Engineering Education and Infinite Superposition

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Abstract: - Attention is focused on continuous-space shift-invariant systems with continuous system maps and inputs and outputs that are elements of $L_\infty(I\!R^d)$. It is shown that infinite superposition can fail in this important setting. It is also shown that continuous shift-invariant linear mappings need not commute with the operation of integration (even when the two composite operations are well defined).

Key-Words: - linear systems, superposition, commutativity, shift-invariant systems, bounded measurable inputs

1 Introduction

This paper is a continuation of this writer's study of the general problem of linear system representation. We begin with a short survey of background material, and then give further related results concerning engineering education, superposition, and commutativity.

In the signal-processing literature, $x(\alpha)$ typically denotes a function. In the following we distinguish between a function x and $x(\alpha)$, the latter meaning the value of x at the point (or time) α. Sometimes a function x is denoted by $x(\cdot)$, and also we use Hx to mean $H(x)$. This notation is often useful in studies of systems in which signals are transformed into other signals.

A recent paper [1] considers continuous-time linear time-invariant systems governed by a relation $y = Hx$ in which x is an input, y is the corresponding output, and H is the system map that takes inputs into outputs. It was assumed that inputs and outputs are complex-valued functions defined on the set $I\!R$ of real numbers. As is well known, it is a widely-held belief of long standing that the input-output properties of H are completely described by its impulse response. Using a standard interpretation of what is meant by a system's impulse response, it is shown in [1] that this belief is incorrect in a simple setting in which x is drawn from the linear space $\mathcal{C}$ of bounded uniformly-continuous complex-valued functions defined on $I\!R$. More specifically, it was shown that there is an H of the kind described above, even a causal H, whose impulse response is the zero function, but which takes certain inputs into nonzero outputs.[1] This contradicts the conclusion of a familiar engineering argument using the so-called sifting property of Dirac's impulse function. An important role in [1] is played by inputs that do not approach zero at infinity in a pointwise or certain average sense, and a similarly important role is played by such inputs in connection with related discrete-time results (see, e.g., [3]). This observation served as the motivation for the study reported on in [4], in which it is shown that the members of a certain large family of linear systems are in fact completely characterized by their (suitably defined) impulse responses – which exist as certain limits. These limits are functions in the usual sense (as opposed to generalized functions). Reference [4] addresses the case in which inputs belong to the space $L_p(I\!R^d)$ of pth-power integrable complex-valued functions x defined on $I\!R^d$, in which p satisfies $1 \leq p < \infty$, and d is an arbitrary positive integer. Outputs are taken to be elements of the space $B(I\!R^d)$ of bounded complex-valued

[1] A similar result [2] holds for maps whose domain is the whole space of bounded Lebesgue-measurable signals. Each of these propositions holds also for maps whose domain and range are the corresponding spaces of real-valued functions.

functions y defined on $I\!R^d$, with the norm given by

$$\|y\| = \sup_{\alpha \in I\!R^d} |y(\alpha)|. \quad (1)$$

For the reason that [4] contains the first in a series of recent representation results, and for the reader's convenience, a detailed description of the main result in [4] is given in the Appendix.

In [5] we consider the case in which outputs are elements of $B(I\!R^d)$, but inputs belong to the normed linear space $C_0(I\!R^d)$ of continuous complex-valued functions x defined on the set $I\!R^d$ of real d-vectors such that $x(\alpha) \to 0$ as $\|\alpha\|_{(d)} \to \infty$, in which $\|\cdot\|_{(d)}$ stands for a norm on $I\!R^d$. As is widely known, such multidimensional systems are of interest in connection with, for example, image processing. The theory concerning these systems is in some respects more interesting than for the $L_p(I\!R^d)$-input-$B(I\!R^d)$-output case in [4]. It is assumed that H is continuous and shown that H has a representation given by

$$(Hx)(\alpha) = \lim_{\epsilon \to 0} \int_{I\!R^d} (Hq_\epsilon)(\alpha-\beta)x(\beta)\, d\beta \quad (2)$$

