Multiple noise-enhanced stability versus temperature in asymmetric bistable potentials

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Abstract: We explore the problem of noise-enhanced stability occurring in an asymmetric double well potential when Brownian particles are driven by trichotomous noise and thermal noise in a dynamical regime where inertial effects can safely be neglected. In the stationary state, we exactly calculate the spatial density profile of the particles and the occupancy ratio between two potential wells. We show that, by conveniently choosing the system parameters, the occupancy of a metastable state is a double peaked function of thermal noise intensity. Thus, thermal noise may facilitate the occupation of the potential minima with an energy above the absolute minimum at certain finite values of temperature. The effect is more pronounced in case the kurtosis of the trichotomous noise tends to -2, i.e., in the case of dichotomous noise.

Key-words: Noise-enhanced stability, stochastic dynamics, trichotomous noise, metastable state, thermal noise, bistable potential

1 Introduction

For a long time, noise was considered to be just a source of disorder, a nuisance to be avoided [1], [2], [3]. However, intensive investigation performed in the last three decades, have revealed some positive aspects of noise.

The idea that noise, via interaction with the nonlinearity of the system, can give rise to some counterintuitive results, has led to many important discoveries: stochastic resonance [4], resonant activation [5], nonequilibrium phase transitions and noise-induced pattern formation [6], [7], [8], and stochastic ratchets (Brownian motors) [9], [10], to name but a few. Active analytical and numerical studies of various models in this field have been stimulated by their possible applications in chemical physics, molecular biology, nanotechnology, and for separation techniques of nanoobjects [9], [11].

The problem of noise-driven barrier crossing dynamics of a Brownian particle in a double-well potential coupled to a heath bath, represented by an additive Gaussian noise with negligible correlation time (white noise), was formulated and solved by Kramers [12] more than half a century ago. Since then the model and many of its variants have been addressed by a large number of works at various levels of description. Although, white noise as a model for studying thermal activation is very useful for physical applications, at practical physical systems treatment of colored noises with finite correlation times is also popular enough. Therefore, it seems important to investigate systems driven by colored noises. The most frequently used model of a colored noise is the Gaussian noise generated by the Ornstein-Uhlenbeck process. Unfortunately, it is a rather limited class of noise-driven model systems that admits exact solutions in the presence of

Gaussian colored noise [13]. Other noises popular because of their tractability are dichotomous noise, also called random telegraphic noise [14], and trichotomous noise [15]. It is notable that, in the case of dichotomous and trichotomous noises, exact formulas for the steady-state probability distributions can be found for a rather broad class of dynamical models [14], [15].

The recent years have witnessed an increasing interest in the dependence of the mean exit time of metastable and unstable systems on noise intensity [16], [17]. Noise can modify the stability of a system in a counterintuitive way such that the system remains in a metastable state for a longer time than in the deterministic case [16]. Related investigations involving noise-induced stability [18] or noise-enhanced stability [19], [20] belong to a highly topical interdisciplinary realm of studies, ranging from condensed matter physics to molecular biology, or to cancer growth dynamics [16], [21], [22].

Motivated by investigations into the effect of a periodic electric field on cell membrane proteins [23], [24] the author of [18] has considered the overdamped motion of a Brownian particle in an asymmetric bistable potential fluctuating according to a dichotomous noise. This biologically motivated model clearly demonstrates the effect of noise-induced stability, as for intermediate fluctuation rates the mean occupancy of minima with an energy above the absolute minimum is enhanced.

In the present paper we consider a model similar to the one presented in [18], except that the dichotomous noise is replaced with a trichotomous noise. Although both dichotomous and trichotomous noises may be useful in modeling natural colored fluctuations, the latter is more flexible, including all cases of dichotomous noises [15]. Furthermore, it is remarkable that for trichotomous noises the kurtosis φ, unlike the Gaussian colored noise, where $\varphi = 0$, and symmetric dichotomous noise, where $\varphi = -2$, can be anything from -2 to ∞. This extra degree of freedom can prove useful in modeling actual fluctuations.

The main contribution of this paper is as follows. We provide an exact formula for the analytic treatment of the dependence of the occupancy probability of a metastable state on various system parameters: viz. temperature, potential asymmetry, correlation time, kurtosis, and noise amplitude. We establish a new thermal fluctuations-induced phenomenon, namely, for certain values of the system parameters there exist three ranges of temperature values where the occupancy of the metastable state is enhanced.

We also show that such a behavior of the system is quite robust, and the mentioned phenomenon occurs within a broad range of trichotomous noise parameters (kurtosis, correlation time).

It is remarkable that one of the temperature regimes where the enhancement of stability occurs is relevant for cell biology. Thus, in the case of living cells, the result may reveal a possibility to control the stability of metastable states by varying the temperature.

The structure of the paper is as follows. Section 2 presents the basic model investigated. A master equation description of the model is given and the formula for the occupancy probability of the metastable state is found. Section 3 analyzes the behavior of the occupancy probability. The phenomenon of double enhanced stability of the metastable state versus temperature is established. Section 4 contains some brief concluding remarks.

2 Model and the exact solution

As a model for systems with a metastable state, which are strongly coupled with noisy environment, we consider one-dimensional overdamped Brownian motion in a fluctuating sawtooth-like asymmetric bistable potential well with the width L

$$\tilde{U}(\tilde{X}, \tilde{Z}) = \tilde{U}(\tilde{X}) + \tilde{X} \cdot \tilde{Z}(\tilde{t}), \qquad (1)$$

where $\tilde{X}(\tilde{t})$ is the displacement of a Brownian particle at the time $\tilde{t}$. The archetypal examples, exhibiting bistable (double-well) potentials are nonequilibrium chemical reactions (e.g., the second Schlögl model [25], [26]), nonequilibrium Ginzburg-Landau-type bistable stochastic dynamics [25], and optical bistability in laser devices [27], [28]. The variable $\tilde{Z}(\tilde{t})$ in Eq. (1) is a Markovian trichotomous noise [15], which consists of jumps between three values: $\tilde{z}_1 = \tilde{a}$, $\tilde{z}_2 = 0$, $\tilde{z}_3 = -\tilde{a}$, $\tilde{a} > 0$. The jumps follow, in time, the pattern of a Poisson process, the values occurring with the stationary probabilities $p_s(\tilde{a}) = p_s(-\tilde{a}) = q$ and $p_s(0) = 1 - 2q$, where $0 < q < 1/2$. In a stationary state the fluctuation process satisfies

$$\begin{aligned} \langle \tilde{Z}(\tilde{t}) \rangle &= 0\,, \\ \langle \tilde{Z}(\tilde{t}+\tilde{\tau})\tilde{Z}(\tilde{t}) \rangle &= 2q\tilde{a}^2 \exp(-\tilde{\nu}\tilde{\tau}), \end{aligned} \qquad (2)$$

where the switching rate $\tilde{\nu}$ is the reciprocal of the noise correlation time $\tilde{\tau}_c = 1/\tilde{\nu}$, i.e., $\tilde{Z}(\tilde{t})$ is a symmetric zero-mean exponentially correlated noise. The probabilities $W_n(\tilde{t})$ that $\tilde{Z}(\tilde{t})$ is in the state $n \in \{1, 2, 3\}$ at the time $\tilde{t}$ evolve according to the master equation

$$\frac{d}{d\tilde{t}} W_n(\tilde{t}) = \tilde{\nu} \sum_{m=1}^{3} S_{nm} W_m(\tilde{t}), \qquad (3)$$

where

$$S_{nm} = \begin{pmatrix} q-1 & q & q \\ 1-2q & -2q & 1-2q \\ q & q & q-1 \end{pmatrix} . \quad (4)$$

