Numerical- analytical BEM for elliptic problems

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Abstract-This work is devoted to the development of the algorithms of solutions of boundary problems of mathematical physics based on the boundary elements method (BEM). The main advantages of the boundary elements method are decrease of dimension of a problem on unit, carrying discretization on the border of investigated area, and also obtaining a continuous decision in the interior of domain. As a result the quantity of calculations is reduced and accuracy of the decision rises. Distinctive features of the approach offered by the author are a use of analytical integration and ideology of parallel calculations at algorithm level.

Key-Words: - Boundary Elements Method, theory of elasticity, strain, stress, integral equation.

1 Introduction

The method of boundary elements had been started to practice in the sixties of the XX century. At that time the works of Kupradze [1], Mikhlin [2] and Rizzo [3] had been published. The fundamental BEM algorithms of the decision of various types of problems were generalized in books [4 – 6]. Recently other approach to the solution of the boundary integrated equations has expanded – the approach on the basis of a method of Galerkin [7]. The works [8-10] are devoted to analytical integration for the Galerkin approach to BEM. Thus only singular integrals can be calculated analytically whereas integrals that have not features are calculated numerically. In the works [11 – 12] formulas for analytical calculation of all integrals are received. Last tendency is perspective for increase a speed of calculations and accuracy of the decision of problems.

2 Modified of boundary element method (MBEM)

2.1 BEM for two-dimensional static problems of the theory of elasticity

Let's take a two-dimensional problem of elasticity for plane area Ω of undefined form. We will consider that the mass forces are absent. Taking into account this assumption, and also considering boundary conditions of the 1st and 2nd types, the expression for any internal point ξ of area Ω can be transformed to an integral:

$$u_i(\xi)=\int_{S_f}\left[u_{ij}^*(\xi,x)f_j^*(x)-f_{ij}^*(\xi,x)u_j(x)\right]dS(x)+\int_{S_u}\left[u_{ij}^*(\xi,x)f_j(x)-f_{ij}^*(\xi,x)u_j^*(x)\right]dS(x) \tag{1}$$

Here $x\in S$ – a boundary point of area Ω, ξ – an internal point of area influence function $u_{ij}^*(\xi,x)$

$$u_{ij}^*(\xi,x)=-\frac{1}{8\pi(1-\nu)\mu}\left[(3-4\nu)\ln(r)\delta_{ij}-r_{,i}r_{,j}\right]$$ and

function $f_{ij}^*(\xi,x)$ for a two-dimensional problem takes the following form:

$$f_{ij}^*(\xi,x)=\frac{-1}{8\alpha\pi(1-\nu)\mathbf{r}^{\alpha}}\times\left\{\left[(1-2\nu)\delta_{ij}+\beta\nabla_i\mathbf{r}\nabla_j\mathbf{r}\right]\frac{\partial r}{\partial n}-(1-2\nu)\left(\nabla_i\mathbf{r}n_j-\nabla_j\mathbf{r}n_i\right)\right\}. \tag{2}$$

From the equation (1) follows that the movement at any point ξ can be interpreted as the result of influence of displacements or surface stresses applied to the boundary.

Let the boundary S of the area Ω in some way is divided into boundary elements.

Various cases of approximation of unknown boundary values were considered. Linear approximation was appeared to be the most cost-effective without degradation of accuracy:

$$u_i=\sum_{k=1}^{2}u_i^{(k)}N_k(x),\quad f_i=\sum_{k=1}^{2}f_i^{(k)}N_k(x),$$

where $u_i^{(k)}$ and $f_i^{(k)}$ - nodal values of the components of the vectors of displacement and surface stress, $N_k(x)$ is a linear functions of the shape. With such interpolation for calculating the coefficients of the resolving system of linear equations and then for calculating the displacement within the area it is necessary to calculate the following integrals:

$$I_k\left(u_{ij}^*\right)=\int_{AB} u_{ij}^*(\xi,x)N_k(x)dS(x), \tag{3}$$

$$I_k\left(f_{ij}^*\right)=\int_{AB} f_{ij}^*(\xi,x)N_k(x)dS(x), \tag{4}$$

Under such conditions nodes P_1 and P_2 are inside the element symmetrically about its center (Fig.1). The distance l from the node to the end of the element is a parameter of the element. The investigated magnitudes are discontinuous at the junctions of the elements.

