Bispectral resolution and leakage effect of the indirect bispectrum estimate for different types of 2D window functions

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Abstract: - An important practical problem in the area of the higher-order statistical signal processing is to estimate the cumulants and polyspectra of the analyzed signal when a finite sequence of time samples is available. We cannot use the theoretical formula because they are based on the assumption that an infinite sequence of time samples is available, but this is not true in practice. In order to obtain a better estimate for bispectrum of the signal, different types of 2D window functions are used. Also, these windows are investigated in terms of the resolution and leakage effect of the indirect bispectrum estimate.

Key-Words: - higher-order statistics, bispectrum estimation, 2D window functions, bispectral resolution

1 Introduction

It is known that ***high order statistics*** is a new useful tool in the signal processing area. The ***cumulants*** and ***polispectra*** have some interesting properties that allow in certain situations to obtain more information about the analyzed signals, [3], [4], [5], [9]. These properties also make them very powerful in many other applications. If the second order statistics (autocorrelation function and power spectrum) are not capable to resolve a particular problem (e.g., detection of quadratic phase coupling), then sometime it is possible to apply successfully high order statistics, [1].

An important problem in practice is to estimate the cumulants and polyspectra of the analyzed signal using a finite number of runs (realisations) and respectively, a finite number of samples for each run. The definitions of high order statistics for random signals are based on the assumption that an infinite sequence of time samples is available. Whereby this approach is not true in practice, a direct theoretical formula cannot be used for computing the values of high order statistics.

According to this idea, the indirect calculus procedure is one of the most used conventional methods for bispectrum estimation. The estimated cumulant is multiplied by a suitable 2D window function in order to obtain an improved estimation for the bispectrum of the signal. Using this approach, the variance of the estimate decreases and the estimate will become consistent.

The 2D window functions should satisfy some important conditions, [4], [5], [6]. Thus, they must have the same symmetry properties like bicorrelation. A lot of 2D window functions are described in the high order statistics literature, [5], [6], but most of these are derived from 1D standard windows, e.g., Daniell, Hamming, Parzen, Priestley or Sasaki (minimum bispectrum bias supremum) windows. Another important 2D window is ***mean-squared error*** (MSE) ***optimal*** window derived by Rao and Gabr theory, [5]. The advantages of these windows are analyzed in terms of variance and bias of bispectrum estimate and respectively, in terms of MSE between the true value and estimated bispectrum.

The aim of this paper is to study the influences of the different types of 2D window functions on the indirect bispectrum estimate. Consequently, this study is done using the bispectral resolution and leakage effect as points of view. Therefore, in the first part of the paper, the definitions and some properties of cumulants and polyspectra are presented. Also, the indirect method for bispectrum estimation is shown. In the second part of the paper, the 2D windows used in the indirect bispectral estimation are presented and their 2D Fourier transforms are analyzed. Thus, some sections through main lobe and side lobes and their interpretation are presented. In the last part of the paper, the experimental results that confirm the broached theoretical aspects from beginning are shown. Finally, some conclusions and future research directions in this action field are included.

2 Cumulants and polyspectra of a stationary random process

If $x(n)$, $n = 0, \pm 1, \pm 2, \mathbf{K}$ is a zero-mean stationary random process, then the *second-order cumulant* (*autocorelation*), *third-order cumulant* (*bicorelation*) and the *fourth-order cumulant* (*tricorelation*) are given by equations, [2], [5]:

$$c_{2,x}(k,l) = E\{x(n)x(n+k)\}, \quad (1)$$

$$c_{3,x}(k,l) = E\{x(n)x(n+k)x(n+l)\}, \quad (2)$$

$$\begin{aligned} &c_{4,x}(k,l,m) = \\ &= E\{x(n)x(n+k)x(n+l)x(n+m)\} - \\ &- c_{2,x}(k)c_{2,x}(l-m) - c_{2,x}(l)c_{2,x}(m-k) - \\ &- c_{2,x}(m)c_{2,x}(k-l), \end{aligned} \quad (3)$$

where $k = 0, \pm 1, \mathbf{K}$, $l = 0, \pm 1, \mathbf{K}$, $m = 0, \pm 1, \mathbf{K}$.

Assuming that the above cumulants are absolutely summable, then the *second-order polyspectrum* (*power spectrum*), *third-order polyspectrum* (*bispectrum*) and the *fourth-order polyspectrum* (*trispectrum*) for the same process $x(n)$, are given by equations, [3]:

$$S_{2,x}(w) = \sum_{k=-\infty}^{+\infty} c_{2,x}(k) \cdot e^{-jwk}, \quad (4)$$

$$S_{3,x}(w_1, w_2) = \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{\infty} c_{3,x}(k,l) \cdot e^{-j(w_1 k + w_2 l)}, \quad (5)$$

$$\begin{aligned} &S_{4,x}(w_1, w_2, w_3) = \\ &= \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{+\infty} c_{4,x}(k,l,m) \cdot e^{-j(w_1 k + w_2 l + w_3 m)}, \end{aligned} \quad (6)$$

where $|w_1| \le p$, $|w_2| \le p$, $|w_3| \le p$, $|w_1 + w_2| \le p$, $|w_1 + w_2 + w_3| \le p$.

The main property which differentiates higher-order cumulants from correlation is that cumulants and polyspectra of order greater than two are identically zero for gaussian signals. Accordingly, the signal-to-noise ratio will become higher in the cumulants or polyspectra space when higher-order statistics based methods are applied to non-gaussian signals processing and the signals are corrupted by gaussian noises, [3].

