_{452}$ system vector $\boldsymbol{p}^{\xi,\zeta}_{452}=[X^{\xi,\zeta}_{452}, Y^{\xi,\zeta}_{452}, Z^{\xi,\zeta}_{452}]^T$ in order to define the sign of the $Z^{\xi,\zeta}_{452}$ coordinate. Coordinate vectors in spherical coordinate system can be expressed as follows:

$$\boldsymbol{q}^{\xi,\varsigma}_{153} = (R_{153}, \psi^{\xi,\varsigma}_{153}, \varphi^{\xi,\varsigma}_{153})^T ;$$
$$\boldsymbol{q}^{\xi,\varsigma}_{452} = (R_{452}, \psi^{\xi,\varsigma}_{452}, \varphi^{\xi,\varsigma}_{452})^T . \qquad (24)$$

Furthermore the mean values of the Cartesian coordinates of the object in the carrier coordinate system are calculated. Throughout the calculation of the mean value the angle between the plane of the measuring antenna location and the direction to the object (latitude angle to object) is analyzed. If the values of the corresponding colatitude angles of both systems are found within [90°-$\Delta_{lat.\ angle}$, 90°+$\Delta_{lat.\ angle}$] diapason, the values of the system with greater latitude angles should be kept for further consideration. If the value of the colatitude angle of one of the USBL systems belongs to the diapason [90°-$\Delta_{lat.\ angle}$, 90°+$\Delta_{lat.\ angle}$] and the value of the colatitude angle of the other system lies outside the diapason [90°-$\Delta_{lat.\ angle}$, 90°+$\Delta_{lat.\ angle}$] the value of the first system is not taken into account and the value obtained from another system is utilized.

The final step in the coordinate determination is the calculation of the Cartesian coordinates of the object in the coordinate system of the carrier. These calculations for the $USBL_{153}$ and $USBL_{452}$ systems are carried out according to the formulas (22).

In second stage of the algorithm we calculate the means of the spherical coordinates of the object based on the mean values of the Cartesian coordinates obtained on the first stage:

$$\boldsymbol{q}_{USBL\_153\_452} = (R_{USBL\_153\_452}, \psi_{USBL\_153\_452}, \varphi_{USBL\_153\_452})^T . \qquad (25)$$

Furthermore he inclination angle Ω of the horizontal USBL systems (plane of location of the antennas of the $USBL_{123}$, $USBL_{234}$, $USBL_{341}$, $USBL_{412}$ systems) is computed using the measurements of pitch and roll angles. The altitude angle *Ψ* of the object relative to the absolute horizontal plane (horizontal plane of the carrier coordinate system) is calculated as follows:

$$\Psi = |\,90° - \psi_{USBL\_153\_452}\,| . \qquad (26)$$

If $\Psi > (\Omega + \Delta_{lat.\ angle})$ the object coordinates are computed using the values obtained by the horizontal measuring systems ($USBL_{123}$, $USBL_{234}$, $USBL_{341}$, $USBL_{412}$) applying formulas (22). In case $\Psi \le (\Omega + \Delta_{lat.\ angle})$ one faces problems of Z-coordinate sign and poor precision determination (with the three element USBL antenna). In order to resolve the aforementioned problems a special procedure has to be applied.

We consider the $USBL_{123}$ system in order to describe this procedure; for the other USBL systems this part of the algorithm functions in the same way. We are supposed to determine the sign of the coordinate $Z^{\xi,\zeta}_{123}$. First we calculate the coordinates of the object on the base of the delays measured with the $USBL_{123}$ system ($\mathbf{p}^{\xi,\zeta}_{123}=[X^{\xi,\zeta}_{123}, Y^{\xi,\zeta}_{123}, Z^{\xi,\zeta}_{123}]^T$). The sign of the $Z^{\xi,\zeta}_{123}$ coordinate is obtained using the values n_{15}, n_{35} of the $USBL_{153}$ system and the values n_{45}, n_{25} of the $USBL_{452}$ system. The values of the components of the vector $\mathbf{p}^{\xi,\zeta}_{123}=[X^{\xi,\zeta}_{123}, Y^{\xi,\zeta}_{123}, Z^{\xi,\zeta}_{123}]^T$ allow us to calculate the values of delays n'_{15} and n'_{35} for the $USBL_{153}$ system and value of delays n'_{45} and n'_{25} for the $USBL_{452}$ system. Next, we calculate the absolute values of the following differences: $\delta 1=|n'_{15}-n_{15}|$, $\delta 2=|n'_{35}-n_{35}|$, $\delta 3=|n'_{45}-n_{45}|$ y $\delta 4=|n'_{25}-n_{25}|$ (the values n_{15}, n_{35}, n_{45} and n_{25} are obtained through the measurement. Now we calculate the geometric mean $\delta_{media\_geom\_1}$ of the δ1, δ2, δ3 and δ4. The same procedure is repeated with the designation of the opposite sign for the coordinate $Z^{\xi,\zeta}_{123}$. The geometric mean in this case will be $\delta_{media\_geom\_2}$. The coordinates corresponding to a less geometric mean are taken as correct. The same procedure is applied for the $USBL_{234}$, $USBL_{341}$, $USBL_{412}$ systems.