in which the convergence is uniform with respect to α, and q_ϵ is a certain type of function that depends on the parameter ϵ. Also given is a necessary and sufficient condition under which there is a function h such that the right side of (2) can be written as a convolution

$$\int_{I\!R^d} h(\alpha - \beta)x(\beta)\, d\beta.$$

Related results are given in [6] for the case in which inputs belong to $L_p(I\!R^d) \cup L_1(I\!R^d)$, where again $1 \leq p < \infty$ (of course, the case in which $p = 2$ is of particular interest).

For reasons closely related to the material in [1] outlined above, a difficulty arises in attempts to obtain corresponding results for the important case in which the space of inputs is $L_\infty(I\!R^d)$, the normed linear space of complex-valued bounded Lebesgue-measurable functions defined on $I\!R^d$, with the norm given by (1). In [7], we indicate how this difficulty can be circumvented. More explicitly, in [7] we describe a representation theorem proved along the general lines of the theorem in [5] for the case in which both the range and domain of H is $L_\infty(I\!R^d)$, but under an additional (typically reasonable) assumption of the form

$$Hx = \lim_{\sigma \to \infty} (HW_\sigma x) \quad (3)$$

for each x, in which $\{W_\sigma : \sigma > 0\}$ is a certain set of weighting operators.[2] Of particular interest is the observation in [7] that for the large family of inputs and maps H addressed, the Dirac impulse-response concept is in fact not the key concept concerning the representation of H, and that instead the input-output properties of H are determined in general by a certain type of *family* $\{Hq_\epsilon : \epsilon \in (0, \rho)\}$ of responses.

Here we consider a different but closely related aspect of the general problem of linear-system representation. One can give a very long list of books – many of them basically very good books – in which superposition is said to hold in the case of any linear system with an excitation that can be written as a sum of a countably infinite number of excitations. As is well known, this conclusion – which has been taught to decades of students in several fields – plays a central role in textbook material concerning the origins of both discrete-time and continuous-time convolution representations. In [8], and also in the brief note [9], attention is directed to the fact that the conclusion is not correct. One consequence of the oversight is that the usual representation for linear discrete-time systems has had to be corrected by adding an additional term [8].[3] In [9], in the setting of linear spaces with metrics and a standard definition of convergence, a simple criterion is given for (infinite) superposition to hold. A discrete-space example is given in [9] to illustrate that superposition can fail. However, the linear system map in the example is not continuous, and it is defined on only a certain unusual input space. In addition, the simple criterion given for superposition to hold involves a strong assumption on the convergence of the input sum representation of the input. With that assumption, superposition can fail only for system maps that lack continuity.

In the following section, attention is focused on shift-invariant continuous-space

[2] A similar result is given in [10] for the special (and less complex) case in which inputs are bounded functions that are continuous.

[3] As mentioned in [8], this writer does not claim that cases in which the extra term is nonzero are necessarily of importance in applications, but he does feel that the existence of these cases illustrates that the analytical ideas in the books are flawed.

systems with continuous system maps. Inputs and outputs are elements of the familiar space $L_\infty(\mathbb{R}^d)$. We show that superposition can fail in this important setting. Also given – and again in contrast with what is said in texts on linear systems – is a related result showing that continuous shift-invariant linear mappings need not commute with the operation of integration (even when the two composite operations are well defined).[4] Our main results are Theorems 1 and 2 of the following section. The section contains in addition results that provide sufficient conditions under which commutativity is valid.