It is well known that the autocorrelation function and the power spectrum contain only the amplitude information about a process, and they are phase blind, whereas cumulants and polyspectra reveal amplitude and phase information. Therefore, it isn't possible to reconstruct the correct phase of a non-minimum phase system from its output data using some algorithms based on the second-order statistics, [3].

The higher-order statistics can be also used to detect and characterize the type of nonlinearity in the time series and to identify nonlinear systems, [3], [5].

3 The indirect method for bispectrum estimation

In practice, we cannot use the theoretical formula (5) to determine the real value of bispectrum because we don't know all the values of the signal. We know only a finite set of observation measurements. Therefore, the third-order polyspectrum must be estimated from available data.

According to special literature, two of the most conventional methods used for the higher-order statistics estimation are the *direct* and *indirect* methods. In fact, these are the natural extensions of the power spectrum estimators.

In the direct method, the estimate is known as the *higher-order periodogram* and it is based on the calculus of the discrete time Fourier transform of the observed data.

In the indirect method, the estimate is based on the computing of the multidimensional discrete time Fourier transform of the data estimated cumulant samples. We assume that N samples of a strictly stationary random process $x(n)$, $n = \overline{0, N-1}$ are known. In the third-order case, first of all, the bicorrelation $\hat{c}_{3,x}(k,l)$, $|k| \le L$, $|l| \le L$, $L \le N$ must be estimated. Note that L determines the region support of the estimated cumulant. At last, using the above cumulant samples, the indirect bispectrum estimate $\hat{S}_{3,x}(w_1, w_2)$ is given by equation, [5], [6]:

$$\hat{S}_{3,x}(w_1, w_2) = \sum_{k=-L}^{L} \sum_{l=-L}^{L} w(k,l) \cdot \hat{c}_{3,x}(k,l) \cdot e^{-j(w_1 k + w_2 l)}, \quad (7)$$

where $w(k,l)$ is a proper 2D window function.

As it can be seen in the above relation, in order to obtain a decreasing of the bispectrum variance the estimated bicorrelation is multiplied by an appropriate 2D window function. Another technique used to reduce this variance is to segment the data into many records. It was shown that this estimate is asymptotically unbiased and consistent, [4], [5].

In the equation (7), the unbiased $\hat{c}^{u}_{3,x}(k,l)$ or the asymptotically unbiased $\hat{c}^{a}_{3,x}(k,l)$ estimates of the third-order cumulant can be used:

$$\hat{c}^{u}_{3,x}(k,l) = \frac{1}{C}\sum_{n=N_1}^{N_2} x(n)x(n+k)x(n+l), \quad (8)$$

$$\hat{c}^{a}_{3,x}(k,l) = \frac{1}{N}\sum_{n=N_1}^{N_2} x(n)x(n+k)x(n+l), \quad (9)$$

where $N_1 = 1 + \max(0,-k,-l)$, $N_2 = 1 - \max(0,k,l)$ and $C = N - \max(0,k,l) - \max(0,-k,-l)$.

4 Two-dimensional window functions for bispectrum estimation

According to [4], [5] and [6], the 2D window functions used for bispectrum estimation must have the following important *properties*:

a) They should satisfy the symmetry properties of the third-order cumulants:

$$w(k,l) = w(l,k) = w(-k,l-k) = w(k-l,-l); \quad (10)$$

b) They should be zero outside the region of support of the estimated third-order cumulant:

$$w(k,l) = 0, \quad (11)$$

for $|k| > L$, $|l| > L$;

c) They should be equal to one at the origin (normalizing condition):

$$w(0,0) = 1; \quad (12)$$

d) They should have a real positive two-dimensional Fourier transforms:

$$W(w_1,w_2) \geq 0, \quad (13)$$

for $|w_1| \leq \pi$, $|w_2| \leq \pi$. Also, they should have a finite energy.

The 2D window functions satisfying the above properties can be derived from 1D windows as follows, [2], [4], [6]:

$$w(k,l) = f(k)\,f(l)\,f(l-k), \quad (14)$$

where the 1D window $f(k)$ satisfies the following properties:

$$f(k) = f(-k), \quad (15)$$

$$f(k) = 0 \text{ for } |k| > L, \quad (16)$$

$$f(0) = 1, \quad (17)$$

$$F(w) \geq 0 \text{ for } |w| \leq \pi. \quad (18)$$

The function $F(w)$ is the time discrete Fourier transform of the window $f(k)$.

Note that not all 1D windows used in classical power spectrum estimation satisfy the last condition (18) for all w, [2], [4], [6].

The standard 1D windows used to generate 2D windows for bispectrum estimation are:

a) Daniell window:

$$f(p) = \begin{cases} \dfrac{\sin(\pi p)}{\pi p}, & |p| \leq 1 \\ 0, & |p| > 1 \end{cases}; \quad (19)$$

b) Hamming window:

$$f(p) = \begin{cases} 0.54 + 0.46\cos(\pi p), & |p| \leq 1 \\ 0, & |p| > 1 \end{cases}; \quad (20)$$

c) Parzen window:

$$f(p) = \begin{cases} 1 - 6p^2 + 6|p|^3, & |p| \leq 0.5 \\ 2(1-|p|)^3, & 0.5 \leq |p| \leq 1 \\ 0, & |p| > 1 \end{cases}; \quad (21)$$

d) Priestley window:

$$f(p) = \begin{cases} \dfrac{3}{(\pi p)^2}\left[\dfrac{\sin(\pi p)}{\pi p} - \cos(\pi p)\right], & |p| \leq 1 \\ 0, & |p| > 1 \end{cases}; \quad (22)$$

e) Sasaki window:

$$f(p) = \begin{cases} \dfrac{1}{\pi}|\sin(\pi p)| + (1-|p|)\cos(\pi p), & |p| \leq 1 \\ 0, & |p| > 1 \end{cases}, \quad (23)$$

where $p = \dfrac{k}{L}$ and $k = -L, \ldots, 0, \ldots L$.