The transition probabilities $T_{ij} = p(\tilde{z}_i, \tilde{t}+\tilde{\tau} \mid \tilde{z}_j, \tilde{t})$ between the states $\tilde{z}_n$, $n = 1, 2, 3$, can be represented by means of the transition matrix (T_{ij}) of the trichotomous process as follows

$$(T_{ij}) = (\delta_{i,j}) + (1 - e^{-\tilde{\nu}\tilde{\tau}})S_{ij},$$

where $\delta_{i,j}$ is the Kronecker symbol.

The trichotomous process is a particular case of the kangaroo process [29] with the kurtosis

$$\varphi = \frac{\left\langle \tilde{Z}^4(\tilde{t}) \right\rangle}{\left\langle \tilde{Z}^2(\tilde{t}) \right\rangle^2} - 3 = \frac{1}{2q} - 3 . \quad (5)$$

We describe the overdamped motion of Brownian particles by the Langevin equation

$$\begin{aligned} \varkappa \frac{d\tilde{X}}{d\tilde{t}} &= \tilde{h}(\tilde{X}) - \tilde{Z}(\tilde{t}) + \tilde{\xi}(\tilde{t}), \\ \tilde{h}(\tilde{x}) &= -\frac{d\tilde{U}(\tilde{x})}{d\tilde{x}}, \end{aligned} \quad (6)$$

where $\varkappa$ is the friction coefficient. The thermal fluctuations $\tilde{\xi}(\tilde{t})$ are modeled by the zero-mean Gaussian white noise with the correlation function

$$\langle \tilde{\xi}(\tilde{t}_1)\, \tilde{\xi}(\tilde{t}_2) \rangle = 2\varkappa k_B T \delta(\tilde{t}_1 - \tilde{t}_2), \quad (7)$$

where k_B is the Boltzmann constant and T is the temperature.

By applying a scaling of the following form:

$$\begin{aligned} X = \frac{\tilde{X}}{L}, \quad U(x) = \frac{\tilde{U}(\tilde{x})}{\tilde{U}_0}, \\ t = \frac{\tilde{t}}{t_0}, \quad \xi = \frac{L\tilde{\xi}}{\tilde{U}_0}, \quad Z = \frac{L\tilde{Z}}{\tilde{U}_0}, \end{aligned} \quad (8)$$

where $\tilde{U}_0 = \tilde{U}_{\max} - \tilde{U}_{\min}$ is the barrier height of the left potential well (cf. Fig. 1), we obtain a dimensionless formulation of the dynamics. Choosing $t_0 = \varkappa L^2/\tilde{U}_0$, the dimensionless friction coefficient turns to unity and the quantities determining the rescaled noises are reduced to

$$\nu = \frac{\varkappa L^2 \tilde{\nu}}{\tilde{U}_0},\ a = \frac{\tilde{a}L}{\tilde{U}_0},\ D = \frac{k_B T}{\tilde{U}_0} , \quad (9)$$

where $2D$ is the strength of the rescaled zero-mean Gaussian white noise $\xi(t)$. For brevity, in what follows we shall call D temperature.

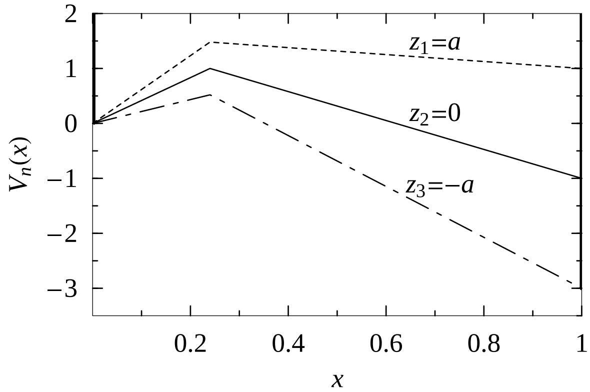

Fig. 1. Representation of different states of the net potentials $V_n(x) = U(x) + z_n x$ with $z_1 = a$, $z_2 = 0$, $z_3 = -a$. The potential $U(x)$ is given by Eq. (11) at the parameter values $k = 0.24$, $a = 2$, and $\varepsilon = 1$. All quantities are dimensionless.

As an example of overdamped dynamics (see Eq. (6)), following Ref. [30], we consider kinesin, which moves along microtubules inside cells. At the temperature $T = 310$ K the typical parameter values of the system are: $\varkappa = 2 \cdot 10^{-8}$ kg/s, $L = 8 \cdot 10^{-9}$ m, $\tilde{m} = 6 \cdot 10^{-22}$ kg, and $\tilde{U}_0 = 5k_B T$. From the scaling (8) we obtain that the dimensionless mass of the kinesin molecule is $m = \tilde{m}\tilde{U}_0/(\varkappa^2 L^2) \approx 5 \cdot 10^{-10}$. Hence, the acceleration term is 10 orders smaller than the dimensionless friction term. Thus, inertial effects can be neglected.

The dimensionless dynamics is described by the stochastic differential equation

$$\begin{aligned} \frac{dX}{dt} &= h(X) - Z(t) + \xi(t) , \\ h(x) &= -\frac{dU(x)}{dx} . \end{aligned} \quad (10)$$

Thus we consider a model similar to the one presented in [18], except for some details of the potential profile and for the dichotomous noise being replaced with a trichotomous noise. As the results of [18] show that the phenomenon of noise-induced stability is quite universal and manifests itself for arbitrary bistable potential landscapes, we decided to study overdamped motion of Brownian particles in an asymmetric, bistable, piecewise linear potential subjected to both a trichotomous noise and a thermal one. The piecewise linear potential is important for at least two reasons. First, it can be used as a first approximation of the shape of an arbitrary potential, and second, it is sufficiently simple to allow an analytic treatment of the relevant quantities, being at the same time physically rich enough to provide most of the effects characteristic of two-well potentials. The asymmetric bistable potential considered has the profile

$$U(x) = \begin{cases} \frac{1}{k}x, & x \in (0,k); \\ 1 + \frac{1+\varepsilon}{1-k}(k-x), & x \in (k,1); \\ U(0) = U(1) = \infty. \end{cases} \quad (11)$$

A schematic representation of the three configurations assumed by the "net potentials" $V_n(x) = U(x) + z_n x$, $n = 1,2,3$, associated with the right-hand side of Eq. (10), is shown in Fig. 1. In this work, we restrict ourselves to the system parameters region where the net potentials $V_n(x)$ for all states $n = 1,2,3$ of the non-equilibrium noise Z have two minima. More precisely, we assume that

$$a < \frac{1+\varepsilon}{1-k}, \quad a < \frac{1}{k}, \quad 0 < k < \frac{1}{2}, \\ 0 < \varepsilon < a(1-2k). \quad (12)$$

