# Wavelet Filter Design based on the Lifting Scheme and its Application in Lossless Image Compression 

TILO STRUTZ<br>Deutsche Telekom AG, Hochschule für Telekommunikation<br>Institute of Communications Engineering<br>Gustav-Freytag-Str. 43-45, 04277 Leipzig<br>GERMANY<br>tilo.strutz@hft-leipzig.de


#### Abstract

The description of filter banks using lifting structures does not only benefit low-complexity implementation in software or hardware, but is also advantageous for the design of filter banks because of the guaranteed perfect reconstruction property. This paper proposes a new design method for wavelet filter banks, which is explained based on a single lifting structure suitable for $9 / 7$ filter pairs. The filters are derived directly, the factorisation of known filters is not necessary. In addition, it is shown that the signal boundaries can be treated with little computational efforts. The modification of the standard design constraints leads to families of related filter pairs with varying characteristics. It includes a filter bank that can be implemented in integer arithmetic without divisions, shows better performance than the standard 9/7 filter bank for lossless image compression and competitive performance when applied in lossy compression.


Key-Words: Lifting scheme, Filter design, Wavelet transform, Image coding

## 1 Introduction

The standard 9/7 wavelet filters, initially constructed for conventional two-channel filter banks and applied straightaway in image compression [1], became, within a family of wavelets [2], the most successful wavelet filter pair in lossy compression for a broad range of natural images such as photographs. This success was manifested in the compression standard JPEG2000 [3].

Daubechies and Sweldens [4] have shown that each two-channel filter bank consisting of filters with finite impulse response (FIR) can be implemented with the so-called lifting scheme, first introduced in [5]. The filter coefficients of single lifting steps can be determined via the factorisation of the polyphase matrix of the filter bank.

The description of filter banks using lifting structures leads to a distinct reduction in computational complexity and is especially attractive for filter design, since it guarantees perfect reconstruction of the filter bank to be constructed. This fact has inspired researchers, for example, to approximate the lifting coefficients of the well-known 9/7 wavelet filter bank by rational values for low-complexity hardware and software implementation $[6,7,8]$ and to design similar 9/7 filter banks [9].

This contribution proposes a new design method directly yielding wavelet filter banks in lifting repre-
sentation. No factorisation of known filter banks is required.

The property of perfect reconstruction holds only true in the domain of real values. Lossless data compression, however, can only deal with values of limited precision, i.e. it must be possible to represent the sub-band values with a possibly small number of bits. That means, the filter bank must be able to map the integer signal samples to integer sub-band values. Luckily, the lifting principle is suitable for such integer-tointeger mapping. The corresponding signal decomposition is also known as integer wavelet transform (IWT) [10]. The IWT is also applicable for twodimensional signals in a non-separable fashion [11].

The new design approach is explained using a lifting structure suitable for the representation of a 9/7 wavelet filter bank. Relaxing the constraints of the original filters from [2], the design can be parameterised leading to families of related wavelet filters. The properties of these filters are analysed, their performance when applied to lossless image compression investigated and compared to the $5 / 3$ wavelet filters used in JPEG2000.

The outline is as follows. The next section repeats the foundations of the lifting principle and introduces the flow chart used for the new approach. Section 3 explains the filter design, first with conventional constraints, followed by different parameterisa-


Figure 1: Principle of lifting with alternating steps: a) analysis, b) synthesis.
tions resulting in two families of related filter banks. Section 4 discusses the compression results. A general interpretation of the derived filter banks is given in Section 5. The last Section summarises the article.

## 2 The lifting scheme

### 2.1 General structure

The conventional structure of two-channel filter banks based on lifting steps is shown in Figure 1a). The incoming signal ( $z$ domain) is split into two paths (polyphase transform) containing the values at even or odd sample positions. In order to devise the filter properties, alternating lifting and dual lifting steps are applied, in which samples from one path are filtered by $L_{i}(z)$ and added to a sample of the other path. The synthesis stage (Fig. 1b) applies the same lifting steps in reverse order but using subtraction instead of addition. Finally, the two paths are reassembled to a single signal. That is, the synthesis stage undoes each operation of the analysis stage and reconstructs the original signal $X(z)$.

### 2.2 Flow chart for 9/7 filter bank

With respect to the design method to be proposed, however, the illustration with a different flow chart is more helpful. Figure 2 depicts a lifting cascade suitable for representing a 9-tap low-pass and a 7 -tap high-pass filter pair. Essentially, it shows the signal flow for processing a signal with eight samples $x_{0}$ to $x_{7}$. A pair of samples at even positions is weighted by (typically negative) coefficients $\alpha$ and added to the sample in between. The next lifting step combines the results of the summations in pairs using the coefficients $\beta$. The third and fourth lifting step act in the same manner using the weights $\gamma$ and $\delta$.


Figure 2: Signal flow in signal decomposition with 9/7-tap filters (input signals with even length)

The property of integer-to-integer mapping, which will be essential for lossless compression, is simply imposed by properly rounding the intermediate values to integer values [10] (not shown in the flow diagram).