A particular 2D window function is the mean-squared error optimal window derived by Rao and Gabr. It is given by the equation:

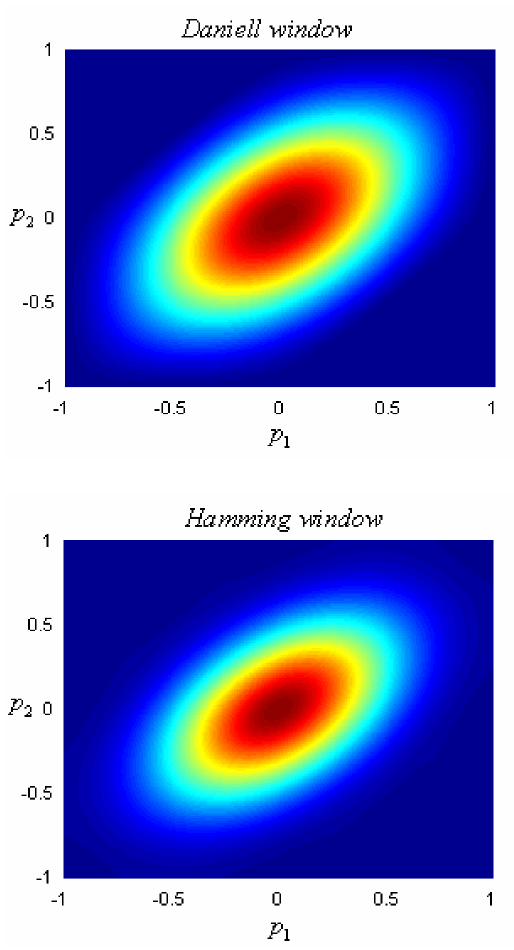
$$w(p_1,p_2)=\frac{8}{7p^3}\left[g(p_1,p_2)+g(-p_1,p_2-p_1)+ \right. \\ \left. +g(p_1-p_2,-p_2)\right], \quad (24)$$

where:

$$g(p_1,p_2)=\frac{2p_1^2+2p_2^2+p_1p_2}{p\,p_1^3p_2^3}\cdot\cos(p_2-p_1)p- \\ -\frac{p_2-p_1}{p_1^2p_2^2}\cdot\sin(p_2-p_1)p \quad ,(25)$$

and $p_1=\frac{k}{L}$, $p_2=\frac{l}{L}$, $k=-L,\mathbf{K},0,\mathbf{K}L$, $l=-L,\mathbf{K},0,\mathbf{K}L$.

The projections on the reference plane of the above mentioned 2D windows are presented in Fig.1.



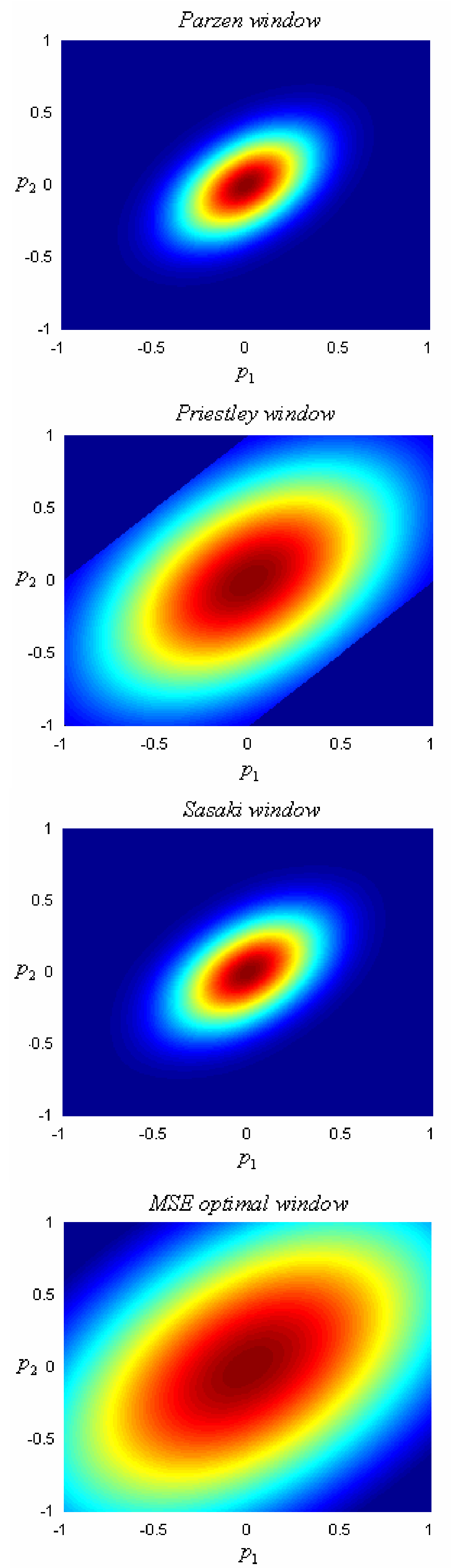


Fig.1: 2D windows for bispectrum estimation

In the special literature, these 2D windows are analyzed in terms of bispectrum bias supremum (J), bispectrum variance (V) and respectively, in terms of MSE between the true value and estimated bispectrum. It is demonstrated that the MSE is proportional to an index of efficiency $E = V \cdot B$, the variance of the estimator is approximately proportional to the index V and finally, the bispectrum bias supremum is proportional to the index J , where, [5]:

$$B = -\frac{1}{(2p)^2} \int_{-p}^{p} \int_{-p}^{p} w_1 w_2 W(w_1, w_2) dw_1 dw_2 , \quad (26)$$

$$V = \frac{1}{(2p)^2} \int_{-p}^{p} \int_{-p}^{p} |W(w_1, w_2)|^2 dw_1 dw_2 , \quad (27)$$

$$J = \frac{1}{(2p)^2} \int_{-p}^{p} \int_{-p}^{p} (w_1 - w_2)^2 W(w_1, w_2) dw_1 dw_2 . \quad (28)$$

In the Table 1 the values of these indexes obtained in [5] are presented.

As one can see from this table, the bispectrum estimator obtained by using of MSE optimal window assures the smallest MSE between the true value and the estimated bispectrum. Also, it has good values for the bias and the variance.

Table 1

Window	Index			
	J	V	B	E
Daniell	99468.5	0.1199	8990	1078.5
Parzen	8392.43	0.0409	1324.78	54.2
Hamming	60664.8	0.9067	6261.80	567.76
Priestley	288002	0.2032	10909.3	2216.91
Sasaki	1315.2	0.0486	2007.43	97.29
MSE optimal	2220.74	0.0691	458.69	31.68

Indexes J , V , B and E of the analyzed 2D window functions

In order to obtain the smallest variance of the estimator, Parzen window is indicated to be used. This window has also a good value of the MSE and a moderate value for the bias. Also, for the smallest bispectrum bias supremum, Sasaki window is recommended to be used. It has a very good value for the variance and a moderate value of the MSE.

A compromise between these three indexes is given by Daniell window. The Hamming window has the largest variance and the Priestley window has the largest bispectrum bias supremum.

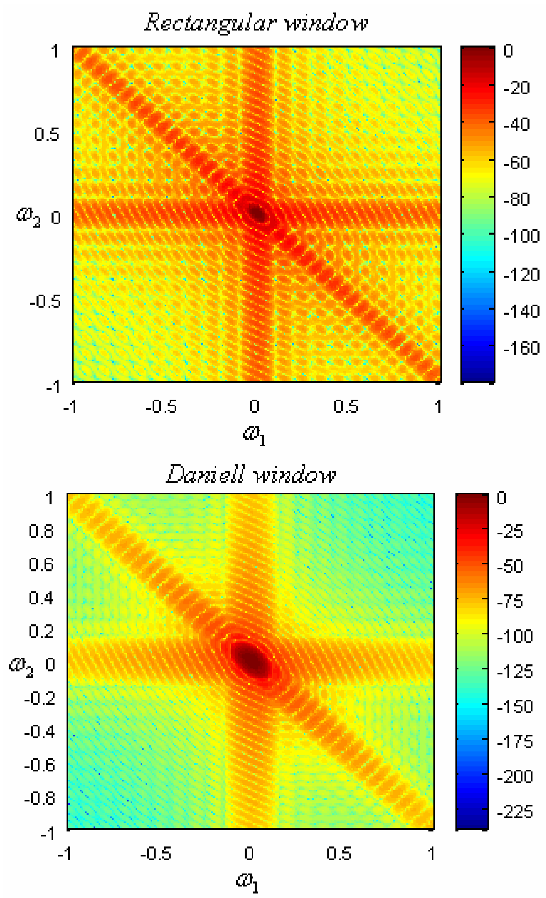
5 Bispectral resolution and leakage effect of 2D windows

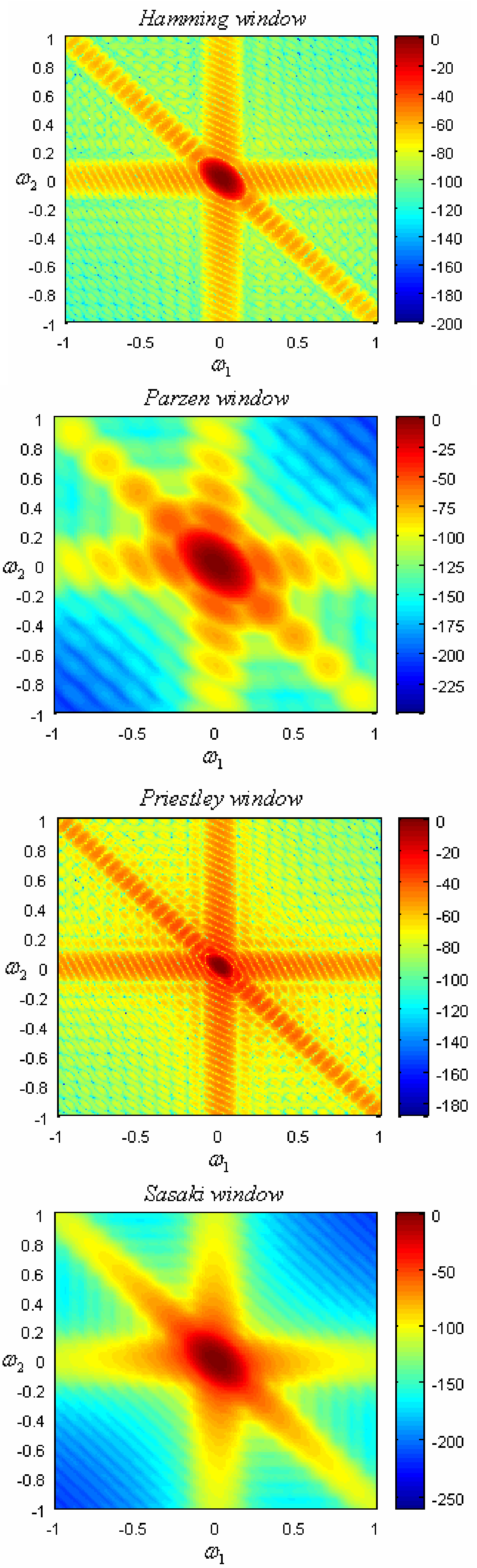
In some higher-order statistical signal processing applications (e.g., quadratic phase coupling detection problem, birange profile reconstruction), it is important to calculate the bispectral resolution and leakage effect of the indirect bispectrum estimate for different types of 2D window functions.