The master equation corresponding to Eq. (10) reads

$$\frac{\partial}{\partial t}P_n(x,t) \\ = -\frac{\partial}{\partial x}\left\{[h(x) - z_n]\,P_n(x,t) - D\frac{\partial}{\partial x}P_n(x,t)\right\} \\ + \nu\sum_{m=1}^{3} S_{nm}P_m(x,t)\,, \quad (13)$$

where $P_n(x,t)$ is the joint probability density for the position variable $x(t)$ and the fluctuation variable $z(t)$; while $S_{nm} = q + (1-3q)\delta_{n,2} - \delta_{n,m}$. Here $\delta_{n,m}$ is the Kronecker symbol. The stationary probability density in the x space $P^s(x)$ is then evaluated via the stationary probability densities $P_n^s(x)$ for the states (x, z_n):

$$P^s(x) = \sum_{n=1}^{3} P_n^s(x). \quad (14)$$

As the "force" $h(x) = -dU(x)/dx$ is piecewisely constant,

$$h(x) = h_0 = -\frac{1}{k}\,, \quad x \in (0,k)\,, \\ h(x) = h_1 = \frac{1+\varepsilon}{1-k}\,, \quad x \in (k,1), \quad (15)$$

Eq. (13) splits up into two linear differential equations with constant coefficients for two vector functions of $\boldsymbol{P}_i^s(x) = (P_{1i}^s, P_{2i}^s, P_{3i}^s)$, $i = 0,1$, defined on the intervals $(0,k)$ and $(k,1)$, respectively. The solution reads

$$P_{ni}^s(x) = p(z_n)\sum_{j=1}^{5} Y_{ij}A_{nij}e^{-\lambda_{ij}x/D}, \quad (16)$$

where

$$p(z_n) = (1-2q)\delta_{n,2} + q(\delta_{n,1} + \delta_{n,3}),$$

$$A_{nij} = \frac{\nu D}{\nu D - \lambda_{ij}(h_i - z_n + \lambda_{ij})},$$

Y_{ij} are constants of integration, and $\{\lambda_{ij}, j = 1,\ldots,5\}$ is the set of roots of the algebraic equation

$$\lambda_i^5 + 3\lambda_i^4 h_i + \lambda_i^3(3h_i^2 - a^2 - 2\nu D) \\ + \lambda_i^2 h_i(h_i^2 - a^2 - 4\nu D) \\ + \lambda_i \nu D\left[\nu D + 2(qa^2 - h_i^2)\right] + h_i\nu^2 D^2 = 0\,, \\ i = 0,1\,. \quad (17)$$

Nine independent conditions for the ten constants of integration Y_{ij} can be determined at the points of discontinuity, by requiring continuity for the quantities $P_{ni}^s(x)$ and for the stationary current densities

$$j_{ni}(x) := (h_i - z_n)P_{ni}^s(x) - D\frac{d}{dx}P_{ni}^s(x) \quad (18)$$

at the point $x = k$ and the vanishing of the current densities $j_{ni}(x)$ at the boundary points $x = 0,1$, i.e.,

$$P_{n0}^s(k) = P_{n1}^s(k), \quad j_{n0}(k) = j_{n1}(k), \\ j_{n0}(0) = j_{n1}(1) = 0, \quad n = 1,2,3. \quad (19)$$

It follows from Eq. (13) that the system of linear algebraic equations (19) contains only nine linearly independent equations for Y_{ij}. By including the normalization condition

$$\sum_{n=1}^{3}\int_0^1 P_n^s(x)dx = 1 \quad (20)$$

a complete set of conditions is obtained for ten constants of integration Y_{ij}. Now, the constants Y_{ij} can be expressed as quotients of two determinants of the tenth degree:

$$Y_{ij} = \frac{\det[B_{lr}(1-\delta_{r,j+5i}) + \delta_{l,10}\delta_{r,j+5i}]}{\det(B_{lr})}, \quad (21)$$

where the matrix (B_{lr}), $l, r = 1, \dots, 10$ is defined as follows:

$$\begin{aligned} B_{6\,j+5} &= B_{7\,j+5} = B_{8\,j} = B_{9\,j} = 0, \\ B_{n\,j+5i} &= (-1)^i A_{nij} \exp\left(-\frac{k\lambda_{ij}}{D}\right), \\ B_{m+3\,j+5i} &= (h_i - z_{2m-1} + \lambda_{ij}) B_{2m-1\,j+5i}, \\ B_{m+5+2i\,j+5i} &= \left[\delta_{0,i} \exp\left(\frac{k\lambda_{ij}}{D}\right)\right. \\ &+ \left.\delta_{1,i} \exp\left(\frac{(k-1)\lambda_{ij}}{D}\right)\right] \\ &\times B_{m+3\,j+5i}, \\ B_{10\,j+5i} &= \frac{(-1)^i D}{\lambda_{ij}} \left[\exp\left(-\frac{\lambda_{ij}\delta_{1,i}}{D}\right)\right. \\ &- \left.\exp\left(-\frac{k\lambda_{ij}}{D}\right)\right], \qquad (22) \end{aligned}$$

with $n = 1, 2, 3$; $m = 1, 2$; $j = 1, \dots, 5$; $i = 0, 1$;

The stationary probability density in the x space $P_i^s(x)$, with $i = 0$ for $x \in (0, k)$ and $i = 1$ for $x \in (k, 1)$, and the occupancy probabilities W_0 and $W_1 = 1 - W_0$ of the left and right potential wells, respectively, are given by

$$P_i^s(x) = \sum_{j=1}^{5} Y_{ij} \exp\left(-\frac{\lambda_{ij} x}{D}\right), \qquad (23)$$

$$W_0(x) = \int_0^k P_0^s(x)dx = \sum_{j=1}^{5} B_{10\,j} Y_{0j}. \qquad (24)$$

The behavior of W_0 at different system parameters regimes will be considered in Sec. 3. All numerical calculations are performed by using the software Mathematica 5.0.

3 Enhancement of the stability of the metastable state

Of central interest to us are the occupancy probability W_0 of the left potential well (see Eq. (24)) and its responses to the switching rate ν and to the temperature D. Figure 2 exhibits the ratio W_0/W_1 as a function of the switching rate ν at different values of the temperature. It can be seen that the functional dependence of W_0/W_1 on the switching rate ν is of a bell-shaped form. Notably, at low temperatures for intermediate values of ν the mean occupancy of the metastable state (the left potential well) is much larger than the mean occupancy of the stable state, i.e., such fluctuations enhance the occupancy of the left minimum, although most of the time it is not the absolute minimum of the potential. Thus, we observe a noise-induced stability for the metastable state (cf. also Table 1).