The arithmetic calculations are (with $m=$ $0,1,2, \ldots$ )

$$
\begin{align*}
d_{m}^{\prime} & =x_{2 m+1}+\left\lfloor\alpha \cdot\left(x_{2 m}+x_{2 m+2}\right)+0.5\right\rfloor \\
a_{m}^{\prime} & =x_{2 m}+\left\lfloor\beta \cdot\left(d_{m-1}^{\prime}+d_{m}^{\prime}\right)+0.5\right\rfloor \\
d_{m} & =d_{m}^{\prime}+\left\lfloor\gamma \cdot\left(a_{m}^{\prime}+a_{m+1}^{\prime}\right)+0.5\right\rfloor  \tag{1}\\
a_{m} & =a_{m}^{\prime}+\left\lfloor\delta \cdot\left(d_{m-1}+d_{m}\right)+0.5\right\rfloor
\end{align*}
$$

The result after all lifting steps is an interleaved sequence of low-pass filter output $a_{m}$ (approximation signal) and the high-pass filter output $d_{m}$ (detail signal). Figure 2 also shows the reconstruction of the original signal $x_{n}$ by performing the lifting steps in reverse order and using the opposite signs

$$
\begin{align*}
a_{m}^{\prime} & =a_{m}-\left\lfloor\delta \cdot\left(d_{m-1}+d_{m}\right)+0.5\right\rfloor \\
d_{m}^{\prime} & =d_{m}-\left\lfloor\gamma \cdot\left(a_{m}^{\prime}+a_{m+1}^{\prime}\right)+0.5\right\rfloor \\
x_{2 m} & =a_{m}^{\prime}-\left\lfloor\beta \cdot\left(d_{m-1}^{\prime}+d_{m}^{\prime}\right)+0.5\right\rfloor  \tag{2}\\
x_{2 m+1} & =d_{m}^{\prime}-\left\lfloor\alpha \cdot\left(x_{2 m}+x_{2 m+2}\right)+0.5\right\rfloor .
\end{align*}
$$

### 2.3 Handling of signal boundaries

The flow diagram in Figure 2 simply explains the exception handling at the signal boundaries. When using conventional filter banks, the input signal has to be


Figure 3: Signal flow in signal decomposition with 9/7-tap filters (input signals with odd length)


Figure 4: Signal flow in signal decomposition with 9/7-tap filters (input signals with odd length and alternative signal extension)
symmetrically extended at both boundaries by a number of samples depending on the length of the filter impulse responses. The structure in Figure 2 shows clearly that in the lifting representation only one single sample at both ends of the signal is required at the most. When applied to signals with an odd length, the handling has to be changed slightly (Fig. 3). Since the filter design is typically aiming at detail values close to zero $d_{m} \approx 0$, the required extensions at signal boundaries could be simply set to zero in the last lifting step (Fig. 4).

## 3 Filter design

### 3.1 Derivation of filter impulse responses

The flow diagram, as depicted in Figure 4, allows the derivation of the analysis filters of the corresponding two-channel filter bank simply by considering all paths from the input samples to a particular approximation sample $a_{m}$ or detail sample $d_{m}$, respectively. For the moment, we will disregard the rounding operations.

First, let us concentrate on the sub-band value $d_{1}$.

Following back the paths to the samples of the input signal $x[n]$, it becomes obvious that $d_{1}$ is dependent on the seven values $x_{0}$ to $x_{6}$. On the path from $x_{0}$ to $d_{1}$ we find the factors $\alpha, \beta$ and $\gamma$. Thus, the corresponding coefficient of the impulse response is the product of these three values $\alpha \cdot \beta \cdot \gamma$. The path from $x_{1}$ to $d_{1}$ contains only two factors $\beta$ and $\gamma$. The corresponding coefficient of the impulse response is accordingly $\beta \cdot \gamma$. All other coefficients can be derived in the same manner leading to the symmetric 7-tap impulse response of the analysis high-pass filter

$$
\left.\left.\begin{array}{rl}
h_{1}[n]= & \left\{\begin{array}{lll}
\alpha \beta \gamma & \beta \gamma & {[\gamma \cdot(2 \alpha \beta+1)+\alpha \cdot(1+\gamma \beta)]}
\end{array}\right. \\
(2 \beta \gamma+1) & {[\gamma \cdot(2 \alpha \beta+1)+\alpha \cdot(1+\gamma \beta)]}
\end{array} \beta \gamma \quad \alpha \beta \gamma\right\}\right\} \text { (3) }=\left\{\begin{array}{llll}
\alpha \beta \gamma & \beta \gamma & {[3 \alpha \beta \gamma+\alpha+\gamma]} & (2 \beta \gamma+1)
\end{array}\right\}
$$

Note that there are sometimes multiple paths. From $x_{3}$ to $d_{1}$, for instance, there is a direct one and two paths via $\beta$ and $\gamma$.