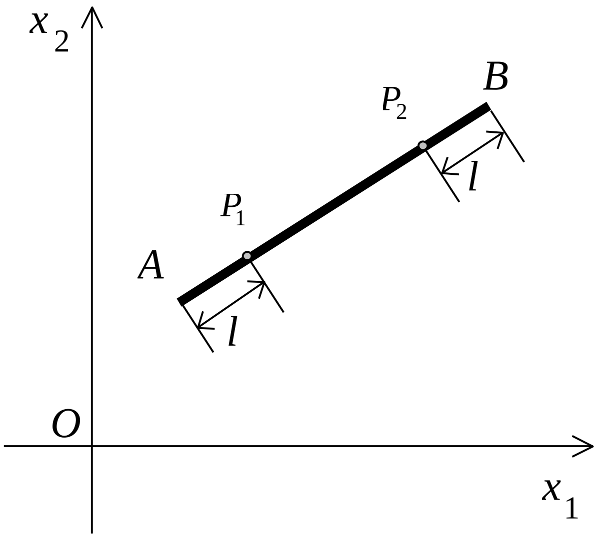

Fig.1. Linear discontinuous element

Because each node is located on the smooth side of the border, the resolving system of linear equations in this case will be the following:

$$\frac{1}{2}u_i^{(p_k)}=\sum_{q=1}^{N}\Big(f_j^{(q_1)*}I\left(u_{ij}^*N_1^{(q)}\right)+f_j^{(q_2)*}I\left(u_{ij}^*N_2^{(q)}\right)-$$

$$u_j^{(q_1)}I\left(f_{ij}^*N_1^{(q)}\right)-u_j^{(q_2)}I\left(f_{ij}^*N_2^{(q)}\right)+$$

$$\sum_{q=N+1}^{N+M}\Big(f_j^{(q_1)}I\left(u_{ij}^*N_1^{(q)}\right)+f_j^{(q_2)}I\left(u_{ij}^*N_2^{(q)}\right)-$$

$$u_j^{(q_1)*}I\left(f_{ij}^*N_1^{(q)}\right)-u_j^{(q_2)*}I\left(f_{ij}^*N_2^{(q)}\right)\Big),$$

$$i=1,2,\ k=1,2,\ p=1,\ldots,M+N, \tag{5}$$

here $\int_{S^{(q)}} u_{ij}^*\left(x^{(p_k)},x\right)N_1^{(q)}(x)dS(x)=I\left(u_{ij}^*N_1^{(q)}\right)\ldots$

$$\int_{S^{(q)}} f_{ij}^*\left(x^{(p_k)},x\right)N_1^{(q)}(x)dS(x)=I\left(f_{ij}^*N_1^{(q)}\right)..$$

Where $x^{(p_1)}$ and $x^{(p_2)}$ are nodal points of the element with index p, $N_1^{(p)}(x)$ and $N_2^{(p)}(x)$ are the corresponding shape function, $u_j^{(q_k)}$ and $f_j^{(q_k)}$ are nodal values of displacements and surface stress. Here and further the nodes on each element are numbered from 1 to 2 in the direction of traversal of the boundary of counter-clockwise.

After solving the system (5) $\left\{f_j^*(x),u_j^*(x)\right\}$ a displacement and a strain within the region can be calculated as follows:

$$u_i(\xi)=\int_{S_f}\left[u_{ij}^*(\xi,x)f_j^*(x)-f_{ij}^*(\xi,x)u_j^*(x)\right]dS(x)+$$

$$\int_{S_u}\left[u_{ij}^*(\xi,x)f_j^*(x)-f_{ij}^*(\xi,x)u_j^*(x)\right]dS(x) \tag{6}$$

here $x\in S$, $\xi\in\Omega$.