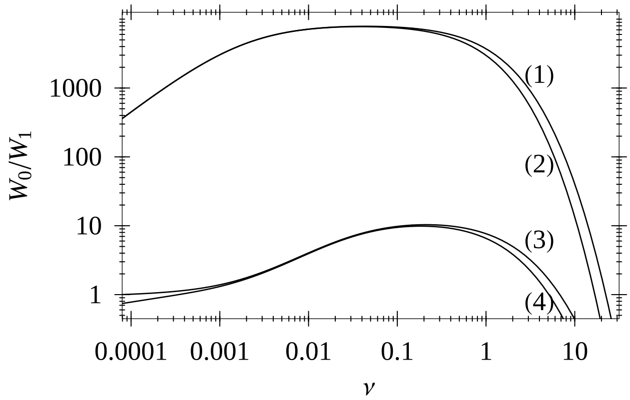

Fig. 2. The ratio W_0/W_1 vs the noise switching rate ν at various temperatures D. The occupancy probabilities W_0 and W_1 of the left and right potential wells, respectively, are computed by means of Eq. (24). The parameter values: $a = 2$, $\varepsilon = 1$, and $k = 0.125$. The different curves correspond to the different values of the parameter q and temperature D: (1) $q = 0.48$, $D = 0.035$; (2) $q = 0.33$, $D = 0.035$; (3) $q = 0.48$, $D = 0.065$; (4) $q = 0.33$, $D = 0.065$.

The tendency that is apparent in Figure 2, namely, an increase of the occupancy probability W_0 as the temperature D decreases, also appears at lower values of D. Moreover, decrease of the kurtosis $\varphi = 1/2q - 3$ of the trichotomous noise Z also enhances the stability of the metastable state [cf. curves (1) and (2) in Fig. 2].

In the case of dichotomous noise, the phenomenon of noise-induced stability in models similar to Eq. (10) has already been examined in Ref. [18], where analogous results of Fig. 2 are presented and a comprehensive physical interpretation of the effect is given. So our result exposed in Fig. 2 shows that the phenomenon of noise correlation time induced stability is robust enough to survive a modification of the noise as well as of the potential profile.

It is of interest to examine the behavior of the exact expression of W_0 (Eq. (24)) versus temperature. In Fig. 3 we have plotted the occupancy probabilities W_0 and W_1 as functions of the dimensionless temperature D for an intermediate value of the correlation time $\tau_c = 200$. For increasing values of D the probability W_0 starts from the value $W_0 \approx 1$ and decreases to the minimum. Next it grows to the local maximum and decreases to the other minimum. Finally, at high temperatures, it grows to the value k.

The interesting peculiarity of Fig. 3 is that there are three temperature regimes where thermal fluctua-

Table 1. The occupancy probability W_0 of the left potential well

$1/D \equiv \tilde{U}_0/k_BT$	q	ν	Occupancy probability W_0
28.6	0.48	10^9	$1.116 \cdot 10^{-13}$
28.6	0.48	0.05	0.99987
28.6	0.33	10^9	$1.116 \cdot 10^{-13}$
28.6	0.33	0.05	0.99987
15.4	0.48	10^9	$5.949 \cdot 10^{-8}$
15.4	0.48	0.4	0.90831
15.4	0.33	10^9	$5.949 \cdot 10^{-8}$
15.4	0.33	0.4	0.90174

The other parameter values in model (10) are $a = 2$, $\varepsilon = 1$, and $k = 0.125$. All quantities are dimensionless with scaling (8) and (9). Here we emphasize that in the high frequency limit, $\nu \to \infty$, the Brownian particle is subjected to the average potential $V_2(x)$ and hence, the result for W_0 is the same as for the non-fluctuating potential $V_2(x)$.

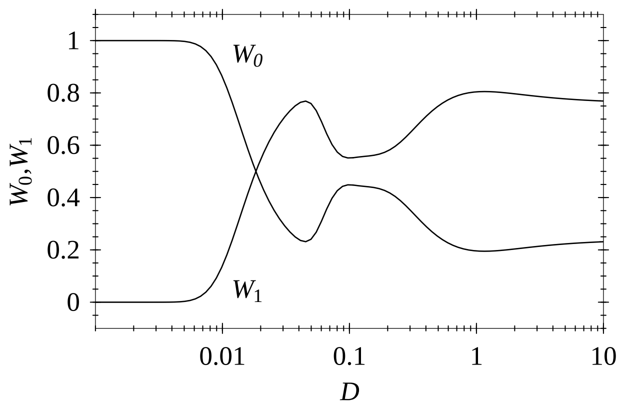

Fig. 3. The occupancy probabilities W_0 and W_1 of the left and right potential wells, respectively, versus the temperature D (Eq. (24)). The parameter values are $a = 2$, $\varepsilon = 1$, $k = 0.24$, $q = 0.45$, and $\nu = 5 \times 10^{-3}$. At large values of the temperature, $D > 10$, the probabilities W_0 and W_1 saturate to the values k and $1 - k$, respectively.

tions cause an enhancement of the occupancy of the metastable state: (i) At high temperatures the effect is trivial. In this case the Brownian particles "fail to see" the structure of the potential profile and move like in a simple rectangular potential well (cf. Fig. 4). (ii) For low values of the temperature the effect of enhancement is very pronounced, i.e., nearly all particles are concentrated in the left potential well, which has higher energy most of the time. This result is in accordance with the phenomenon of noise correlation time induced stability (see Fig. 2 and [18]). (iii) In the case of moderate values of the temperature a new resonance-like behavior is observed — enhancement of stability also occurs in a finite interval of the temperature, where the lowest depth of the potential wells is comparable with the thermal energy of the particle.

A general feature of the phenomenon of double temperature-enhanced stability is that the effect occurs over a broad range of potential fluctuation rates (cf. Figs. 5 and 6). It is remarkable that the local maximum of $W_0(D)$ disappears at such values of noise correlation time τ_c that are comparable with or lower than the intrawell relaxation time for the right well of the net potential $V_1(x)$, i.e.

$$\tau_c \lesssim \frac{(1-k)^2}{1+\epsilon-a(1-k)} .$$

Let us note that the value of the temperature that maximizes $W_0(D)$ can be estimated from the following equation:

$$\tau_c \approx \exp\left(\frac{V_1(k)-V_1(1)}{D}\right),$$

i.e. the noise correlation time τ_c is comparable with the Kramers escape time from the right potential well (in the noise state $z_1 = a$). This is a remarkable connection that throws some light on the physics of the effect, namely, it relates two characteristic time scales of the dynamical system (10) and demonstrates that all the three agents — colored noise, thermal noise, and potential configuration — act in unison to generate enhancement of the occupancy of the metastable state at moderate temperatures.

A comparison of the above results with calculations for mean passage times shows that the highly nonlinear behavior of W_0 and W_1 at low and moderate temperatures is related to resonant activation [5], [18], [31]. In particular, a general feature of the resonant activation phenomenon for a linear ramp, which is similar to our situation, is that with increasing barrier height (or decreasing temperature) a long flat region of the mean first passage time develops around the resonant switching rate ν_{res} (ν_{res} corresponds to the minimum of the mean first passage time versus ν) [31].