The final part of the algorithm examines the reliability of the calculation of the Z-coordinate for the USBL systems under consideration. For that we calculate the spherical coordinates:

$$\mathbf{q}^{\xi,\varsigma}_{123} = (R_{123}, \psi^{\xi,\varsigma}_{123}, \varphi^{\xi,\varsigma}_{123})^T ;$$
$$\mathbf{q}^{\xi,\varsigma}_{234} = (R_{234}, \psi^{\xi,\varsigma}_{234}, \varphi^{\xi,\varsigma}_{234})^T ;$$
$$\mathbf{q}^{\xi,\varsigma}_{341} = (R_{341}, \psi^{\xi,\varsigma}_{341}, \varphi^{\xi,\varsigma}_{341})^T ;$$
$$\mathbf{q}^{\xi,\varsigma}_{412} = (R_{412}, \psi^{\xi,\varsigma}_{412}, \varphi^{\xi,\varsigma}_{412})^T . \quad (27)$$

If the values of the colatitude angle of some USBL systems lie within the diapason [$90°-\Delta_{lat.\ angle}$, $90°+\Delta_{lat.\ angle}$],the calculated values are discarded and the final values of the coordinates of the object are taken as the mean of the coordinates obtained with the $USBL_{153}$ and $USBL_{452}$ systems. If the values of the colatitude angle of all these USBL systems lay outside of the diapason [$90°-\Delta_{lat.\ angle}$, $90°+\Delta_{lat.\ angle}$], the values of the object coordinates are calculated according to the formulas (22). The last step of the algorithm implies the calculation of the means of the object coordinates in the carrier coordinate system with the reliable data obtained by elemental USBL systems.

3.5 Simulation results

For the algorithm simulation a special computer program was designed. During the simulation of the algorithm, it was assumed that the distance to the object, and the pitch and roll angles were being measured precisely. We assume that the measurement of the time delays is provided by utilizing the binary counters and the signal-to-noise ratio (SNR) on the inputs of receiving elements; and signal reception conditions allow us to measure the time delays without errors. It is also supposed that the accuracy of measurement of the time delays is limited by the clock drive frequency of the time delay counters. The different values of the horizontal distance to the object, the depth of the object, the azimuth angle, the pitch and roll angles were utilized for modeling the difficult conditions to measure the Z-coordinate with high accuracy (in cases when an object is found in the plane of receiving bases). The computer simulation of algorithm (as it was in the case of three element USBL system) will estimate the instrumental precision of the USBL system. As it was previously let X, Y, Z be the true values of the coordinates of the object in the carrier coordinate system $\Sigma=(0,x,y,z)$. Let R be the true incline distance to the object. Let X_{USBL}, Y_{USBL}, Z_{USBL} be the values of the coordinates obtained by means of the developed algorithm (the coordinates of the object are calculated utilizing the expression (21) for $USBL_{123}$, $USBL_{234}$, $USBL_{341}$, $USBL_{412}$, $USBL_{153}$ and $USBL_{452}$ systems). The values of the true errors of determination of the coordinates are: $\Delta X=X_{USBL}-X$; $\Delta Y= Y_{USBL}-Y$, $\Delta Z= Z_{USBL}-Z$. The values of the relative true errors of the coordinates: ΔX/R; ΔY/R; ΔZ/R. In process of the simulation the azimuth angle φ is changing clockwise (if looking down on the horizontal plane, see Fig.3) from 0° to 360° with 1° step in the *(x,y)* coordinate plane (zero reading is coincided with x-axis of the $\Sigma=(0,x,y,z)$ coordinate system, see Fig.3). The value of the threshold Δ_δ is taken as 1% of n_{max}. The value of threshold $\Delta_{lat.\ angle}$ is taken equal to 10°. The other parameters of algorithm simulation have following values (just as in the section 3.3): the speed of the sound in the water *c*=1500m/s; the size of receiving bases *d*=0.056m; transponder pulse carrier frequency *f*=11KHz (operating frequency of USBL system); the frequency of the time delay counter f_C=160MHz.