The approximation values $a_{m}$ are dependent on nine input samples each. $a_{2}$, for example, is directly derived from $x_{0}$ to $x_{8}$ and the analysis 9-tap low-pass filter reads as

$$
\begin{align*}
h_{0}[n] & =\{\alpha \beta \gamma \delta \quad \beta \gamma \delta \\
& \{\delta \cdot[\gamma \cdot(2 \alpha \beta+1)+\alpha \cdot(1+\gamma \beta)]+\alpha \beta \cdot(1+\gamma \delta)\} \\
& {[\delta \cdot(2 \beta \gamma+1)+\beta \cdot(1+\gamma \delta)] } \\
& \{\alpha \cdot[\delta \cdot(2 \beta \gamma+1)+\beta \cdot(1+\gamma \delta)]+(1+2 \gamma \delta)+ \\
& \alpha \cdot[\delta \cdot(2 \beta \gamma+1)+\beta \cdot(1+\gamma \delta)]\} \\
& {[\delta \cdot(2 \beta \gamma+1)+\beta \cdot(1+\gamma \delta)] } \\
& \{\delta \cdot[\gamma \cdot(2 \alpha \beta+1)+\alpha \cdot(1+\gamma \beta)]+\alpha \beta \cdot(1+\gamma \delta)\} \\
& \beta \gamma \delta \quad \alpha \beta \gamma \delta \quad\} \\
= & \{\alpha \beta \gamma \delta \quad \beta \gamma \delta \quad[4 \alpha \beta \gamma \delta+\alpha \beta+\alpha \delta+\gamma \delta] \\
& {[3 \beta \gamma \delta+\beta+\delta][6 \alpha \beta \gamma \delta+2 \cdot(\alpha \beta+\alpha \delta+\gamma \delta)+1] } \\
& {[3 \beta \gamma \delta+\beta+\delta][4 \alpha \beta \gamma \delta+\alpha \beta+\alpha \delta+\gamma \delta] } \\
& \beta \gamma \delta \quad \alpha \beta \gamma \delta \quad\} . \tag{4}
\end{align*}
$$

The synthesis filters are constructed by following all paths from a particular approximation (or detail) sample to the reconstructed signal values $x_{n}$. In this particular lifting structure, it turns out that they are directly related to the analysis filters by

$$
\begin{align*}
g_{0}[n] & =(-1)^{n+1} \cdot h_{1}[n] \quad n=0,1,2, \ldots  \tag{5}\\
g_{1}[n] & =(-1)^{n} \cdot h_{0}[n] \tag{6}
\end{align*}
$$

### 3.2 Filters with maximum number of vanishing moments

The frequency response (in $z$-domain) of a $t$-tap filter $h[n]$ is

$$
\begin{equation*}
H(z)=\sum_{n=0}^{t-1} h[n] \cdot z^{n} \tag{7}
\end{equation*}
$$



Figure 5: Spectra of three typical photographs (semilogarithmic chart)

Since $H_{0}(z)=\mathcal{Z}\left\{h_{0}[n]\right\}$ should be a real low-pass filter, its magnitude response at sampling frequency must be equal to zero $\left.H_{0}(z)\right|_{z=-1}=0$. This leads to the following condition

$$
\begin{align*}
0= & \alpha \beta \gamma \delta-\beta \gamma \delta+[4 \alpha \beta \gamma \delta+\alpha \beta+\alpha \delta+\gamma \delta] \\
& -[3 \beta \gamma \delta+\beta+\delta] \\
& +[6 \alpha \beta \gamma \delta+2 \cdot(\alpha \beta+\alpha \delta+\gamma \delta)+1] \\
& -[3 \beta \gamma \delta+\beta+\delta]+[4 \alpha \beta \gamma \delta+\alpha \beta+\alpha \delta+\gamma \delta] \\
& -\beta \gamma \delta+\alpha \beta \gamma \delta \tag{8}
\end{align*}
$$

$H_{1}(z)=\mathcal{Z}\left\{h_{1}[n]\right\}$ is the corresponding highpass filter. Its magnitude response must be equal to zero at frequency $f=0$, i.e. $\left.H_{1}(z)\right|_{z=1}=0$. This leads to the condition

$$
\begin{align*}
0= & \alpha \beta \gamma+\beta \gamma+[3 \alpha \beta \gamma+\alpha+\gamma]+(2 \beta \gamma+1) \\
& +[3 \alpha \beta \gamma+\alpha+\gamma]+\beta \gamma+\alpha \beta \gamma . \tag{9}
\end{align*}
$$

The original aim of filter design in [2] was to create low-pass filters with frequency responses that are as flat as possible at sampling frequency by imposing a maximum number of so-called vanishing moments, i.e. multiple zeros at $\left.H_{0}(z)\right|_{z=-1}$ and $\left.H_{1}(z)\right|_{z=1}$. The positive effect of the latter one is simply based on the frequency content of typical photographs (Fig. 5).
Figure 6 shows the photographs, which were taken from [16]. The main image information is covered by very low spatial frequencies. If the magnitude response of the high-pass filter is maximally flat around $f=0$, then it propagates minimal signal energy into the high-pass band.