There are several important time scales in our system: six mean first passage times $T_n^{(i)}$ for the two minima of $V_n(x)$ ($i = 0$ and $i = 1$ correspond to the left and right minima respectively); the intrawell relaxation times for $V_n(x)$, and the correlation time $\tau_c = 1/\nu$ for the fluctuations of the potential. If τ_c

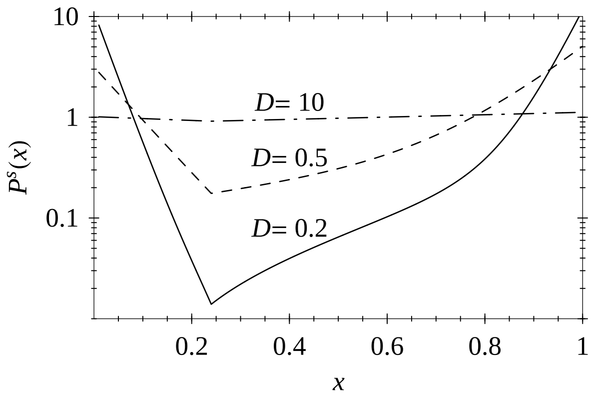

Fig. 4. The stationary probability density $P^s(x)$ at various temperatures D (Eq. (23)). The parameter values are $k = 0.24$, $a = 2$, $\nu = 0.1$, $\varepsilon = 1$, and $q = 0.45$. At large values of the temperature, $D > 10$, the probability density $P^s(x)$ tends to a uniform distribution.

is long enough compared to the intrawell deterministic relaxation times of the net potential $V_n(x)$, i.e. $\tau_c \gg \max(L_i^2/\Delta V_n^{(i)})$, where $L_0 = k$, $L_1 = 1 - k$, and $\Delta V_n^{(i)}$ are the depths of the net potential wells $V_n^{(i)}$, the condition $D < \min(\Delta V_n^{(i)})$ guarantees a sharp occupancy distribution in the minima of the net potentials. In this case the probability flux J_i from the left (right) potential well to the right (left) one is given by

$$J_0 = \frac{W_0}{T_0}\ ,\ J_1 = -\frac{W_1}{T_1}\ ,$$

where T_0 and T_1 are the mean first passage times from the bottom of the left potential well to the bottom of the right potential well and vice versa, respectively. In the stationary case, the total probability flux between the left and right potential wells must vanish, implying

$$\frac{W_0}{W_1} = \frac{T_0}{T_1}\ . \qquad (25)$$

Now, we will briefly consider the behavior of the probability W_0 in the high frequency regime, $\nu > \min[\Delta V_n^{(i)}/L_i^2]$. A general feature of our solution is that with an increasing switching rate ν the resonance-like phenomenon, i.e., the local maximum [cf. Figs. 3 and 6], becomes less and less sharp until it disappears at $\nu \sim \max[\Delta V_n^{(i)}/L_i^2]$. For large values of the switching rate ν two characteristic regions can be discerned for the temperature D. First, the region of low intrawell diffusion levels $D\nu < \min[(\Delta V_n^{(i)}/L_i)^2]$, for which the characteristic distance of intrawell thermal diffusion $\sqrt{D\tau_c}$ is much smaller than the typical deterministic distances of the driven particles dur-

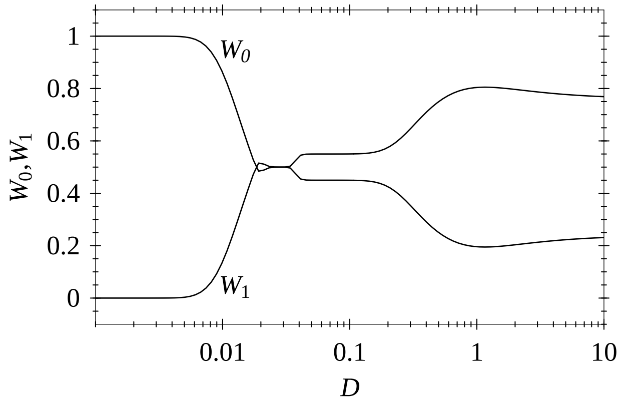

Fig. 5. The occupancy probabilities W_0 and W_1 versus the dimensionless temperature D, calculated by means of equations (21) - (24). The noise switching rate $\nu = 5 \times 10^{-10}$, the other parameter values are the same as in Figure 3.

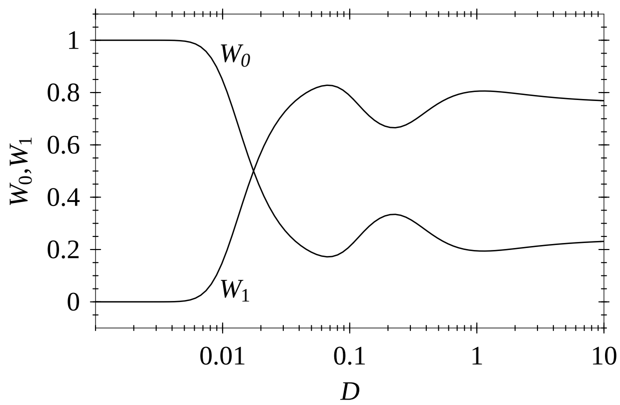

Fig. 6. The occupancy probabilities W_0 and W_1 versus the dimensionless temperature D, calculated by means of equations (21) - (24). The case of $\nu = 0.2$. The other parameter values are the same as in Figure 3.

ing the noise correlation time $\tau_c = 1/\nu$, and second, the regime $D\nu \gg \min[(\Delta V_n^{(i)}/L_i)^2]$, where thermal diffusion dominates. In the regime of low diffusion the behavior of W_0 is similar to that presented at low temperatures in Fig. 6 (the temperature is lower than the temperature $D_{\min}$ corresponding to the first minimum of W_0), but the "critical" temperature D_{cr}, at which $W_0 = W_1 = 1/2$, decreases as ν increases. In the region of strong diffusion the Brownian particle is subject to the average potential $V_2(x)$ in the case of fast fluctuations. Hence, in this regime the occupancy probability W_0 depends on temperature in the same way as in the case of the non-fluctuating potential $V_2(x)$, i. e. with increasing the temperature, W_0 increases monotonically up to the value k.

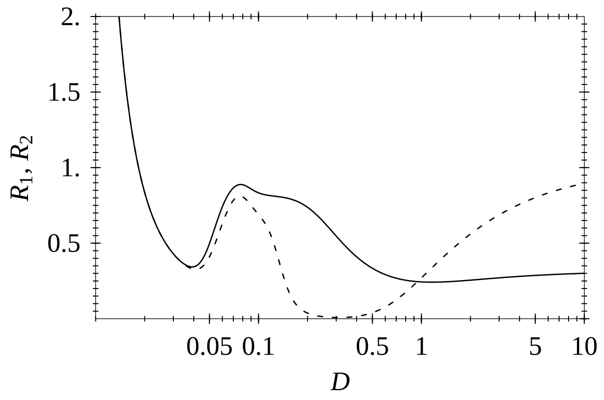

Fig. 7. The ratios $R_1 \equiv W_0/W_1$ and $R_2 \equiv T_0/T_1$ versus the dimensionless temperature D in the case of $q = 0.45$. Solid line: the function $R_1(D)$ computed from Eq. (24). Dashed line: the function $R_2(D)$ computed from Eqs. (34) - (36). The parameter values: $a = 2,\ \varepsilon = 1,\ k = 0.24,\ \nu = 10^{-3}$.