In the case of continuous interpolation the ends of the elements are taken as the nodes of interpolation. The total number of nodes in this case is twice less than in a discontinuous interpolation, while the investigated magnitudes are continuous along the border. However, the problem is compounded by the fact that nodes do not lie on a smooth surface but in the corners of the polygon formed by the boundary elements.

Thus, the calculation of integrals (3), (4) is an important part of the solution of two-dimensional elasticity problems with boundary element method. Usually, all these integrals, except those with features are computed numerically. Each influence point is fixed and for this point integration is performed over all elements of the border.

2.2 Modification of the method

The proposed approach selects the most appropriate "basic element" which allows an analytical calculation of the integrals necessary to obtain compact formulas. These formulas are the functions of the coordinates of the influence point. It is shown that for every boundary element and the corresponding influence point there is an equivalent block, which is appeared to be the "basic element", and the point for all the results of the integration will have the equal value relationships. Thus,

instead of making a bypassing of all the elements of the border with the numerical integration "around" the fixation of the influence point suggested to make a bypass of the influence points which determined by the linear transformation for a "basic element". Obtained elementary functions are valid for all elastic problems. In two-dimension case as a basic element a segment lying on a coordinate axis one end of which lies at the origin of the coordinates is chosen. To calculate the integrals on the selected ”basic elements” we obtain simple analytical formulas that are applicable to any problem for any mechanical properties of the material. Calculation of the coefficients of the system of equations, as well as determination of the values of variables at internal points of the field is carrying out by a linear substitution of various coordinates of points of influence. Substitution of numerical integration on an analytic integration yields a significant reduction in computation time. Naturally, the accuracy of the calculation of integrals increases, which leads to a more accurate solution of the original problem. As a result, solutions were obtained in the form of continuous analytic functions. Continuity of the solution by the modified method of boundary elements in the domain means that we can accurately determine the parameters of the stress-strain state at any interior point. In our case, obtaining a solution in the form of analytical formulas allow to calculate the physical and mechanical characteristics, which are defined by differentiation such as gradients of stress.
Coefficients of the system of equations (3) can be calculated independently, so the level of parallelization when filling the system matrix depends only on the number of processors. Formulas (5) (6) are valid for any interior point ξ of this area, and the computation of displacements, strains and stresses at one point does not depend on the calculation of these quantities at other points, therefore the level of parallelization for the calculation of the stress-strain state within this area also depends on the quantity of processors.
It is important to note that once resolved the problem of determining the boundary values we can change many times a grid of internal points and recalculate the necessary values on it. In particular, we can select only the "dangerous" zone inside the area and carry out the calculations only on it, which is important, for example, in the study of complex objects with structural stress concentrators.

2.3 Analytical integration of the functions of influence

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To obtain compact formulas of analytical integration we propose the following procedure. Calculated block of arbitrary rectilinear boundary elements and arbitrary points of influence associated with the block of "basic element" and the corresponding influence point, so that the integrals over them are linked. As a result, for the calculation of the integrals over an arbitrary interval for an arbitrary influence point it would be sufficient to choose a way of calculating coordinates of new influence point and calculating the integrals for coefficients matrix of system of equations according to formulas obtained before.
Let’s take a segment *AB* on the plane, where $A(A_1, A_2)$ and $B(B_1, B_2)$ are arbitrary points and $\xi(\xi_1,\xi_2)$ is the influence point (Fig. 2) when the bypass of the boundary is counter-clockwise,

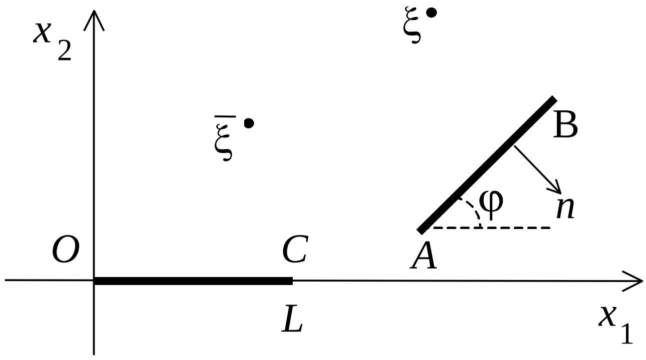

Fig. 2 Coordinate transformation.