Multiple vanishing moments can be incorporated by substituting $z$ with $\sqrt[n]{n^{p}}(p=0,1,2, \ldots)$ in equation (7). The second zero for $H_{1}(z)$ (and accordingly for $G_{0}(z)$ at $z=-1$ ), for example, is included using $z=\sqrt[n]{n}$ leading to the condition

$$
\begin{align*}
0= & 0 \cdot \alpha \beta \gamma+1 \cdot \beta \gamma+2 \cdot[3 \alpha \beta \gamma+\alpha+\gamma]+3 \cdot(2 \beta \gamma+1) \\
& +4 \cdot[3 \alpha \beta \gamma+\alpha+\gamma]+5 \cdot \beta \gamma+6 \cdot \alpha \beta \gamma . \tag{10}
\end{align*}
$$



Figure 6: Test images; a) 'kodim07'; b) 'kodim08'; c) 'kodim09', $768 \times 512$ pixels, only green component each

A different interpretation of this approach is based on the approximation of signal segments by polynomials of increasing order [13]. The condition for the second zero for $H_{0}(z)$ and $G_{1}(z)$ reads as

$$
\begin{align*}
0= & 0 \cdot \alpha \beta \gamma \delta-1 \cdot \beta \gamma \delta+2 \cdot[4 \alpha \beta \gamma \delta+\alpha \beta+\alpha \delta+\gamma \delta] \\
& -3 \cdot[3 \beta \gamma \delta+\beta+\delta] \\
& +4 \cdot[6 \alpha \beta \gamma \delta+2 \cdot(\alpha \beta+\alpha \delta+\gamma \delta)+1] \\
& -5 \cdot[3 \beta \gamma \delta+\beta+\delta]+6 \cdot[4 \alpha \beta \gamma \delta+\alpha \beta+\alpha \delta+\gamma \delta] \\
& -7 \cdot \beta \gamma \delta+8 \cdot \alpha \beta \gamma \delta . \tag{11}
\end{align*}
$$

The conditions (10) and (11) are, however, not independent from (9) and (8). Two more constraints are necessary for the unique determination of the four weights $\alpha \ldots \delta$.

Choosing $z=\sqrt[n]{n^{2}}$ imposes another vanishing
moment. The corresponding conditions are

$$
\begin{align*}
0= & 0 \cdot \alpha \beta \gamma+1 \cdot \beta \gamma+4 \cdot[3 \alpha \beta \gamma+\alpha+\gamma]+9 \cdot(2 \beta \gamma+1) \\
& +16 \cdot[3 \alpha \beta \gamma+\alpha+\gamma]+25 \cdot \beta \gamma+36 \cdot \alpha \beta \gamma \tag{12}
\end{align*}
$$

and

$$
\begin{align*}
0= & 0 \cdot \alpha \beta \gamma \delta-1 \cdot \beta \gamma \delta+4 \cdot[4 \alpha \beta \gamma \delta+\alpha \beta+\alpha \delta+\gamma \delta] \\
& -9 \cdot[3 \beta \gamma \delta+\beta+\delta] \\
& +16 \cdot[6 \alpha \beta \gamma \delta+2 \cdot(\alpha \beta+\alpha \delta+\gamma \delta)+1] \\
& -25 \cdot[3 \beta \gamma \delta+\beta+\delta]+36 \cdot[4 \alpha \beta \gamma \delta+\alpha \beta+\alpha \delta+\gamma \delta] \\
& -49 \cdot \beta \gamma \delta+64 \cdot \alpha \beta \gamma \delta . \tag{13}
\end{align*}
$$

Equations (10) - (13) form a system of non-linear equations resulting in the irrational weights

$$
\begin{array}{ll}
\alpha \approx-1.58613434206 & \gamma \approx 0.88291107553 \\
\beta \approx-0.05298011857 & \delta \approx 0.44350685204 \tag{14}
\end{array}
$$

Due to the inherent structure of the filter bank, each of the conditions impose double zeros, i.e. each filter shows four vanishing moments in total. This is denoted by $(\tilde{N}, N)=(4,4)$ in the following, where $\tilde{N}$ is the number of multiple zeros in $H_{1}(z)$ and $N$ the number in $H_{0}(z)$.

The result is exactly the same as derived from the factorisation of a polyphase matrix presented in [2].

### 3.3 Remarks

The major difference in filter design for lossless compression compared to lossy applications is, apart from the integer-to-integer mapping, the possibility to concentrate on the analysis filters only. The goal of filter design for 2-channel filter banks is to propagate as little signal energy as possible into the high-pass channel (the detail signal) and to amplify the low frequencies as little as possible. In lossy compression systems, one has also to take into account the quantisation errors introduced by the encoder, because the synthesis filters amplify these errors decreasing the signal quality. Here, a balanced design considering both analysis and synthesis is essential.

In the example of the $9 / 7$ filters chosen for the explanation of the design method, however, this difference is not pivotal, since the spectral properties of analysis and synthesis filters are complementary owing to equations (5) and (6).

### 3.4 Parameterisation through relaxing constraints

Starting from the design example above, the design has been modified in different ways in order to find
sets of wavelet filters suitable for lossless image compression.

Table 1 contains several investigated cases with different sets of lifting weights $\alpha \ldots \delta$. The $\tilde{N}$ column denotes the number of vanishing moments in the analysis high-pass filter $H_{1}(z), N$ the number of vanishing moments in the analysis low-pass filter $H_{0}(z)$. The last three columns contain the bitrates for lossless compression of three different grey-scale images (see Fig. 6) using a JPEG2000-like compression system. Only the green component for each has been used for investigations.