In order to see the bispectral resolution of the indirect estimate, it must take into account the 2D discrete-time Fourier transforms of 2D window functions and the cross sections through the main lobes of these at -3 dB level.

In the Fig.2 these transforms are presented. As one can see from this figures, they have a larger or a narrower main lobe and a bigger or a smaller sidelobe level. The shapes of the main lobes and of the sidelobes are also different.

In the Fig.3 the cross sections through the main lobes are indicated. Consequently, we will have very good information and a suggestive visual view on the bispectral resolution of the analyzed 2D window functions.





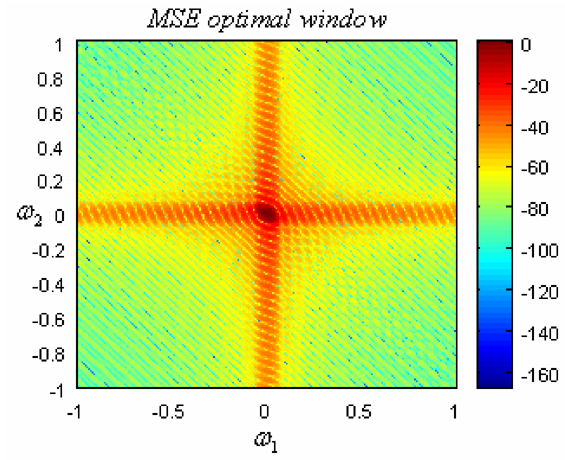


Fig.2: 2D discrete-time Fourier transforms of the analyzed window functions (in dB)

Similar with the case of 1D window functions used in the power spectrum estimation (Blackman-Tukey method, [8]), the bispectral resolution is better when the size L of the 2D window increases. The diagrams presented in the Fig.2 and Fig.3 are obtained for $L = 129$ samples. This value cannot be increasing very much because the variance of the estimator increases. Also, a large value of L means a large number of estimated third-order cumulant samples $\hat{c}_{3,x}(k,l)$, $|k| \le L$, $|l| \le L$. Accordingly, a longer computing time it is necessary to perform a larger number of operations. In the same time, small values of L reduce the bispectral resolution and increase the bias of the estimator.

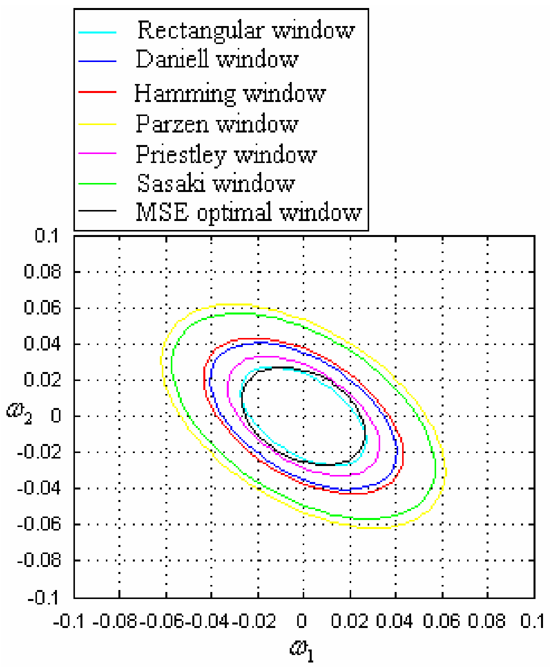


Fig.3: Cross section through the main lobes of 2D discrete-time Fourier transforms of the tested 2D window functions (at -3 dB level)

As it can be seen from Fig.3, the shape of the cross section through the main lobes is not circular, and practically, is one elliptic. Accordingly, the bispectral resolution depends a lot by the chosen direction into frequency plane.

The values of ***relative width*** assigned to the main lobes are important to study the reducing way of the bispectral resolution when another 2D window functions than the rectangular one are used. Consequently, the relative width at -3 dB level is the ratio between the area of the cross section through the main lobe of the investigated window and the area of the cross section through the main lobe for the (reference) rectangular window. The values of relative width are obtained analyzing Fig. 3 and are synthetically presented in Table 2.

As one can see from Table 2, the rectangular window has the smallest width of the main lobe and accordingly, an estimator using this type of window has the best bispectral resolution, [7]. Also, similar results are given by the MSE optimal window use. However, this last window has an additional advantage: the estimator has the smallest MSE.

The Priestley, Daniell and Hamming windows have good values of the relative width compared with the case of rectangular window.

Although the Sasaki and Parzen windows offer the largest main lobes, the Sasaki window has the best bias and respectively, the Parzen window has the best variance of the estimator.