The results of designed algorithm simulation are

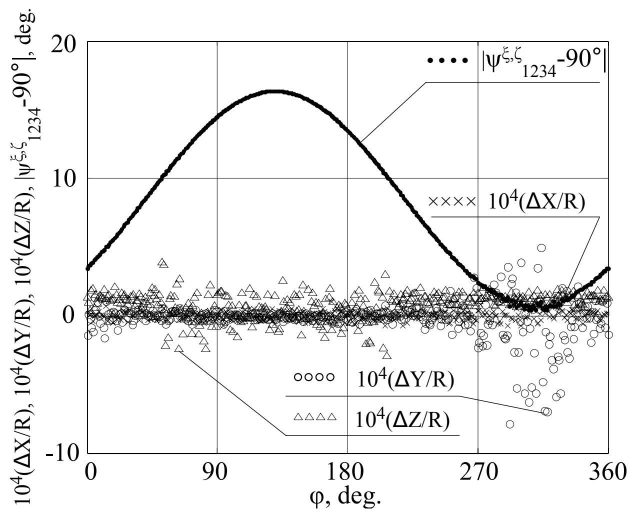


Fig.8. Variation of the relative errors of the object coordinates and modulus of the latitude angle to the transponder (object) relatively the $USBL_{1234}$ plane; R=100m; Z=15 m; ξ=5°; ζ= –6°.

shown in Figs.8-11. From Fig.8 (R=100m; Z=15m; ξ=5°; ζ= –6°) it is seen that the relative errors (ΔX/R, ΔY/R, ΔZ/R) of all three coordinates have not exceeded the threshold of 0.1% of inclined distance to the object.

Some increase of relative error ΔY/R can be observed when the modulus of altitude angle $|\psi^{\xi,\zeta}_{1234}$ -90°| is less than 3° (this corresponds to azimuth angle diapason [270°-360°]). The graphs in Fig.9 illustrate the variation of relative errors of object coordinates for the case when R=100m; Z=40 m; ξ= –20°; ζ=10°. Relative location of measuring system and object (with predetermined spatial orientation of receiving antenna) defines the case of significant inclination of receiving antenna and when the object can be found in the plane of horizontal receiving bases of USBL system. It is seen that the relative errors not exceed the threshold of 0.2% of incline distance in all diapason of changing azimuth angle φ. In wide diapason of values of the azimuth angle φ (from 0° to 120° and from 190° to 360°) the relative coordinate errors are less than 0.05%. Also it is seen that when the modulus of altitude angle $|\psi^{\xi,\zeta}_{1234}$ -90°| is less than 3° the fluctuations of relative errors are increased (see Fig.9, φ takes on the values from 120° to 190°). The results of calculation of the relative errors for the significant inclination angles of receiving antenna are shown in Fig.10 and Fig.11. In Fig.10 the pitch and roll angle have values: ξ= –30°; ζ= 20°; in the Fig.11 the pitch and roll angles are: ξ= –40°; ζ= –40°. From the present curves we can note that as before the noticeable fluctuations of the relative errors are observed when the values of the altitude angle $|\psi^{\xi,\zeta}_{1234}$ -90°| are approximately less than 3° (the object is in the plane of the inclined horizontal receiving bases of USBL antenna or close to this plane). With the latitude angle increasing the fluctuations of the relative errors are rapidly decreased (see Fig.10 and Fig.11 for the values of latitude angle more than 10°).

The designed algorithm has been examined for various relative locations of the measuring system and object (lower hemisphere was considered, the maximum incline distance was assumed to be

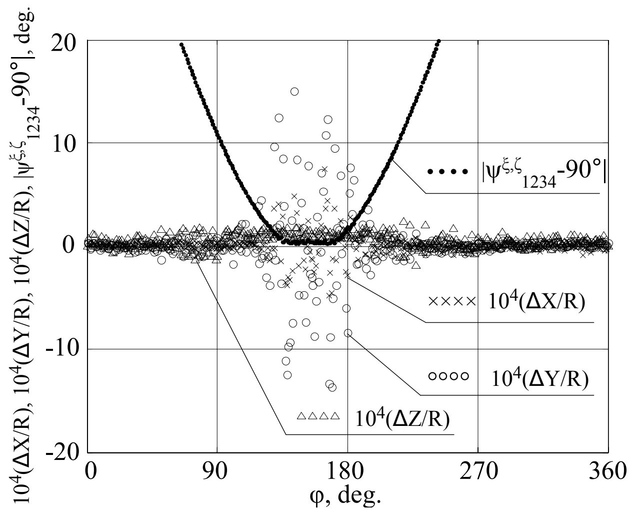


Fig.9. Variation of the relative errors of the object coordinates and modulus of the latitude angle to the transponder (object) relatively the $USBL_{1234}$ plane; R=100m; Z=40 m; ξ= –20°; ζ=10°.