2 On Superposition and Commutativity

2.1 Preliminaries

As in Section 1, $\|\cdot\|_d$ stands for a norm on $\mathbb{R}^d$, and $L_\infty(\mathbb{R}^d)$ denotes the linear space of bounded Lebesgue measurable complex-valued functions defined on $\mathbb{R}^d$, where d is any positive integer. We view $L_\infty(\mathbb{R}^d)$ as a normed space with the norm given by

$$\|x\|_\infty = \sup_{\alpha \in \mathbb{R}^d} |x(\alpha)|. \tag{4}$$

All integrals in this section and in the Appendix are Lebesgue integrals. For $1 \leq p < \infty$, $L_p(\mathbb{R}^d)$ stands for the usual normed linear space space of pth power integrable complex-valued functions defined on $\mathbb{R}^d$. $BL_1(\mathbb{R}^d)$ denotes the linear space of bounded $L_1(\mathbb{R}^d)$ functions.

2.2 Lack of Infinite Superposition

Our main result is the following.

Theorem 1 : There are elements x and $x_1, x_2, \ldots$ in $L_\infty(\mathbb{R}^d)$, and a continuous linear shift-invariant map G from $L_\infty(\mathbb{R}^d)$ into itself, such that

(i) $x = \sum_{n=1}^\infty x_n$ in the sense of pointwise convergence.

(ii) $\sum_{n=1}^\infty Gx_n$ converges in $L_\infty(\mathbb{R}^d)$, and we have

$$Gx \neq \sum_{n=1}^\infty Gx_n.$$

[4]For related material motivated by the fact that there exist continuous shift-invariant linear maps whose input-output behavior is not determined by its impulse response, see [11] – [13].

Proof:

Let x be any element of $L_\infty(\mathbb{R}^d)$ such that $\lim_{\|\alpha\|_d \to \infty} x(\alpha) = c$, in which c is a nonzero number, and define x_n by $x_n(\alpha) = x(\alpha)$ for $n-1 \leq \|\alpha\|_d < n$, and zero otherwise, for each positive integer n. It is clear that $x = \sum_{n=1}^\infty x_n$ in the sense of pointwise convergence. We will show that there is a continuous linear shift-invariant map G from $L_\infty(\mathbb{R}^d)$ into itself, such that (ii) holds, with in fact each Gx_n the zero function in $L_\infty(\mathbb{R}^d)$. We will use the following lemma.

Lemma 1 : There is a continuous linear shift-invariant map H from $L_\infty(\mathbb{R}^d)$ into itself such that

(a) H takes every element of $BL_1(\mathbb{R}^d)$ into the zero function in $L_\infty(\mathbb{R}^d)$.

(b) $(Hy)(\alpha) = \zeta$ for all $\alpha \in \mathbb{R}^d$ and each $y \in L_\infty(\mathbb{R}^d)$ with $\lim_{\|\alpha\|_d \to \infty} y(\alpha) = \zeta$, in which ζ is an arbitrary complex number.

Lemma 1 is all but stated in [14]. For the reader's convenience, a proof is given in the Appendix.

Select G to be any H of the kind described in the lemma, and observe that G satisfies (ii) because of the assumed limit property of x, and the fact that each x_n belongs to $BL_1(\mathbb{R}^d)$. This proves the theorem.

The fact that the series $\sum_{n=1}^\infty x_n$ of the theorem is not required to converge in $L_\infty(\mathbb{R}^d)$ is crucial, in that (see [9]) $\sum_{n=1}^\infty Gx_n$ converges in $L_\infty(\mathbb{R}^d)$, with

$$Gx = \sum_{n=1}^\infty Gx_n$$

when the series converges in $L_\infty(\mathbb{R}^d)$ (and G is as indicated). Also, a result entirely analogous to Theorem 1 holds in the corresponding discrete-space setting. Specifically, direct modifications of the proof show that Theorem 1 holds if $L_\infty(\mathbb{R}^d)$ is replaced with the usual normed linear space $\ell_\infty(\mathcal{Z}^d)$ of complex-valued functions defined on $\mathcal{Z}^d$, where $\mathcal{Z}$ denotes the integers. The theorem holds also if $L_\infty(\mathbb{R}^d)$ is replaced with its corresponding real-valued space, and similarly for $\ell_\infty(\mathcal{Z}^d)$.