Table 2

Window	**Relative widths of the main lobes**
Rectangular	1
Daniell	2.25
Hamming	2.57
Parzen	5.25
Priestley	1.49
Sasaki	4.50
MSE optimal	1.07

The relative widths assigned to the main lobes obtained in case of analyzed 2D window functions

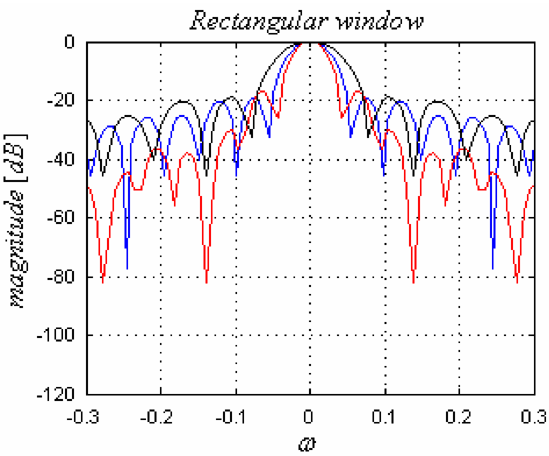
The level of the sidelobes is an important factor to study the effect known as ***spectral leakage***. The cross section acquired at $w_1 = 0$ or $w_2 = 0$ and respectively, the diagonal sections at $w_1 = w_2$ and $w_1 = -w_2$ through the 2D Fourier transforms are shown in Fig. 4. As it can be seen, these diagrams put in evidence a lower sidelobe level of the window functions compared with the case of rectangular window use. Accordingly, the bispectral leakage effect decreases when another 2D window functions than the rectangular one are used.

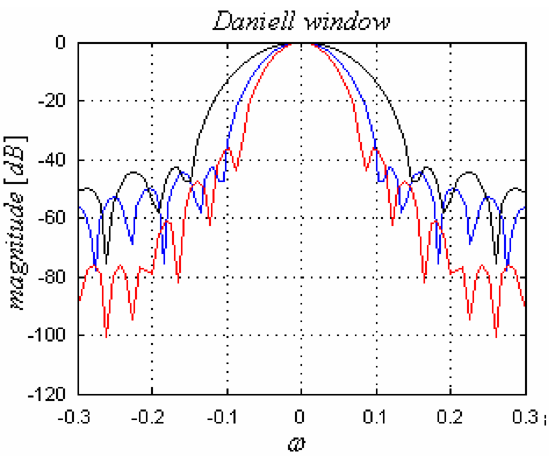
The values assigned to the sidelobe levels can be obtained from Fig. 4 and are synthetically presented in Table 3.

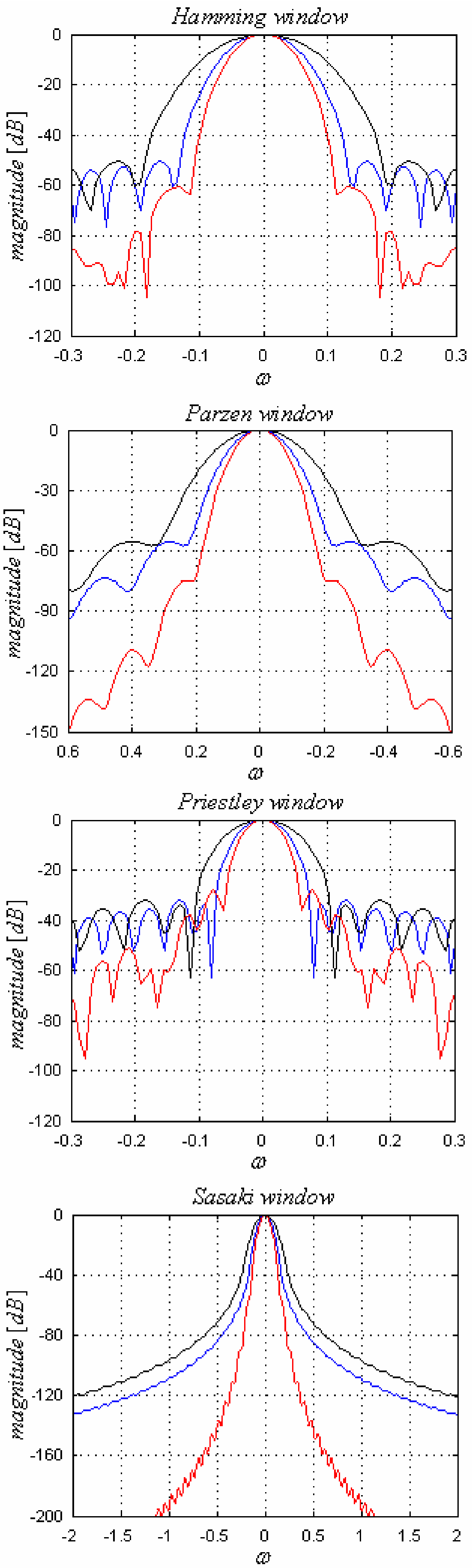
Table 3

Window	**Level of the sidelobes (dB)**		
	$w_1 = 0$ or $w_2 = 0$ sections	$w_1 = w_2$ section	$w_1 = -w_2$ section
Rectangular	-19.03	-16.77	-19.03
Daniell	-42.23	-35.87	-42.23
Hamming	-50.74	-60.72	-50.74
Parzen	-55.37	-74.59	-55.37
Priestley	-33.66	-27.88	-33.66
Sasaki	-107.56	-111.05	-107.56
MSE optimal	-21.38	-24.63	-38.47