The movement along the element *AB* goes from point *A* to point *B* i.e. outward normal is oriented the way as shown in Fig. 2. A displacement $u=(u_1, u_2)$ working on a segment *AB* and surface stress $f=(f_1,f_2)$ cause at the point ξ some displacement $u(\xi)=(u_1(\xi,)\ u_2(\xi,))$. Let’s make a coordinate transformation that remains the distance $L=\sqrt{(B_1-A_1)^2+(B_2-A_2)^2}$ and displays the point *A* to the origin *O(0,0)* and displays the point *B* to the point *C(L,0)* where the movement along the element *AB* goes from *A* point to point *B*, i.e. outward normal is oriented as shown in Fig. 2. A displacement $u(\xi)=(u_1(\xi,)\ u_2(\xi,))$ working on a segment *AB* and surface stress $f(\xi)=(f_1(\xi,)\ f_2(\xi,))$ cause at the point ξ some displacement $u(\xi)=(u_1(\xi,),u_2(\xi,))$
is the length of the segment *AB*. Such transformation is a combination of parallel translation and rotation by an angle φ (Fig. 2). A arbitrary point $x(x_1,x_2)$ on the plane mapped to the

point $\bar{x}(\bar{x}_1, \bar{x}_2)$ and its associated relations are the following:

$$x = Q\bar{x} + A,\ \bar{x} = Q^{-1}(x - A) \tag{7}$$

where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\bar{x} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}$, $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$,

Q is rotation matrix:

$$Q = \begin{pmatrix} q_{11}\ q_{12} \\ q_{21}\ q_{22} \end{pmatrix},\ Q^{-1} = Q^T, \tag{8}$$

$$q_{11} = q_{22} = \frac{B_1 - A_1}{L} = \cos\varphi,$$

$$q_{12} = -q_{21} = -\frac{B_2 - A_2}{L} = -\sin\varphi.$$

Here and further in all relationships one-dimensional arrays are treated as matrix-columns. A arbitrary vector $w = (w_1, w_2)$ in the plane is displayed in the vector $\bar{w} = (\bar{w}_1, \bar{w}_2)$:

$$w = Q\bar{w},\ \bar{w} = Q^{-1}w.$$

It is obvious that such transformation is a rigid displacement of the investigated system of the objects as a whole and does not change the elastic interaction. This means that if the surface displacements $\bar{u} = Q^{-1}u$ and the surface stress $\bar{f} = Q^{-1}f$ effect on the segment OC, they will cause a displacement $u(\bar{\xi}) = Q^{-1}u(\xi)$ at the point $\bar{\xi} = Q^{-1}(\xi - A)$. Using these relations we can establish the following relation between the integrals (3), (410) over the segment AB and the corresponding integrals over the segment OC for an influence point $\bar{\xi}$:

$$\begin{pmatrix} I(u_{11}^*) & I(u_{12}^*) \\ I(u_{12}^*) & I(u_{22}^*) \end{pmatrix} = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \begin{pmatrix} \bar{I}(u_{11}^*) & \bar{I}(u_{12}^*) \\ \bar{I}(u_{12}^*) & \bar{I}(u_{22}^*) \end{pmatrix} \begin{pmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{pmatrix},$$

$$\begin{pmatrix} I(f_{11}^*) & I(f_{12}^*) \\ I(f_{21}^*) & I(f_{22}^*) \end{pmatrix} = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \begin{pmatrix} \bar{I}(f_{11}^*) & \bar{I}(f_{12}^*) \\ \bar{I}(f_{21}^*) & \bar{I}(f_{22}^*) \end{pmatrix} \begin{pmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{pmatrix}$$