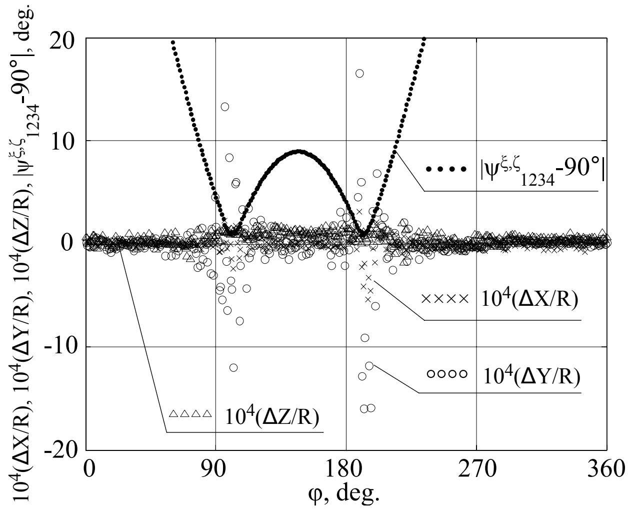


Fig.10. Variation of the relative errors of the object coordinates and modulus of the latitude angle to the transponder (object) relatively the $USBL_{1234}$ plane; R=100m; Z=50 m; ξ= –30°; ζ=20.

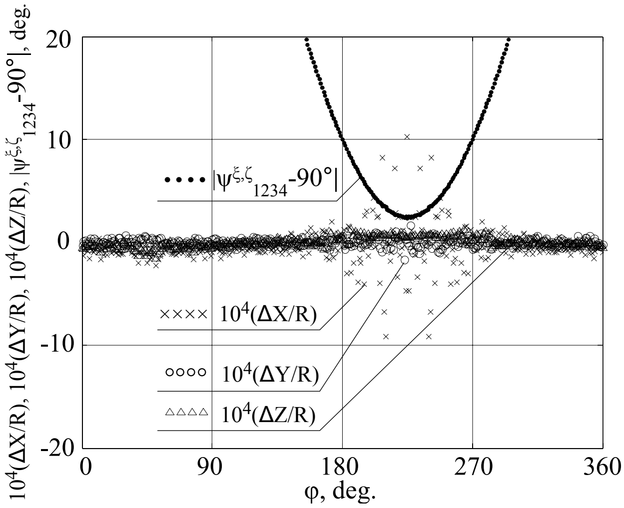


Fig.11. Variation of the relative errors of the object coordinates and modulus of the latitude angle to the transponder (object) relatively the $USBL_{1234}$ plane; R=100m; Z=150 m; ξ= –40°; ζ= –40°.

100m). The values of pitch and roll angles ξ and ζ are assumed to be in the range from –40° to +40°. The computer simulation demonstrated the reliable operation of the algorithm for all tested angular antenna positions and verified locations of object. The results of the calculation of the errors of the determination of the coordinates of the object show that the relative true errors of the coordinates have values less than 0.2%. It is also necessary to mention, that in a wide range of distances, depths, pitch and roll angles the values of true relative errors were less than 0.1%.

4 Conclusion

The problem of the accuracy enhancement of the USBL systems has been considered. In particular the low Z-coordinate determination accuracy problem for the cases when the object is located in the plane (or close to the plane) of the receiving USBL array has been investigated. The proposed approach of the accuracy enhancement is based on the idea of increasing the number of receiving elements (and hence the number of receiving bases) of the USBL antenna and utilization of the information from this additional receiving bases for object position determination. In particular, the fourth element is added to the horizontal array to form the additional three basic USBL systems and the fifth element is added to obtain two additional three element USBL systems with receiving bases placed in the orthogonal vertical planes.

The coordinate determination algorithm was designed for the proposed five elements USBL system. In process of algorithm designing it was supposed that the receiving antenna is fixed on a carrier and may have inclination defined by the pitch and roll angles. The computer simulation of the designed algorithm was realized. Different angular antenna positions and various object locations have been investigated in process of algorithm simulation. Simulation results showed the reliable operation of the designed algorithm for all tested angular antenna positions and verified locations of object. The presented calculations of errors of the object coordinates determination have demonstrated significant improvement of the Z-coordinate determination accuracy. The calculations of errors in determination of the object coordinates show that the relative true errors of the coordinate determination have values less than 0.2% of the measured distance, so the coordinate determination accuracy of the proposed USBL system may be evaluated as 0.2% of slant distance to the object.

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