Next, we consider the regime

$$\min\left(\frac{\Delta V_n^{(i)}}{L_i^2}\right) > \nu > \frac{\left(\Delta V_1^{(1)}\right)^2}{D\,(1-k)^2} e^{-\Delta V_1^{(1)}/D}, \quad (26)$$

which corresponds to temperatures that are lower than the temperature $D_{\max}$ corresponding to the local maximum of W_0. In this case the formula (25) is applicable. The mean first passage time depends on the initial occupancy probabilities $p_n^{(i)}$ of the net potential wells $V_n^{(i)}$. In the case of Eq. (26) the time scale of barrier fluctuations is much faster than the escape times and there is, between two crossings over the barrier, enough time for the particle probability distribution to relax and spend most of the time in a quasi-stationary probability distribution corresponding to the stationary trichotomous process, i.e.

$$\begin{aligned} p_1^{(i)} &= p(z_1) = q\,, \\ p_2^{(i)} &= p(z_2) = 1 - 2q\,, \qquad (27) \\ p_3^{(i)} &= p(z_3) = q\,. \end{aligned}$$

The exact formulas for the mean passage times T_0 and T_1, being complex and cumbersome, are presented in the Appendix (Eqs. (36)).

In Fig. 7 the ratios W_0/W_1 and T_0/T_1 are compared for $q = 0.45$. When comparing the whole curves of $T_0(D)/T_1(D)$ and $W_0(D)/W_1(D)$, one can distinguish two regions. For $D < D_{\max} = 0.1$ the agreement between the curves $T_0(D)/T_1(D)$ and $W_0(D)/W_1(D)$ is surprisingly good at moderate temperature values, and even excellent for small values of D. Thus, in this region formula (25) with Eqs. (27) applies and the occupation process of the potential wells can be characterized by the mean first passage times over the potential barrier. For higher values of D, however, a significant discrepancy occurs. This means that for $D > D_{\max}$, the time scale considered (cf. Eq. (26)) is no more applicable and the conditions for the initial occupancy probabilities (see Eq. (27)) become invalid.

Finally, note that for sufficiently large values of the correlation time, the first minimum and the local maximum disappear and two plateaus occur at moderate temperatures (cf. Fig. 5), i.e., in this parameter region thermal noise is effectively suppressed. To throw some light on the physics of the above-mentioned effect, we shall now consider some physical approximations for the situation:

$$e^{-\Delta V_2^{(0)}/D} \ll \nu \ll e^{-\Delta V_3^{(0)}/D}\,, \qquad (28)$$

which corresponds to the first plateau in Fig. 5. Remember the inequalities $\Delta V_1^{(1)} < \Delta V_3^{(0)} < \Delta V_2^{(0)} < \Delta V_1^{(0)} < \Delta V_2^{(1)} < \Delta V_3^{(1)}$ (cf. Eq. (12) and Fig. 1).

Let us now consider the derivation of an approximate equation for the mean first passage time T_0 (the derivation of T_1 is analogous to that). For the regime (28), the particle locked in the noise state $n = 1$ at the right net potential minimum (cf. Fig. 1) will move, at the initial time $t = 0$, to the left net potential minimum $V_1^{(0)}(0)$. The particle can escape over the potential barrier back to the right potential minimum only in the noise state $z_3 = -a$. In this state the left net potential well is shallow and the corresponding Kramers time is much shorter than the noise correlation time $\tau_c = 1/\nu$. In the case of trichotomous fluctuations $Z(t)$ the probability $\bar{W}(t)$ that in a certain time interval $(0,\ t)$ transition to the noise state $z_3 = -a$ does not occur is given by $\bar{W}(t) = \exp(-q\nu t)$ [15]. The probability that the transition to $z_3 = -a$ occurs within the time interval $(t,\ t + dt)$ is $\nu q\,dt$. Consequently, the mean first passage time from the left potential well to the right one is approximately given by

$$T_0 \approx q\nu \int_0^\infty t e^{-q\nu t} dt = \frac{1}{q\nu}. \qquad (29)$$

As mentioned above the derivation of T_1 is analogous to that for T_0 and the result is also the same as for T_0, i.e., $T_1 \approx T_0$. Thus from Eq. (25) we obtain

$$W_0 \approx \frac{T_0}{T_1 + T_0} \approx \frac{1}{2}\,.$$

A comparison with Fig. 5 shows that for the temperature interval determined by the conditions (28) our approximation (29) with Eq. (25) captures the exact result extremely well.

4 Conclusion

In the present work, we have analyzed the behavior of one-dimensional overdamped Brownian motion in a sawtooth-like asymmetric bistable potential driven by a trichotomous noise and an additive thermal noise. Using the corresponding master equation we have obtained an exact expression for the occupancy probability of the metastable state and demonstrated the phenomenon of noise-induced stability. One should take care not to confuse the terms noise-induced stability and noise-enhanced stability used in this work with the effect of noise-enhanced stability discussed in [17], [19], and [22]. The effect called noise-enhanced stability in [17], [19], [22] is only a postponement of system instability (see also [16]), and is observed in a periodically (or stochastically) driven system with a single metastable minimum of the potential. The system remains in the metastable minimum for some time given by the mean first passage time for the barrier, and the mean first passage time has a maximum at a certain noise intensity. Evidently, in the stationary regime, the occupancy probability of the metastable state is zero. In the present work the potential fluctuates stochastically with a certain correlation time and has two minima. The less stable minimum is the absolute minimum for a certain configuration of the potential, but most of the time this minimum is metastable. Nevertheless, in a stationary regime it can be highly occupied (see also [18]).

Our major novel result is the effect of double enhanced stability of a metastable state versus temperature. Notably, enhancement of the stability also occurs at moderate temperatures, i.e., when the temperature D is such that the lowest barrier height of the system is just a few D, which is relevant for cell biology [32]. For dichotomous noise, which is a special case of trichotomous noise, a qualitatively similar model has been studied in [18]. However, to our knowledge, neither the phenomenon of double temperature-enhanced stability nor the existence of the corresponding resonance-like peak versus temperature at moderate values of D have been noticed or discussed before. The major advantage of this effect is that the control parameter is temperature, which can easily be varied in experiments.

Another important conclusion is that the phenomenon is robust enough to survive a variation of the noise kurtosis, the noise correlation time (over a very broad range) or the potential profile.

Our exact analytical results, concerning enhancement of the stability of a metastable state in a fluctuating bistable potential can be a good starting point to investigating more realistic systems. First, it would be interesting, for example, to investigate the behavior of W_0 at continuous transformation of the piecewise linear potential into a smooth one. Second, for possible experimental realizations the multiple regimes of temperature-enhanced stability of the metastable well versus different time scales should be considered in more detail, especially in the two-dimensional case. Finally, our paper is restricted to the case of a well-defined potential flipping rate determined by noise correlation time. However, in many physical systems fluctuations have power-law correlations (a well-defined noise correlation time is absent). Thus, it is important to investigate, by numerical simulations, the occurrence of the resonant phenomena described in this paper at those, strongly correlated fluctuations.