Here

$$\bar{I}(f_{ij}^*) = \int_{OC} f_{ij}^*(\bar{\xi}, x)\,dS(x);\ \bar{I}(u_{ij}^*) = \int_{OC} u_{ij}^*(\bar{\xi}, x)\,dS(x);$$

$$\bar{I}(u_{11}^*) = c_1\left((c_2 + 1)(\bar{\xi}_2 Q_3 - L) + \frac{c_2}{2}(LQ_1 + \bar{\xi}_1 Q_2)\right)$$

$$\bar{I}(u_{12}^*) = -\frac{1}{2}c_1 \bar{\xi}_2 Q_2,$$

$$\bar{I}(u_{22}^*) = c_1\left((c_2 - 1)\bar{\xi}_2 Q_3 + c_2\left(\frac{1}{2}(LQ_1 + \bar{\xi}_1 Q_2) - L\right)\right),$$

$$\bar{I}(f_{11}^*) = c_3((c_4 + 1)Q_3 + \bar{\xi}_2 d_1),$$

$$\bar{I}(f_{12}^*) = c_3\left(-\frac{1}{2}c_4 Q_2 - \bar{\xi}_2 d_2\right),$$

$$\bar{I}(f_{21}^*) = c_3\left(\frac{1}{2}c_4 Q_2 - \bar{\xi}_2 d_2\right),$$

$$\bar{I}(f_{22}^*) = c_3((c_4 + 1)Q_3 - \bar{\xi}_2 d_1),$$

$$c_1 = -\frac{1}{8\pi\mu(1-\nu)};\ c_2 = 3 - 4\nu;$$

$$c_3 = -\frac{1}{4\pi(1-\nu)};\ c_4 = 1 - 2\nu;$$

$$Q_1 = \ln\left((\bar{\xi}_1 - L)^2 + \bar{\xi}_2^2\right);$$

$$Q_2 = \ln\left(\bar{\xi}_1^2 + \bar{\xi}_2^2\right) - \ln\left((\bar{\xi}_1 - L)^2 + \bar{\xi}_2^2\right)$$

$$Q_3 = \operatorname{arctg}\left(\frac{\bar{\xi}_1}{\bar{\xi}_2}\right) - \operatorname{arctg}\left(\frac{\bar{\xi}_1 - L}{\bar{\xi}_2}\right)$$

$$d_1 = \frac{\bar{\xi}_1 - L}{(\bar{\xi}_1 - L)^2 + \bar{\xi}_2^2} - \frac{\bar{\xi}_1}{\bar{\xi}_1^2 + \bar{\xi}_2^2}$$

$$d_2 = \bar{\xi}_2\left(\frac{1}{\bar{\xi}_1^2 + \bar{\xi}_2^2} - \frac{1}{(\bar{\xi}_1 - L)^2 + \bar{\xi}_2^2}\right)$$

Thus, we found that for the calculation of the integrals over a arbitrary segment *AB* for the point of influence ξ it is sufficient to construct the matrix *Q* (8), define through a point $\bar{\xi}$ and calculate the integrals over the segment *OC* that is less complicated problem.