2.3 Lack of Commutativity

The tools used to prove Theorem 1 also yield the following.

Theorem 2 : Let a be any element of $BL_1(\mathbb{R}^d)$ such that a has a nonzero integral. Then there is an element x in $L_\infty(\mathbb{R}^d)$ and a continuous linear shift-invariant map H from $L_\infty(\mathbb{R}^d)$ into itself such that H maps $BL_1(\mathbb{R}^d)$ into itself and

$$H \int_{\mathbb{R}^d} a(\cdot - \beta)x(\beta)\, d\beta \neq \int_{\mathbb{R}^d} Ha(\cdot - \beta)x(\beta)\, d\beta.$$

Proof:

Choose x to to be an element of $L_\infty(\mathbb{R}^d)$ with $\lim_{\|\alpha\|_d \to \infty} x(\alpha) = c$, in which c is a nonzero number, and notice that the theorem follows at once from Lemma 1 and Proposition 1 (in the Appendix). Here too, we arrive at a case in which the right side, but not the left side, is the zero function in $L_\infty(\mathbb{R}^d)$.

The theorem holds also if $L_1(\mathbb{R}^d)$ and $L_\infty(\mathbb{R}^d)$ are replaced with their corresponding real-valued spaces. Also, a result entirely analogous to Theorem 2 holds in the corresponding discrete-space setting in which $L_\infty(\mathbb{R}^d)$ and $BL_1(\mathbb{R}^d)$ are replaced with the familiar spaces $\ell_\infty(\mathcal{Z}^d)$ and $\ell_1(\mathcal{Z}^d)$, respectively, and the integrals are replaced with the analogous infinite sums.[5]

2.4 Sufficient Conditions for Commutativity

We close Section 2 by stating two results that provide conditions under which a linear map does commute with integration:

Theorem 3: Suppose that $1 \leq p < \infty$, and that M is a linear shift-invariant map whose domain includes $L_p(\mathbb{R}^d) \cup L_1(\mathbb{R}^d)$. Assume that M is defined on $L_p(\mathbb{R}^d) \cup L_1(\mathbb{R}^d)$ such that the restriction of M to $L_p(\mathbb{R}^d)$ is a continuous map into $L_p(\mathbb{R}^d)$, and the restriction of M to $L_1(\mathbb{R}^d)$ is a continuous map into $L_1(\mathbb{R}^d)$. Let $x \in L_p(\mathbb{R}^d)$, and let $g \in L_1(\mathbb{R}^d)$. Then, with ℓ the element of $L_p(\mathbb{R}^d)$ given by

$$\ell = \int_{\mathbb{R}^d} g(\cdot - \beta)x(\beta)\, d\beta$$

we have

$$M\ell = \int_{\mathbb{R}^d} (Mg)(\cdot - \beta)x(\beta)\, d\beta.$$

[5]A classical observation related in a general sense to Theorem 2 is that differentation under the integral sign is not always valid.

Theorem 3 is an extension [6] of Lemma 21.2.2 of [15, pp. 568] where the $p = 1$ case is given. Although not stated in [15], the proof given of the lemma yields also the following.

Theorem 4: Suppose that $1 \leq p < \infty$, and that M is a continuous linear shift-invariant map of $L_p(\mathbb{R}^d)$ into itself. Let $g \in L_p(\mathbb{R}^d)$ and let $x \in L_1(\mathbb{R}^d)$. Then, with ℓ the element of $L_p(\mathbb{R}^d)$ given by

$$\ell = \int_{\mathbb{R}^d} g(\cdot - \beta)x(\beta)\, d\beta$$

we have

$$M\ell = \int_{\mathbb{R}^d} (Mg)(\cdot - \beta)x(\beta)\, d\beta.$$

2.5 Conclusion

It is shown that infinite superposition can fail in a certain important setting. It is also shown that continuous shift-invariant linear mappings need not commute with the operation of integration (even when the two composite operations are well defined).