Level of the sidelobes for the tested 2D window functions (cross and diagonal sections)







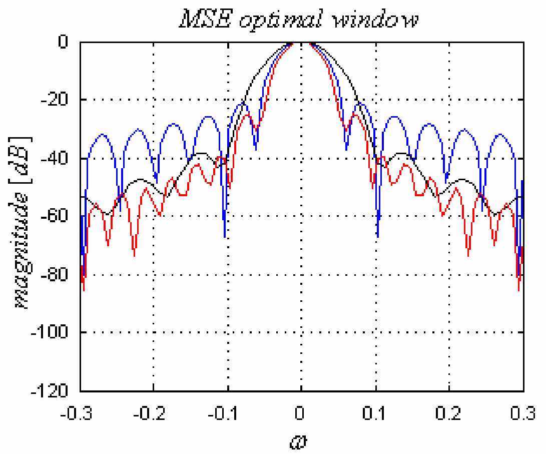


Fig.4: Cross (blue line $w_1 = 0$ or $w_2 = 0$) and diagonal (red line $w_1 = w_2$, black line $w_1 = -w_2$) sections through the Fourier transforms of the tested 2D windows

As one can see from Table 3, the rectangular and the MSE optimal windows have the biggest level of the sidelobes and therefore, they have the largest leakage effect.

In the same direction, a very good compromise between resolution and leakage effect in case of indirect bispectral estimator use are given by Priestley, Hamming, and Daniell windows.

More details regarding theoretical aspects treated in this section can be found in [7].

6 Experimental results

The *main objective* of the experimental part of this paper is to put in evidence the advantages (or disadvantages) resulting after applying of the different types of 2D window functions on the bispectral estimation. In order to implement this goal, it was used a concrete application of signal processing namely, the quadratic phase coupling detection problem, [4], [5].

Let the signal $s(n)$:

$$s(n) = \sum_{i=1}^{6} \cos(w_i n + j_i), \qquad (29)$$

where $w_2 > w_1 > 0$, $w_5 > w_4 > 0$, $w_3 = w_1 + w_2$, $w_6 = w_4 + w_5$, $j_1, j_2, \mathbf{K}, j_5$ are all independent random variables, uniformly distributed over $[-p, p]$ and $j_6 = j_4 + j_5$. The components of the signal $s(n)$ with the frequencies $w_4 = 2p\, f_4$ and $w_5 = 2p\, f_5$ are quadratically phase coupled, [4], [6].

According to [4], the bicorrelation $c_{3,s}(k,l)$ of $s(n)$ is the following:

$$c_{3,s}(k,l)=\frac{1}{4}\big[\cos(w_5k+w_4l)+\cos(w_6k-w_4l)+ \\ +\cos(w_4k+w_5l)+\cos(w_6k-w_5l)+ \\ +\cos(w_4k-w_6l)+\cos(w_5k-w_6l)\big]. \qquad (30)$$

The 2D Fourier transform of the first term from the right side of the above formula is:

$$\sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty}\cos(w_5k+w_4l)\cdot e^{-j(\tilde{w}_1k+\tilde{w}_2l)}= \\ =2p^2d(\tilde{w}_1-w_5)d(\tilde{w}_2-w_4)+ \\ +2p^2d(\tilde{w}_1+w_5)d(\tilde{w}_2+w_4). \qquad (31)$$

From relations (30) and (31), we have the bispectrum $S_{3,s}(\tilde{w}_4,\tilde{w}_5)$ of $s(n)$:

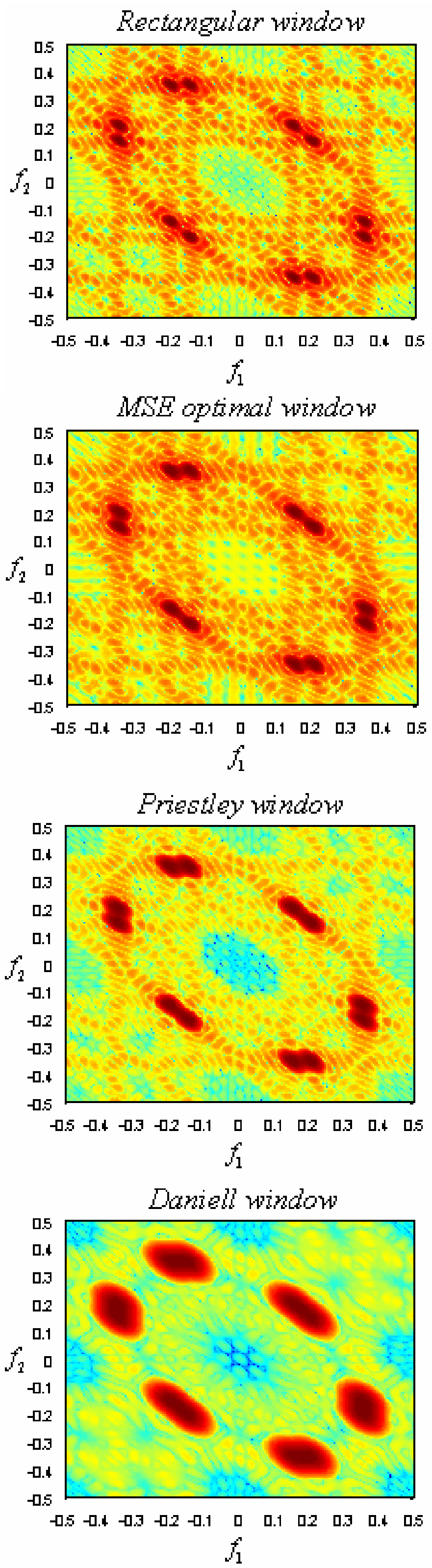
$$\frac{2}{p^2}S_{3,s}(\tilde{w}_1,\tilde{w}_2)=d(\tilde{w}_1-w_5)d(\tilde{w}_2-w_4)+ \\ +d(\tilde{w}_1+w_5)d(\tilde{w}_2+w_4)+d(\tilde{w}_1-w_6)d(\tilde{w}_2+w_4)+ \\ +d(\tilde{w}_1+w_6)d(\tilde{w}_2-w_4)+d(\tilde{w}_1-w_4)d(\tilde{w}_2-w_5)+ \\ +d(\tilde{w}_1+w_4)d(\tilde{w}_2+w_5)+d(\tilde{w}_1-w_6)(\tilde{w}_2+w_5)+ \\ +d(\tilde{w}_1+w_6)d(\tilde{w}_2-w_5)+d(\tilde{w}_1-w_4)d(\tilde{w}_2+w_6)+ \\ +d(\tilde{w}_1+w_4)d(\tilde{w}_2-w_6)+d(\tilde{w}_1-w_5)d(\tilde{w}_2+w_6)+ \\ +d(\tilde{w}_1+w_5)d(\tilde{w}_2-w_6). \qquad (32)$$