### 3.4.1 Design of $\mathbf{5 / 3}$ wavelet filters

Setting the factors $\gamma$ and $\delta$ equal to zero shortens the flow chart of Figure 2 to only two lifting steps. The lengths of the impulse responses are reduced to 5 taps for the low-pass filter and 3 taps for the high pass, respectively

$$
\left.\begin{array}{rl}
h_{0}[n] & =\{\alpha \beta \quad \beta \quad(1+2 \alpha \beta) \beta \alpha \beta
\end{array}\right\}
$$

The required conditions are in $z$-domain

$$
\begin{align*}
\left.H_{0}(z)\right|_{z=-1} & =0 \\
& =\alpha \beta-\beta+(1+2 \alpha \beta)-\beta+\alpha \beta \\
\left.H_{1}(z)\right|_{z=1} & =0=\alpha+1+\alpha \tag{16}
\end{align*}
$$

leading to the unique solution of $\alpha=-1 / 2$ and $\beta=$ $1 / 4$. This is in accordance with the original solution in [5]. This filter bank is listed in Table 1 as case 0.

### 3.4.2 Variation of the $9 / 7$ wavelet filters

Removing the constraint (13) releases one pair of zeros in $G_{1}(z)$ and $H_{0}(z)$ leading to $(\tilde{N}, N)=(4,2)$ filter banks and a dependency of the weights on $\alpha$

$$
\begin{align*}
\beta & =-\frac{1}{4 \cdot(2 \alpha+1)^{2}} \quad \gamma=-\frac{4 \alpha+1+4 \alpha^{2}}{4 \alpha+1} \\
\delta & =\frac{\left(8 \alpha^{2}+6 \alpha+3\right) \cdot(4 \alpha+1)}{16 \cdot(2 \alpha+1) \cdot\left(4 \alpha+1+4 \alpha^{2}\right)} \tag{17}
\end{align*}
$$

The released pair of zeros can now be moved in order to modify the frequency responses (cases 1 to 10 in Table 1 , sorted by $\alpha$ ). The cases $1,3,5$, and 9 have also been found by a different design approach [9] and case 5 also by [7]. Figure 7a) visualises the moving zeros in $H_{0}(z)$. The complex pairs of zeros on the unit circle belong to cases $1 \ldots 5$ leading to a steeper slope of the low-pass filter (Fig. 7b). Case 6 is the original $9 / 7$ filter with $(\tilde{N}, N)=(4,4)$. For the sake of comparison, also the spectra of the $5 / 3$ filter bank are included (case 0).

Table 1: Parameterised design, resulting lifting weights, and results of lossless compression [bpp]

| Case | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ | $\tilde{N}, N \mid$ | kodim07 | kodim08 | kodim09 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | -1/2 | 1/4 | 0 | 0 | 2, 2 | 3.7774 | 5.5307 | 4.0270 |
| 1 | -1 | -1/4 | 1/3 | 15/16 | 4, 2 | 3.8688 | 5.6349 | 4.1060 |
| 2 | $-\frac{\sqrt{2}+3}{4}$ | $2 \cdot \sqrt{2}-3$ | $\frac{2+\sqrt{2}}{8}$ | $\frac{6 \cdot \sqrt{2}-7}{2}$ | 4, 2 | 3.8486 | 5.6065 | 4.0905 |
| 3 | -5/4 | -1/9 | 9/16 | 16/27 | 4, 2 | 3.8660 | 5.5866 | 4.0879 |
| 4 | -4/3 | -9/100 | 25/39 | $\frac{1079}{2000}$ | 4, 2 | 3.8657 | 5.5836 | 4.0917 |
| 5 | -3/2 | -1/16 | 4/5 | 15/32 | 4, 2 | 3.8467 | 5.5762 | 4.0908 |
| 6 | -1.5861.. | -0.05298.. | 0.8829.. | 0.4435. | 4, 4 | 3.8430 | 5.5729 | 4.0945 |
| 7 | -8/5 | -25/484 | 121/135 | $\frac{9369}{21296}$ | 4, 2 | 3.8394 | 5.5731 | 4.0935 |
| 8 | $\frac{-1}{\sqrt{2}}-1$ | $\frac{1}{\sqrt{2}}-\frac{3}{4}$ | 1 | $\frac{1}{2 \sqrt{2}}+\frac{1}{16}$ | 4, 2 | 3.8279 | 5.5713 | 4.0852 |
| 9 | -7/4 | -1/25 | 25/24 | 51/125 | 4, 2 | 3.8525 | 5.5738 | 4.0966 |
| 10 | -2 | -1/36 | 9/7 | 161/432 | 4, 2 | 3.8549 | 5.5705 | 4.1101 |
| 11 | -17/32 | $\frac{164-20 \cdot \sqrt{4249}}{1089}$ | $\frac{2137-5 \cdot \sqrt{4249}}{65536}$ | $\frac{79825+5405 \cdot \sqrt{4249}}{287496}$ | 2, 4 | 3.9343 | 5.5804 | 4.1399 |
| 12 | -3/4 | $\frac{3-2 \cdot \sqrt{21}}{25}$ | $\frac{11-\sqrt{21}}{32}$ | $\frac{32+12 \cdot \sqrt{21}}{125}$ | 2, 4 | 3.8602 | 5.5718 | 4.0707 |
| 13 | -1 | $\frac{7-\sqrt{265}}{72}$ | $\frac{29-\sqrt[3]{265}}{32}$ | $\frac{205+17 \cdot \sqrt{265}}{864}$ | 2, 4 | 3.8231 | 5.5614 | 4.0574 |
| 14 | $\frac{1+\sqrt{6}+\sqrt{15+10 \cdot \sqrt{6}}}{-8}$ | -0.08731.. | 0.5732.. | 1/2 | 2, 4 | 3.8368 | 5.5676 | 4.0775 |
| 15 | -3/2 | $\frac{9-\sqrt{273}}{128}$ | $\frac{23-\sqrt{273}}{8}$ | $\frac{217+15 \cdot \sqrt{273}}{1024}$ | 2, 4 | 3.8426 | 5.5755 | 4.0878 |
| 16 | -8/5 | $\frac{235}{3528}-\frac{128}{3 \cdot \sqrt{849409}} \frac{5808}{}$ | $\frac{13717}{4000}-\frac{8}{81 \cdot \sqrt{849409}} 4$ | $\frac{61679}{296352}+\frac{827 \cdot \sqrt{849409}}{3259872}$ | 2, 4 | 3.8387 | 5.5731 | 4.0945 |
| 17 | -2 | $\frac{11-\sqrt{3880}}{200}$ | $\frac{201-9 \cdot \sqrt{300}}{32}$ | $\frac{783+47 \cdot \sqrt{321}}{4000}$ | 2, 4 | 3.8804 | 5.5854 | 4.1189 |
| 18 | -11/4 | $\frac{21-2 \cdot \sqrt{237}}{507}$ | $\frac{477-27 \cdot \sqrt{237}}{32}$ | $\frac{1188+80 \cdot \sqrt{237}}{6591}$ | 2, 4 | 3.9602 | 5.6233 | 4.1953 |
| 19 | -1 | -1/8 | 2/5 | 35/64 | 2, 2 | 3.8273 | 5.5594 | 4.0579 |
| 20 | $-\sqrt{5} / 2$ | $(\sqrt{5}-3) / 8$ | 1/2 | 1/2 | 2, 2 | 3.8424 | 5.5685 | 4.0776 |
| 21 | -1 | $(2-\sqrt{7}) / 6$ | $(\sqrt{7}-1) / 4$ | 1/2 | 2, 2 | 3.8117 | 5.5537 | 4.0513 |
| 22 | -1 | -33/256 | 64/161 | 1/2 | $2, \approx 2$ | 3.8047 | 5.5539 | 4.0522 |
| 23 | -1 | -33/256 | 51/128 | 1/2 | $\approx 2, \approx 2$ | 3.8085 | 5.5549 | 4.0560 |
| 24 | -1 | -7/64 | 16/39 | 1/2 | $2, \approx 2$ | 3.8115 | 5.5537 | 4.0512 |
| 25 | -1 | -7/64 | 105/256 | 1/2 | $\approx 2, \approx 2$ | 3.8123 | 5.5541 | 4.0512 |