3 Appendix

3.1. The Theorem in [4]: $L_p(\mathbb{R}^d)$ Inputs

As in Section 1, H stands for a system map that takes inputs into outputs. Recall that $B(\mathbb{R}^d)$ denotes the normed linear space of bounded complex-valued functions defined on $\mathbb{R}^d$, with the norm given by (1).

In the following theorem, which is a slightly simplified version of the main result in [4], p satisfies $1 \leq p < \infty$, and $L_p(\mathbb{R}^d)$, the space of inputs, stands for the set of Lebesgue measurable complex-valued functions x defined on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} |x(\alpha)|^p\, d\alpha < \infty.$$

As usual, when $L_p(\mathbb{R}^d)$ is regarded as a metric space with the usual $L_p(\mathbb{R}^d)$ norm $\|\cdot\|_p$ the elements of $L_p(\mathbb{R}^d)$ are understood to be equivalence classes. Of course, the case in which $p = 2$ is of particular importance. By convergence in $L_p(\mathbb{R}^d)$, we mean convergence with respect to the norm in $L_p(\mathbb{R}^d)$. Here $L_\infty(\mathbb{R}^d)$ stands for the normed linear space of essentially bounded Lebesgue measurable complex-valued functions on $I\!R^d$.[6] Given $h \in L_\infty(I\!R^d)$, we say that $g_\epsilon \in L_\infty(I\!R^d)$ for $\epsilon \in (0,1)$ converges in $M_\infty(I\!R^d)$ to h as $\epsilon \to 0$ if we have

$$\int_S |g_\epsilon(\beta) - h(\beta)|\, d\beta \to 0 \quad \text{as} \quad \epsilon \to 0.$$

for every bounded Lebesgue measurable subset S of $I\!R^d$. Also, $m \in (1, \infty]$ and p are related by $1/p + 1/m = 1$ (in other words, m is the exponent conjugate to p for $p > 1$, and $m = \infty$ corresponds to $p = 1$).

$\mathcal{Q}$ denotes the family of bounded $L_1(I\!R^d)$-valued maps q defined on $(0,1)$ such that, with $q(\epsilon)$ denoted by q_ϵ,

$$\int_{I\!R^d} q_\epsilon(\alpha)\, d\alpha = 1 \quad \text{for} \quad \epsilon \in (0,1),$$

$$\sup_\epsilon \int_{I\!R^d} |q_\epsilon(\alpha)|\, d\alpha < \infty,$$

and

$$\lim_{\epsilon \to 0} \int_{\|\alpha\|_{(d)} > \xi} |q_\epsilon(\alpha)|\, d\alpha = 0, \quad \xi > 0.$$

Note that q given by the familiar expression

$$\begin{aligned} q_\epsilon(\alpha) &= 1/\epsilon, \quad |\alpha| \le \epsilon/2 \\ &= 0, \quad \text{otherwise} \end{aligned}$$

is an element of $\mathcal{Q}$ for $d = 1$.

It is reasonable to say, roughly speaking, that H has an impulse response (or what might more accurately be called a "q-response limit") if for every nicely-behaved $q \in \mathcal{Q}$ we have Hq_ϵ well defined for each $\epsilon \in (0,1)$, with $\lim_{\epsilon \to 0} Hq_\epsilon$ existing in a meaningful sense and not dependent on q. The main result in [4], except for a minor simplification, is the following theorem. For the type of H addressed in the theorem, H has an impulse response h in the precise sense that statement (b) of the theorem holds. (The literature says little about the existence of impulse responses for general systems, which typically are simply assumed to exist.)

Theorem: Suppose that H (which is linear and shift invariant) is a continuous map of $L_p(I\!R^d)$ into $B(I\!R^d)$. Let q be an element of $\mathcal{Q}$.

Then

(a) $q_\epsilon \in L_p(I\!R^d)$ for each ϵ.