We believe that the results obtained are also of interest for experimental cell biology, where the proposed model can be applied [18], [23], [24].

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5 Appendix: Formulas for the mean first passage time

Here the exact formulas for the mean first passage times (MFPT) from the bottom of the left potential well to the bottom of the right potential well (T_0) and back (T_1) will be represented. Using standard methods described in [33], from the backward equation of master equation (13) the following set of equations for the MFPT can be deduced:

$$\begin{aligned}(h_i - z_n)\frac{\partial}{\partial x}T_i^{(m)}(x, z_n) \\ + D\frac{\partial^2}{\partial x^2}T_i^{(m)}(x, z_n) \\ - \nu T_i^{(m)}(x, z_n) + \nu T_i^{(m)}(x) = -1,\end{aligned} \tag{30}$$

where $i = 0, 1$, $m = 0, 1$, $n = 1, 2, 3$, $h(x) = h_0 = -1/k$ for $x \in (0, k)$, $h(x) = h_1 = (1 + \epsilon)/(1 - k)$ for $x \in (k, 1)$, and

$$T_i^{(m)}(x) = \sum_{n=1}^{3} p(z_n)T_i^{(m)}(x, z_n). \tag{31}$$

with the initial probabilities

$$p(z_1) = p(z_3) = q,\ p(z_2) = 1 - 2q.$$

As the force h is piecewisely constant, (30) splits up into four linear differential equations with constant coefficients for the four vector functions ($T_i^{(m)}(x, z_1)$, $T_i^{(m)}(x, z_2)$, $T_i^{(m)}(x, z_3)$, $i = 0, 1$, $m = 0, 1$), defined on the intervals $(0, k)$ and $(k, 1)$, separately. The solution reads

$$T_i^{(m)}(x, z_n) = -\frac{1}{h_i}x + \frac{z_n}{h_i\nu} + C_{i0}^{(m)} + \sum_{j=1}^{5} C_{ij}^{(m)} A_{nij} \exp\left(\frac{\lambda_{ij}x}{D}\right), \quad (32)$$

where $C_{i0}^{(m)}$, $C_{ij}^{(m)}$ are constants of integration, and the quantities A_{nij} , λ_{ij} are the same as in Eq. (16).

The twenty-four independent conditions for the constants of integration $C_{ij}^{(m)}$, $C_{i0}^{(m)}$ can be determined at the points of discontinuity, by requiring continuity for the quantities $T_i^{(m)}(x, z_n)$ and for the stationary current densities at the point $x = k$, and be applying the corresponding conditions of absorbing and reflecting boundaries at the potential wells bottoms. The appropriate boundary conditions to $T_i^{(m)}(x, z_n)$ are:

$$T_0^{(m)}(k, z_n) = T_1^{(m)}(k, z_n)\,,$$

$$\frac{d}{dx}T_0^{(m)}(x, z_n)_{\,|x=k} = \frac{d}{dx}T_1^{(m)}(x, z_n)_{|x=k}\,,$$

$$T_0^{(m)}(1, z_n) = T_1^{(m)}(0, z_n) = 0\,, \quad (33)$$

$$\frac{d}{dx}T_0^{(m)}(x, z_n)_{|x=0} = \frac{d}{dx}T_1^{(m)}(x, z_n)_{|x=1} = 0\,.$$

As follows from Eqs. (32) and (33), this procedure leads to an inhomogeneous set of 24 linear algebraic equations for $C_{ij}^{(m)}$ and $C_{i0}^{(m)}$.

Now the constants $C_{ij}^{(0)}$ and $C_{i0}^{(0)}$ are determined by the following system of 12 linear algebraic equations:

$$C_{10}^{(0)} + \sum_{j=1}^{5} C_{1j}^{(0)} A_{n1j} \exp\left(\frac{\lambda_{1j}}{D}\right) = -\frac{z_n}{h_1\nu} + \frac{1}{h_1}\,,$$

$$\sum_{j=1}^{5} C_{0j}^{(0)} \lambda_{0j} A_{nij} = \frac{D}{h_0}\,,$$

$$C_{00}^{(0)} - C_{10}^{(0)} + \sum_{j=1}^{5}\left[C_{0j}^{(0)} A_{n0j} \exp\left(\frac{\lambda_{0j}k}{D}\right) - C_{1j}^{(0)} A_{n1j} \exp\left(\frac{\lambda_{1j}k}{D}\right)\right] = -\frac{z_n}{\nu}\left(\frac{1}{h_0} - \frac{1}{h_1}\right) + k\left(\frac{1}{h_0} - \frac{1}{h_1}\right),$$

$$\sum_{j=1}^{5}\left[C_{0j}^{(0)} \lambda_{0j} A_{n0j} \exp\left(\frac{\lambda_{0j}k}{D}\right) - C_{1j}^{(0)} \lambda_{1j} A_{n1j} \exp\left(\frac{\lambda_{1j}k}{D}\right)\right] = D\left(\frac{1}{h_0} - \frac{1}{h_1}\right), \quad n = 1, 2, 3\,. \quad (34)$$

The corresponding equations for $C_{ij}^{(1)}$ and $C_{i0}^{(1)}$ read:

$$C_{00}^{(1)} + \sum_{j=1}^{5} C_{0j}^{(1)} A_{n0j} = -\frac{z_n}{h_0\nu}\,,$$

$$\sum_{j=1}^{5} C_{1j}^{(0)} \lambda_{1j} A_{nij} \exp\left(\frac{\lambda_{1j}k}{D}\right) = \frac{D}{h_1}\,,$$

$$C_{00}^{(1)} - C_{10}^{(1)} + \sum_{j=1}^{5}\left[C_{0j}^{(1)} A_{n0j} \exp\left(\frac{\lambda_{0j}k}{D}\right) - C_{1j}^{(1)} A_{n1j} \exp\left(\frac{\lambda_{1j}k}{D}\right)\right] = -\frac{z_n}{\nu}\left(\frac{1}{h_0} - \frac{1}{h_1}\right) + k\left(\frac{1}{h_0} - \frac{1}{h_1}\right),$$

$$\sum_{j=1}^{5}\left[C_{0j}^{(1)} \lambda_{0j} A_{n0j} \exp\left(\frac{\lambda_{0j}k}{D}\right) - C_{1j}^{(1)} \lambda_{1j} A_{n1j} \exp\left(\frac{\lambda_{1j}k}{D}\right)\right] = D\left(\frac{1}{h_0} - \frac{1}{h_1}\right), \quad n = 1, 2, 3\,. \quad (35)$$

Now, a straightforward calculation gives for the mean passage times $T_0 = T_0^{(0)}(0)$ and $T_1 = T_1^{(1)}(1)$:

$$T_0 = C_{00}^{(0)} + \sum_{j=1}^{5} C_{0j}^{(0)},$$

$$T_1 = C_{10}^{(1)} - \frac{1}{h_1} + \sum_{j=1}^{5} C_{1j}^{(1)} \exp\left(\frac{\lambda_{1j}}{D}\right). \quad (36)$$

Hence, the problem set has been solved and we can see that exact evaluation of the MPFT can be handled by linear algebra.