2.4 Deformation is calculated by the formulas

Strains $\{\varepsilon_{ij}\}$ and stresses $\{\sigma_{ij}\}$ are determined by known differential relations and Hooke's law

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})\quad \sigma_{ij} = 2\mu\varepsilon_{ij} + \frac{2\mu\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}.$$

Here $\mu;\nu$ - elastic modules

$$\varepsilon_{ij}(\xi)=\sum_{q=1}^{N+M}\left(f_k^{(q)}\left(\int_{S^{(q^-)}} w_{ijk}^*(\xi,x)N_2^{(q^-)}(x)dS(x)+\int_{S^{(q^+)}} w_{ijk}^*(\xi,x)N_1^{(q^+)}(x)dS(x)\right)-u_k^{(q)}\left(\int_{S^{(q^-)}} g_{ijk}^*(\xi,x)N_2^{(q^-)}(x)dS(x)+\int_{S^{(q^+)}} g_{ijk}^*(\xi,x)N_1^{(q^+)}(x)dS(x)\right)\right)$$

$$\bar I\left(u_{ij,k}^*\right)=\int_{OC}\frac{\partial u_{ij}^*\left(\bar\xi,x\right)}{\partial\bar\xi_k}dS(x)$$

$$\bar I\left(u_{11,1}^*\right)=c_1\left(\frac{1}{2}c_2Q_2+\bar\xi_2d_2\right)$$

$$\bar I\left(u_{11,2}^*\right)=c_1\left((c_2+1)Q_3+\bar\xi_2d_1\right)$$

$$\bar I\left(u_{12,1}^*\right)=c_1\bar\xi_2d_1,\quad \bar I\left(u_{12,2}^*\right)=c_1\left(-\frac{1}{2}Q_2-\bar\xi_2d_2\right)$$

$$\bar I\left(f_{ij,k}^*\right)=\int_{OC}\frac{\partial f_{ij}^*\left(\bar\xi,x\right)}{\partial\bar\xi_k}dS(x)$$

$$\bar I\left(f_{11,1}^*\right)=c_3\left(c_4d_2+d_3\right)\quad \bar I\left(f_{11,2}^*\right)=c_3\left((c_4+2)d_1+d_4\right)$$

$$\bar I\left(f_{12,1}^*\right)=c_3\left(c_4d_1+d_4\right)\quad \bar I\left(f_{12,2}^*\right)=c_3\left(-(c_4+2)d_2+d_5\right)$$

$$\bar I\left(f_{21,1}^*\right)=c_3\left(-c_4d_1+d_4\right)\quad \bar I\left(f_{21,2}^*\right)=c_3\left((c_4-2)d_2+d_5\right)$$

$$\bar I\left(f_{22,1}^*\right)=c_3\left((c_4+2)d_2-d_3\right)\quad \bar I\left(f_{22,2}^*\right)=c_3\left(c_4d_1-d_4\right)$$

here

$$d_3=2\bar\xi_2\left(\frac{\bar\xi_1^2}{\left(\bar\xi_1^2+\bar\xi_2^2\right)^2}-\frac{\left(\bar\xi_1-L\right)^2}{\left(\left(\bar\xi_1-L\right)^2+\bar\xi_2^2\right)^2}\right)$$

$$d_4=2\bar\xi_2^2\left(\frac{\bar\xi_1}{\left(\bar\xi_1^2+\bar\xi_2^2\right)^2}-\frac{\bar\xi_1-L}{\left(\left(\bar\xi_1-L\right)^2+\bar\xi_2^2\right)^2}\right)$$

3 Examples

As an illustration, let's show a comparison our numerical solution with analytical solutions of the simplest problem for tension an elastic rectangular plate with elliptical hole to uniform stress. We assume that the deformation occurs under the condition state of plane stress. Plate size M=100,H=1 m, ratio of the semi axes m elastic parameters were as follows: E=2·10^{11} Pa, $\nu=0.33$.

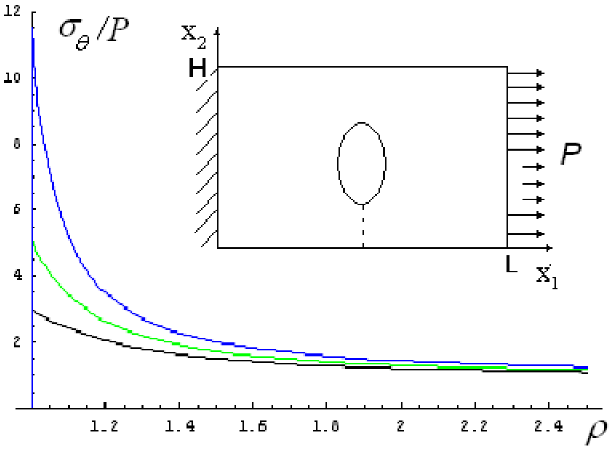

Fig.3. Tension a long strip with an elliptic hole.
_____ – *a/b=1* , _____ – *a/b=2*, _____ – *a/b=5*