(b) Hq_ϵ is an element of $L_m(I\!R^d)$ for each ϵ, Hq_ϵ converges in $L_m(I\!R^d)$ if $p > 1$, or in $M_\infty(I\!R^d)$ if $p = 1$, to an element h of $L_m(I\!R^d)$ as $\epsilon \to 0$, and h is independent of q.

(c) We have

$$(Hx)(\alpha) = \int_{I\!R^d} h(\alpha - \beta) x(\beta)\, d\beta, \quad \alpha \in I\!R^d$$

for all $x \in L_p(I\!R^d)$, and each Hx is a continuous function on $I\!R^d$.

(d) $(Hq_\epsilon)(\cdot)$ converges almost everywhere to h as $\epsilon \to 0$, provided that q_ϵ is given on $I\!R^d$ by

$$q_\epsilon(\alpha) = (1/\epsilon^d) r(\alpha/\epsilon), \quad \epsilon \in (0,1) \qquad (5)$$

where $r \in L_1(I\!R^d)$ has unit integral, is essentially bounded, and has compact support.[7]

Statement (c) of the theorem shows that, for the family of H's considered, the input-output properties of each H are completely described by its impulse response. As discussed at the beginning of this paper, not all linear shift-invariant systems are characterized by their impulse responses.

3.2 Proof of Lemma 1

We use the following three propositions, in which F denotes the map defined on $L_\infty(I\!R^d)$ by

$$(Fy)(\alpha) = \int_{I\!R^d} f(\alpha - \beta) y(\beta)\, d\beta, \quad \alpha \in I\!R^d$$

where $f \in BL_1(I\!R^d)$ with unit integral. Also, $\mathcal{C}$ stands for the normed linear space of bounded uniformly-continuous complex-valued functions defined on $I\!R^d$, with the norm described by (4).[8]

Proposition 1: If $y \in L_\infty(I\!R^d)$ satisfies

$$\lim_{\|\alpha\|_d \to \infty} y(\alpha) = \zeta \quad \text{for some} \quad \zeta,$$

[6] in Section 2, $L_\infty(I\!R^d)$ stands for the corresponding space of bounded (not essentially bounded) functions.

[7] The statement that $(Hq_\epsilon)(\cdot)$ converges almost everywhere to (the equivalence class) h means that $(Hq_\epsilon)(\alpha) \to g(\alpha)$ for almost every α, in which g is any individual function belonging to the class h. Also, it is not difficult to check that q defined by (5) is an element of $\mathcal{Q}$.

[8] A complex-valued function x defined on $I\!R^d$ is uniformly continuous if for each $\epsilon > 0$ there is a $\delta > 0$ for which $|x(\alpha_1) - x(\alpha_2)| < \epsilon$ whenever $\|\alpha_1 - \alpha_2\|_d < \delta$.

then

$$\lim_{\|\alpha\|_d \to \infty} (Fy)(\alpha) = \zeta.$$

Proof of Proposition 1:

Suppose that y is as indicated. We have

$$(Fy)(\alpha) = \int_{\mathbb{R}^d} f(\alpha-\beta)y(\beta)\, d\beta = \zeta \int_{\mathbb{R}^d} f(\beta)\, d\beta$$

$$+ \int_{\mathbb{R}^d} f(\alpha-\beta)[y(\beta)-\zeta]\, d\beta$$

for each α. Consider the last integral, and let any $\epsilon > 0$ be given. Choose a positive c_1 so that

$$\sup_{\|\alpha\|_d > c_1} |y(\alpha)-\zeta| \int_{\mathbb{R}^d} |f(\beta)|\, d\beta < \epsilon/2$$

and then, using the hypothesis that $f \in L_1(\mathbb{R}^d)$, select a $c_2 > 0$ for which

$$\sup_{\alpha \in \mathbb{R}^d} |y(\alpha)-\zeta| \int_{\|\beta\|_d \le c_1} |f(\alpha-\beta)|\, d\beta < \epsilon/2$$

for $\|\alpha\|_d > c_2$. Observe that for $\|\alpha\|_d > c_2$,

$$\left| \int_{\mathbb{R}^d} f(\alpha-\beta)[y(\beta)-\zeta]\, d\beta \right|$$

$$\le \int_{\|\beta\|_d > c_1} |f(\alpha-\beta)[y(\beta)-\zeta]|\, d\beta$$

$$+ \int_{\|\beta\|_d \le c_1} |f(\alpha-\beta)[y(\beta)-\zeta]|\, d\beta < \epsilon/2+\epsilon/2 = \epsilon,$$

which proves the proposition.