$$
\begin{align*}
r & =\sqrt{64 a^{4}-224 a^{3}+52 a^{2}+60 a-15} \\
\beta & =-\frac{4 a^{2}-a-1.5-0.5 \cdot r}{4 \cdot(1+2 a) \cdot\left(4 a^{2}-4 a+1\right)} \quad \gamma=\left(-\frac{a^{2}}{4}+\frac{7 a}{16}+\frac{r-7}{32}\right) \cdot(1+2 a) \\
\delta & =\frac{-a^{2}-\frac{a}{4}+\frac{r+1}{8}}{\left(-4 a^{2}+7 a+8+\frac{r-7}{2}\right) \cdot a^{3}-\left(-5 a^{2}+\frac{35 a}{4}+\frac{5 r-35}{8}\right) \cdot a^{2}+\left(-\frac{23 a^{2}}{4}+\frac{17 a}{16}+\frac{7 r-17}{32}\right)} \tag{18}
\end{align*}
$$



Figure 7: Cases $0 \ldots$. 10: a) movement of the zeros of the analysis low-pass filter dependent on $\alpha, \beta, \gamma$ and $\delta$; b) frequency responses of analysis filters.

Keeping constraint (13) and ignoring (12) instead releases a pair of zeros in $H_{1}(z)$ and $G_{0}(z)$ leading to $(\tilde{N}, N)=(2,4)$ filter banks. The dependency of $\beta$, $\gamma$, and $\delta$ on $\alpha$ is shown in equation (18). Owing to the complex relations between the weights, their dependency is expressed utilising the intermediate variable $r$.

Based on the variation of $\alpha$, different sets of lifting weights have been derived (Cases $11 \ldots 18$ ). The domain of definition of $\alpha$ is, however, much more restricted than for the $(4,2)$ filter banks due to the square root in $r$.

Figure 8 shows the spectral properties of some designed ( 2,4 )-filters. The complex pairs of zeros on the unit circle in Figure 8a) belong to cases 17 and 18. In contrast to the $(4,2)$ filter banks, now the analysis high-pass filter is more affected by the parameterisation than the low-pass filter, as can be seen in Figure $8 b)$.