On figure 3 shows the solution of the Kirsch problem with different ratios of semi-diameter of the ellipse. Ratio σ_ϑ/P computed along the defect mouth is identical with the analytical solution. At the beginning of the defect mouth for a round hole (a/b = 1) ratio σ_ϑ/P = 3. The decision was carried out by means of parallel programs

Tension of a plate with a crack.

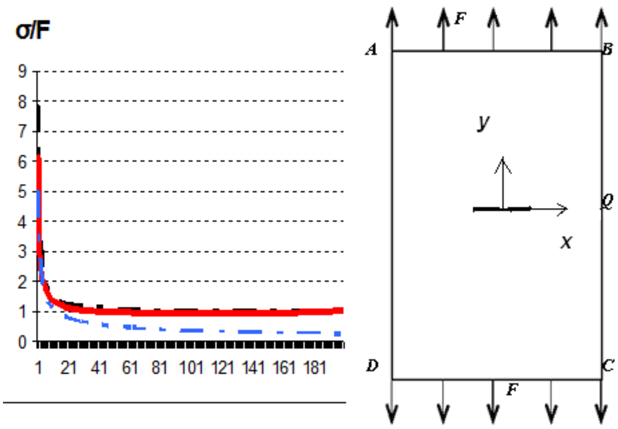

Fig. 4 The problem with the crack
FEM; MBEM analytical solution

Plate size (AB) M=200, (AD) H=200, the size of a crack l=2. Elastic parameters were as follows: E=2·10^{11} Pa, $\nu=0.33$.

Here the results of calculations by MBEM fully coincide with the analytical

From figure 4 visible that the decision by FEM and MBEM coincide but analytic solution agrees with these decisions only in the immediate vicinity of the mouth of the crack, and it has been continued by $\sigma/F=1$

4 Effectiveness of the algorithm for multiprocessor

Accelerating the computation time during the transition from numerical integration to analytical integration for the elastic problem.

Table 1

n/p	k,M	t_1^i, c	t_1^a, c	k_1^y, c
1	200, 4	4737	1.472	3218
2	200, 4	2370	0.736	3220
4	200, 4	1210	0.367	3297
8	200, 4	614	0.221	2778
32	200, 4	148	0.112	1325
2	500, 10	15157	5.0	3031
4	500, 10	7491	2.4	3121
8	500, 10	3746	1.2	3121
16	500, 10	1874	0.6	3123
n/p	k,M	t_3^i, c	t_3^a, c	k_3^y, c
1	200, 4	418	0.220	1900
2	200, 4	224	0.218	1027
4	200, 4	108	0.246	440
32	200, 4	16	0.047	345
2	500, 10	3292	0.165	4039
4	500, 10	1657	0.739	2242
8	500, 10	829	0.708	1171
16	500, 10	433	0.844	1513

n/p - order of system / number of processors
k - number of partition points of each of the M parts of the boundary region
t_1^i - time for computing integrals of Green's tensor for the boundary element
t_3^i- time for computing integrals of Green's tensor and their derivatives for domestic points
t_1^a- the numerical integration
t_3^a - compute integrals using analytic formulas
k_1^{ac}-,k_3^{ac} the corresponding coefficients speedup.

5 Conclusion

In the paper gives a numerical-analytical BEM, eliminating the numerical integration, driven to the calculation of elementary functions and is available for solving engineering problems for students and researchers. Method significantly reduces the computation time without loss of accuracy of the solution.

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