Proposition 2:

$$\lim_{\|\alpha\|_d \to \infty} \int_{\mathbb{R}^d} f(\alpha-\beta)y(\beta)\, d\beta = 0$$

for each $y \in BL_1(\mathbb{R}^d)$.

Proof of Proposition 2:

Both f and y belong to $L_2(\mathbb{R}^d)$. Using a version of the Parseval identity [16, p. 119],

$$(2\pi)^d \int_{\mathbb{R}^d} f(\alpha-\beta)y(\beta)\, d\beta$$

$$= \int_{\mathbb{R}^d} \exp\{j(\omega \cdot \alpha)\} \hat{f}(\omega)\hat{y}(\omega)\, d\omega, \quad \alpha \in \mathbb{R}^d \tag{6}$$

in which $\omega \cdot \alpha$ stands for the dot product of ω and α, $j = \sqrt{-1}$, and $\hat{f}$ and $\hat{y}$ denote the Fourier transforms of f and x, respectively. These Fourier transforms belong to $L_2(\mathbb{R}^d)$. By the Schwarz inequality, z given by $z(\omega) = \hat{f}(\omega)\hat{y}(\omega)$ for all ω belongs to $L_1(\mathbb{R}^d)$. Therefore, by the Riemann-Lebesgue lemma for functions in $L_1(\mathbb{R}^d)$ [16, p. 57], the right side of (6) approaches zero as $\|\alpha\|_d \to \infty$, proving the proposition.

Proposition 3: There exists a continuous linear shift-invariant map $E : \mathcal{C} \to \mathcal{C}$ such that $(Ey)(\alpha) = \zeta$ for all $\alpha \in \mathbb{R}^d$ and each $y \in \mathcal{C}$ with $\lim_{\|\alpha\|_d \to \infty} y(\alpha) = \zeta$, in which ζ is an arbitrary complex number.

For a proof of this proposition, see the proof of the theorem in [17].

Continuing with the proof of the lemma, observe that

$$|(Fy)(\alpha_1)-(Fy)(\alpha_2)| \le \|y\|_\infty \int_{\mathbb{R}^d} |f(\alpha_1-\beta)$$

$$-f(\alpha_2-\beta)|\, d\beta$$

$$= \|y\|_\infty \int_{\mathbb{R}^d} |f(\beta)-f(\alpha_2-\alpha_1+\beta)|\, d\beta$$

in which the last integral approaches zero as $\|\alpha_2-\alpha_1\|_d \to 0$. Thus, because $f \in L_1(\mathbb{R}^d)$, we see that F in fact maps into $\mathcal{C}$.

Set $H = EF$ with E as described in Proposition 3. We see that H is a linear shift-invariant continuous map of $L_\infty(\mathbb{R}^d)$ into $\mathcal{C}$, and therefore of $L_\infty(\mathbb{R}^d)$ into itself. By Propositions 2 and 3, the range of H restricted to $BL_1(\mathbb{R}^d)$ is the zero function, showing that part (a) of the lemma holds. Using Propositions 1 and 3, we see that part (b) also holds. This proves the Lemma.

It is of interest to note that we have proved a stronger result than is stated in the lemma, in that the range $H[L_\infty(\mathbb{R}^d)]$ of H can be taken to be contained in (the relatively simple space) $\mathcal{C}$.[9] There are several variations of Lemma 1. For example, for $d = 1$, one can show (using the result in [1]) that H can be taken to be causal.

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