Owing to the good compression results for case 13, which has one integer factor ( $\alpha=-1$ ), experiments have been carried out, whether the release of even more vanishing moments can positively in-


Figure 8: Cases $11 \ldots$ 18: a) movement of the zeros of the analysis high-pass filter dependent on $\alpha, \beta, \gamma$ and $\delta ; \mathrm{b}$ ) frequency responses of analysis filters.
fluence the compression results, while keeping the lengths of the impulse responses. Accordingly, both the constraint (12) and the condition (13) were removed. The analysis filters (cases 19 to 21) show only two vanishing moments each.

Finally, it was investigated, whether the irrational numbers can be approximated by rational numbers (cases 22 and 24) or even by fractions in which the denominator is a power of two (cases 23 and 25). The latter avoids any divisions in computation, since they can be substituted by bitshift operations. This simplifies, for instance, the implementation in hardware [14]. Figure 9 shows the spectra of selected cases in comparison to the original $(\tilde{N}, N)=(4,4)$ analysis filters. It can be seen that the magnitude responses are still similar to the original one.

The zero-pole-diagrams in Figure 10 reveal the non-existence of multiple zeros in case 25 . They are merely close to $z=-1$ and $z=1$, respectively.

## 4 Compression results

The performance of the designed filter banks has been tested using the green component of three typical images (see Fig. 6). Albeit the usage of only three samples may not allow a full statistical analysis, it nevertheless gives a very useful impression.

The different wavelet filter banks have been im-


Figure 9: Cases $19 \ldots 25$ : frequency responses of analysis filters in comparison to case 6.


Figure 10: Case 25: zeros of the analysis filters.
plemented in a JPEG2000-like compression system. The last three columns of Table 1 contain the bitrates resulting from the lossless compression of the three images.

Case 0 corresponds to the 5/3 filter bank, which is used in the JPEG2000 Standard for reversible compression. As can be seen, the bitrates are much lower compared to the result achieved using the $9 / 7$ filters with a maximum number of vanishing moments (case $6)$.

Releasing two vanishing moments in the analysis low-pass filter enables the design of filter banks with better performance. Case 8 yields the best results on
average for $(4,2)$ filter banks.
If the number of multiple zeros is reduced in the analysis high-pass filter instead, even lower bitrates can be obtained (case 13). Please note that in both cases at least one factor out of $\alpha, \beta, \gamma$, and $\delta$ is equal to one.

The compression performance can be further improved by relaxing the conditions for multiple zeros in both the high-pass and the low-pass analysis filters. Case 21 has been found the best for $(2,2)$ filter banks with 9 and 7 taps. Interestingly, the approximation of the irrational numbers $\beta$ and $\gamma$ of case 21 by rational numbers does not influence the performance very much (cases 22-25).

## 5 Discussion

Using the flow structure for $9 / 7$ wavelet filter banks and setting either $\alpha=0$ or $\delta=0$ would lead to a lifting scheme with three lifting steps and analysis filters having impulse responses with 7 (low pass) and 5 taps (high pass) or 5/7 filters, respectively. As the corresponding compression results are distinctly worse compared to the $9 / 7$ filters, they are not considered here.

The bitrates listed in Table 1 indicate the superiority of the simple $5 / 3$ filter bank. The rounding of intermediate values at each single lifting step, which must be introduced to enable the mapping of integer signal samples to integer sub-band values, affects the characteristic of the filters. They are no longer necessarily true low- or high-pass filters. The more lifting steps are involved, the higher is the degradation of the filter properties. Figure 11 shows the deviation of the magnitude response (cases 0 and 6 ) depending on the value of a single impulse functioning as input signal. The smaller the input value the higher the effect of rounding.

The reasons for the superiority of the $5 / 3$ filters lie in the minimal accumulated influence of rounding at each lifting step leading to a minimal change in the filter characteristics (Fig. 11a). Not only the number of steps is lower (only two instead of four in all other cases), but also the rounding error in its first step is small, since the factor of $\alpha=-1 / 2$ only leads to errors when the sum of $x_{2 m}$ and $x_{2 m+2}$ is odd (see eq.(1), first equation). The degradation of the magnitude response of the $9 / 7$ filter, case 6 , is distinctly higher (Fig. 11b). This might lead to sub-band signals with less inherent correlation adversely affecting the signal decomposition cascade and the coding stage. More investigations are needed for the exploration of the exact reasons.

Comparing the compression results of cases 1 to


Figure 11: Change of magnitude response caused by rounding to integer values: a) case 0 ; b) case 6 . (solid: transfer function without rounding; dash, dashdot, and dot indicate signal values of 63,31 , and 15 used for determination of the impulse responses with rounding)

18 , it is interesting to see that case 13 is the best $9 / 7$ filter bank for all three test images. One reason could be the absence of rounding errors in the first lifting step ( $\alpha=-1$ ), but this also holds true for case 1 having a distinctly worse performance. Figure 12 shows the affected filter characteristics. It is likely that the relatively high gain of the low-pass filter of case 1 leads to an unfavourable propagation of signal energy from one decomposition level to the next.

Owing to the non-linear rounding effects, the filter spectra are changing when applied to varying signals and do not show multiple zeros in many cases. That is why the vanishing moments are less important in the filter design as long as the general spectral characteristics are maintained.