Consequently, the theoretical bispectrum of this signal is made by peaks only at frequencies of quadratically phase coupled components: (w_4,w_5), (w_5,w_4), $(w_6,-w_4)$, $(w_6,-w_5)$, $(w_5,-w_6)$, $(w_4,-w_6)$, $(-w_4,-w_5)$, $(-w_5,-w_4)$, $(-w_6,w_4)$, $(-w_6,w_5)$, $(-w_5,w_6)$ and $(-w_4,w_6)$.

It was generated a synthetic signal consisting of three cosinusoidal components with normalized frequencies: $f_1=1.5$, $f_2=2$ and $f_3=f_1+f_2=3.5$, and respectively, phases: j_1, j_2, $j_3=j_1+j_2$. The phases are independent random variables, uniformly distributed over $[-p,p]$.

In order to obtain the estimated bispectrum of this signal, 64 independent realizations were used, and each realization contained 128 samples. It was estimated 41 samples of the bicorrelation using the biased cumulant estimator.

The experimental results are shown in Fig. 5 and they confirm the theoretical aspects presented in previous chapters of the paper. Also, as one can see from this figure, if other types of windows different by rectangular window case are used in simulations, the bispectral resolution is weaker and the levels of the sidelobes are smaller one.



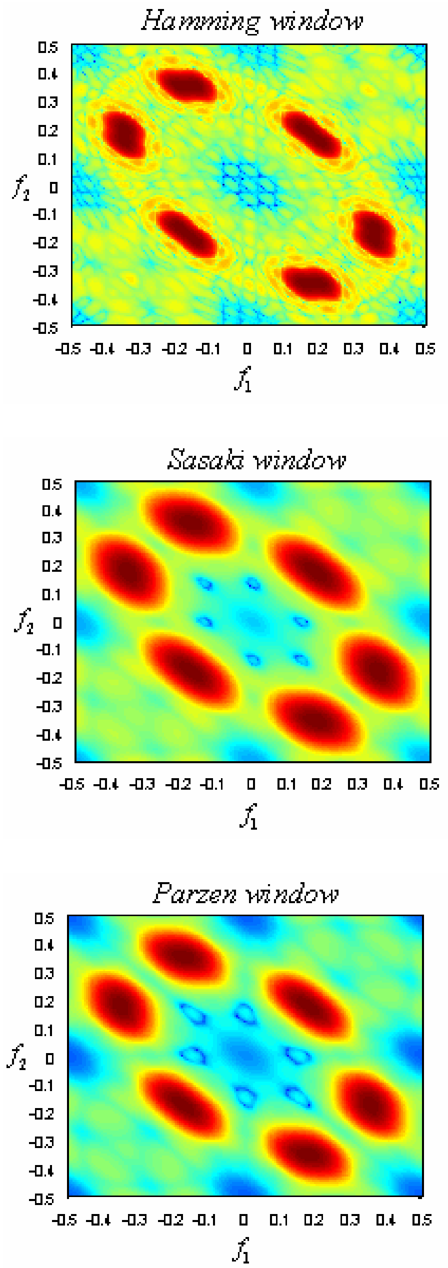


Fig.5: Estimated bispectrum of three quadratically phase coupled cosinusoidal components (in dB)

More details regarding experimental aspects treated in this section can be found in [7].

7 Conclusions and Future Work

The theoretical and experimental results presented in this paper leads to the following *remarks* concerning the influence of 2D window functions on the performances assigned to the estimated bispectrum:

- when a finite set of measured data is available, the problem of higher-order statistics estimation for a signal is a *sensible* problem. In order to obtain good results, it must be known all the variables which are involved in the estimation process;

- in order to obtain the best estimates, the suitable 2D window functions must be used in the bispectral estimation (using indirect method). Also, these windows have particular advantages and disadvantages, and none of them is perfect.

Consequently, when we choose a proper 2D window, it must take into account the bias, the variance and others quality measures of the estimator, the bispectral resolution and finally, the leakage effect. In this sense, the authors of the paper offer some important theoretical and experimental details that can lead to the right decision.

The main goal for a future work in this action field refers to the design and implementation of other *improved* 2D window functions. Also, it will be interesting to make a generalization of these 2D windows to the multidimensional case.

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