References:

[1] J. M. G. Vilar and J. M. Rubí, Noise Suppression by Noise, *Phys. Rev. Lett.*, Vol. 86, No. 6, 2001, pp. 950–953.

[2] S.-E. Qian, J. Lévesque, and R. A. Neville, Effect of Removing Random Noise of Radiance Data Using Smoothing on Data Compression Onboard a Hyperspectral Satellite, *WSEAS Transactions on Systems*, Vol. 5, No. 1, 2006, pp. 219–224.

[3] H. Wang, L. Y. Wang, H. Zheng, R. Haladjian, and M. Wallo, Lung Sound, Noise Separation for Anesthesia Respiratory Monitoring, *WSEAS Transactions on Systems*, Vol. 3, No. 4, 2004, pp. 1839–1844.

[4] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Stochastic resonance, *Rev. Mod. Phys.*, Vol. 70, No. 1, 1998, pp. 223–287.

[5] C. R. Doering and J. C. Gadoua, Resonant activation over a fluctuating barrier, *Phys. Rev. Lett.*, Vol. 69, No. 16, 1992, pp. 2318–2321.

[6] J. García-Ojalvo and J. M. Sancho, *Noise in Spatially Extended Systems*, Springer Verlag, New York, 1999.

[7] R. Mankin, A. Sauga, T. Laas and E. Soika, Environmental-fluctuations-induced slow-fast cycles in ratio-dependent predator-prey systems, *WSEAS Transactions on Systems*, Vol. 6, No. 5, 2007, pp. 934–941.

[8] R. Mankin, A. Sauga, A. Ainsaar, A. Haljas and K. Paunel, Colored-noise-induced discontinuous transitions in symbiotic ecosystems, *Phys. Rev. E*, Vol. 69, No. 6, 2004, pp. 061106(1)–(8).

[9] P. Reimann, Brownian motors: noisy transport far from equilibrium, *Phys. Rep.*, Vol. 361, No. 2-4, 2002, pp. 57–265.

[10] D. Martila, R. Mankin, R. Tammelo, A. Sauga, and E. Reiter, Constructive influence of noise-flatness in correlation ratchets, *Eur. Phys. J. B*, Vol. 54, No. 3, 2006, pp. 375–383.

[11] H. Linke (ed), Special issue on ”Ratchets and Brownian motors: basics, experiments and applications”, *Appl. Phys. A*, Vol. 75, 2002, pp. 167–352.

[12] H. A. Kramers, Brownian motion in a field of force and the diffusion model of chemical reactions, *Physica*, Vol. 7, No. 4, 1940, pp. 284–304.

[13] K. Lindenberg and B. J. West, *The Nonequilibrium Statistical Mechanics of Open and Closed Systems*, John Wiley & Sons, 1990.

[14] W. Horsthemke and R. Lefever, *Noise-induced transitions*, Springer-Verlag, New York, 1984.

[15] R. Mankin, A. Ainsaar, and E. Reiter, Trichotomous noise-induced transitions, *Phys. Rev. E*, Vol. 60, No. 2, 1999, pp. 1374–1380.

[16] R. A. Ibrahim, Excitation-induced stability and phase transition: A review, *Journal of Vibration and Control*, Vol. 12, No. 10, 2006, pp. 1093–1170.

[17] B. Spagnolo, N. Agudov, and A. Dubkov, Noise enhanced stability, *Acta Phys. Pol. B*, Vol. 35, No. 4, 2004, pp. 1419–1436.

[18] A. Mielke, Noise induced stability in fluctuating, bistable potentials, *Phys. Rev. Lett.*, Vol. 84, No. 5, 2000, pp. 818–821.

[19] R. N. Mantegna and B. Spagnolo, Noise enhanced stability in an unstable system, *Phys. Rev. Lett.*, Vol. 76, No. 4, 1996, pp. 563–566.

[20] M. Gitterman, Stabilization of metastable states, *Phys. Rev. E*, Vol. 65, No. 3, 2002, pp. 031103 (1)–(3).

[21] P. D’Odorico, F. Laio, and L. Ridolfi, Noise-induced stability in dryland plant ecosystems, *Proc. Natl. Acad. Sci. USA*, Vol. 102, No. 31, 2005, pp. 10819–10822.

[22] B. Spagnolo, A. Dubkov, A. Pankratov, E. Pankratova, A. Fiasconaro, and A. Ochab-Marcinek, Lifetime of metastable states and suppression of noise in interdisciplinary physical models, *Acta Phys. Pol., B*, Vol. 38, No. 4, 2007, pp. 1925–1950.

[23] R. D. Astumian and B. Robertson, Nonlinear effect of an oscillating electric field on membrane proteins, *J. Chem. Phys.*, Vol. 91, No. 8, 1989, pp. 4891–4901.

[24] D. Petracchi, M. Pellegrini, M. Pellegrino, M. Barbi, and F. Moss, Periodic forcing of a K+ channel at various temperatures, *Biophys. J.*, Vol. 66, No. 6, 1994, pp. 1844–1852.

[25] P. Hänggi, P. Talkner, and M. Borkovec, Reaction-rate theory: fifty years after Kramers, *Rev. Mod. Phys.*, Vol. 62, No. 2, 1990, pp. 251–341.

[26] F. Schlögl, Chemical reaction models for non-equilibrium phase transitions, *Z. Phys.* Vol. 253, No. 2, 1972, pp. 147–161.

[27] E. Abraham and S. D. Smith, Optical bistability and related devices, *Rep. Progr. Phys.*, Vol. 45, 1982, pp. 815–886.

[28] B. McNamara, K. Wiesenfeld, and R. Roy, Observation of Stochastic Resonance in a Ring Laser, *Phys. Rev. Lett.*, Vol. 60, No. 25, 1988, pp. 2626–2629.

[29] C. R. Doering, W. Horsthemke, and J. Riordan, Nonequilibrium fluctuation-induced transport, *Phys. Rev. Lett.*, Vol. 72, No. 19, 1994, pp. 2984–2987.

[30] J. Łuczka, Application of statistical mechanics to stochastic transport, *Physica A*, Vol. 274, No. 1–2, 1999, pp. 200–215.

[31] M. Bier and R.D. Astumian, Matching a diffusive and a kinetic approach for escape over a fluctuating barrier, *Phys. Rev. Lett.*, Vol. 71, No. 10, 1993, pp. 1649-1652.

[32] R. Astumian and M. Bier, Mechanochemical coupling of the motion of molecular motors to ATP hydrolysis, *Biophys. J.*, Vol. 70, No. 2, 1996, pp. 637–653.

[33] C. W. Gardiner, *Handbook of Stochastic Methods*, Springer-Verlag, Berlin, 1985.