The original aim was to design wavelet filter banks for lossless compression. However, it also may be of interest how the filter bank according to case 25 performs in lossy image compression, since its implementation in hardware does not need any division. The results are depicted in Figure 13. In comparison to the $9 / 7$ filter bank used in JPEG2000 (case 6), there is a drop in compression for the image 'kodim07', no difference for 'kodim09', and even a slight improve-


Figure 12: Change of magnitude response caused by rounding to integer values: a) case $1 ; b$ ) case 13 .

a)


Figure 13: Comparison of lossy compression using filter banks cases 0,6 , and 25 ; a) images kodim07 and kodim09, b) image kodim08
ment for 'kodim08'. Therefore, case 25 seems to be competitive to the standard $9 / 7$ wavelet filter bank to a certain extent.

## 6 Summary

A new lifting-based method for filter design has been proposed which has the advantage of determining coefficients directly instead of factorising the known filter banks, as introduced in [4] and extended for Mchannel filter banks in [15]. The method was demonstrated based on $9 / 7$ wavelet filters, but is applicable when designing arbitrary lifting-based filter banks. Even keeping the structure of Figure 2 and exchanging the constraints, impulse responses with different lengths can be derived, as, for example, in the $5 / 3$ wavelet filter bank.

The application of the newly designed filters to lossless image compression did not lead to an improvement compared to the $5 / 3$ wavelet filter used in JPEG2000 due to the disadvantageous non-linear effects of rounding. It has been demonstrated, however, that the relaxation of design constraints can lead to an increased compression performance. Multiple zeros, also known as vanishing moments, play a less important role compared to applications in lossy compression, since the rounding distorts the spectral characteristic in any case.

The derived filters, however, may also be applied in lossy compression systems, where rounding to integers is not required. As a side result of the investigations made, a filter bank has been found that can be implemented in integer arithmetic without divisions and still shows competitive compression performance compared to the $9 / 7$ filter bank having a maximum number of vanishing moments.

## References:

[1] Antonini, M.; Barlaud, M.; Mathieu, P.; Daubechies, I.: Image Coding Using Wavelet Transform. IEEE Trans. on Image Proc., Vol.1, No.2, April 1992, 205-220
[2] Cohen, A.; Daubechies, I.; Feauveau, J.-C.: Biorthogonal Bases of Compactly Supported Wavelets. Comm. on Pure and Applied Mathematics, Vol.45, 1992, 485-560
[3] ISO/IEC JTC1/SC29/WG11 N1890, Information technology - JPEG 2000 Image Coding System. JPEG 2000 Part I, Final Draft Intern. Standard 15444, 25 Sep. 2000
[4] Daubechies, I.; Sweldens, W.: Factoring Wavelet Transform into Lifting Steps. J. Fourier Anal. Appl., Vol.4, No.3, 1998, 247-269
[5] Sweldens, W.: The Lifting Scheme: A New Philosophy in Biorthogonal Wavelet Construction. Proc. of SPIE, Vol.2569, San Diego, USA, July 1995, 68-79
[6] Barua, S.; Kotteri, K.A.; Bell, A.E.; Carletta, J.E.: Optimal quantized lifting coefficients for the 9/7 wavelet. ICASSP '04, Vol.5, 17-21 May 2004, 193196
[7] Guangjun, Z.; Lizhi, C.; Huowang, C.: A simple 9/7tap wavelet filter based on lifting scheme. IEEE Int. Conf. on Image Processing, Vol.2, 2001, 249-252.
[8] Tay, D.B.H.: Rationalizing the coefficients of popular biorthogonal wavelet filters. IEEE Trans. Circ. and Sys. for Video Techn., Vol.10, No.6, Sep. 2000, 9981005
[9] Tay, D.B.H.: A class of lifting based integer wavelet transform. IEEE Int. Conf. on Image Processing, Vol.1, 07-10 Oct. 2001, Thessaloniki, Greece, 602605
[10] Calderbank, A.R.; Daubechies, I.; Sweldens, W.; Yeo, B.L.: Wavelet transform that maps integers to integers. Applied Computational and Harmonic Analysis, Vol.5, No.3, 1998, 332-369
[11] Uytterhoeven, G.; Roose, D.; Bultheel, A.: Integer wavelet transforms using the lifting scheme. Proc. of the 3rd World Multiconference on Circuits, Systems, Communications and Computers, Athens, Greece, July 4-9, 1999, 6251-6253
[12] ISO/IEC 14495-1, Information technology - Lossless and near-lossless compression of continuoustone still images: Baseline (JPEG-LS). International Standard, corrected and reprinted version, 15 September 2000
[13] Strutz, T.; Müller, E.: Wavelet filter design for image compression. IEEE-SP Int. Symposium on TimeFrequency and Time-Scale Analysis, Paris, 18-21 June 1996, 273-276
[14] Sung, Tze-Yun: Memory-efficient and highperformance parallel-pipelined architectures for $5 / 3$ forward and inverse discrete wavelet transform. Proc. of the 7th WSEAS Int. Conf. on Multimedia Systems \& Signal Processing, 2007, 1-6
[15] Vijayakumar, A.; Abhilash, G.: Jordan Representation of Perfect Reconstruction Filter Banks using Nilpotent Matrices. Proc. of the 5th WSEAS Int. Conf. on Signal Processing, Istanbul, Turkey, May 27-29, 2006, 704-709
[16] http://r0k.us/graphics/kodak/, Oct. 2008

