APPLICATION OF ANALYTIC FUNCTIONS TO THE GLOBAL SOLVABILTY OF THE CAUCHY PROBLEM FOR EQUATIONS OF NAVIER-STOKES

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Abstract: The interrelation between analytic functions and real-valued functions is formulated in the work. It is shown such an interrelation realizes nonlinear representations for real-valued functions that allows to develop new methods of estimation for them. These methods of estimation are approved by solving the Cauchy problem for equations of viscous incompressible liquid.

Keywords: Shrödinger, Cauchy problem, Navier-Stokes', inverse, analytic functions, scattering theory.

1 Introduction

The work of L. Fadeyev dedicated to the many-dimensional inverse problem of scattering theory inspired the author of this article to conduct this research. The first results obtained by the author are described in the works [2,3,4]. This problem includes a number of subproblems which appear to be very interesting and complicated. These subproblems are thoroughly considered in the works of the following scientists: R. Newton [6], R. Faddeyev [1], R. Novikov and G. Khenkin [5], A. Ramm [4] and others. The latest advances in the theory of SIPM(Scattering Inverse Problem Method) were a great stimulus for the author as well as other researchers. Another important stimulus was the work of M. Lavrentyev on the application of analytic functions to Hydrodynamics. Only one-dimensional equations were integrated by SIPM. The application of analytic functions to Hydrodynamics is restricted only by bidimensional problems. The further progress in applying SIPM to the solution of nonlinear equations in R3 was hampered by the poor development of the three-dimensional inverse problem of scattering in comparison with the progress achieved in the work on the one-dimensional inverse problem of scattering and also by the difficulties the researchers encountered building up the corresponding Lax' pairs. It is easy to come to a conclusion that all the success in developing the theory of SIPM is connected with analytic functions, i.e., solutions to Schrodinger's equation. Therefore we consider Schrodinger's equation as an interrelation between real-valued functions and analytic functions, where real-valued functions are potentials in Schrodinger's equation and analytic functions are the corresponding eigenfunctions of the continuous spectrum of Schrodinger's operator. The basic aim of the paper is to study this interrelation and its application for obtaining new estimates to the solutions of the problem for Navier-Stokes' equations. We concentrated on formulating the conditions of momentum and energy conservation laws in terms of potential instead of formulating them in terms of wave functions. As a result of our study, we obtained non-trivial nonlinear relationships of potential. The effectiveness and novelty of the obtained results are displayed when solving the notoriously difficult Chauchy problem for Navier-Stokes' equations of viscous incompressible fluid.

2 Basic Notions and Subsidiary Statement

Letusconsider Shrödingerse equation

$$-\Delta_x \varphi + q\varphi = |k|^2 \varphi \qquad (1)$$

where q - is a bounded fast-decreasing function,

$$k \in R^3, \qquad |k|^2 = \sum_{j=1}^{3} k_j^2.$$

Definition 1. *Rolnik's Class* $\boldsymbol{R}$ *is a set of measurable functions* q,

$$||q||_{\mathbf{R}} = \int_{R^6} \frac{q(x)q(y)}{|x-y|^2} dxdy < \infty.$$

It is considered to be a general definition ([8], p. 110).

Theorem 1. *Suppose that* $q \in \boldsymbol{R}$*; then a exists a unique solution of equation (1), with asymptotic form (2) as* $|x| \to \infty$

$$\varphi_{\pm}(k,x) = e^{i(k,x)} + \\ + \frac{e^{\pm i|k||x|}}{|x|} A_{\pm}(k,k^{'}) + 0\left(\frac{1}{|x|}\right), \qquad (2)$$

where

$$x \in R^3, k^{'} = |k|\frac{x}{|x|}, (k,x) = \sum_{j=1}^{3} k_j x_j,$$

$$A_{\pm}(k,\lambda) = \frac{1}{(2\pi)^3} \int_{R^3} q(x)\varphi_{\pm}(k,x) e^{-i(\lambda,x)} dx.$$

The proof of this theorem is in [8], p. 110.

Consider the operators $H = -\Delta_x + q(x)$, $H_0 = -\Delta_x$ defined in the dense set $W_2^2(R^3)$ in the space $L_2(R^3)$. The operator H is called Schrodinger's operator. Povzner [9] proved that the functions $\varphi_{\pm}(k,x)$ form a complete orthonormal system of eigenfunctions of the continuous spectrum of the operator H, and the operator fills up the whole positive semi-axis. Besides the continuous spectrum the operator H can have a finite number N of negative eigenvalues Denote these eigenvalues by $-E_j^2$ and conforming normalized egenfunctions by $\psi_j(x, -E_j^2)(j = \overline{1,N})$, where $\psi_j(x,-E_j^2) \in L_2(R^3)$.

Theorem 2 ***(About Completeness).*** *For any vector-function* $f \in L_2(R^3)$ *and eigenfunctions of the operator* H*, we have Parseval's identity*

$$|f|_{L_2}^2 = \sum_{j=1}^{N} |f_j|^2 + \int_{R^3} |\bar{f}(s)|^2 ds,$$

where f_j *and* $\bar{f}$ *are Fourier coefficients in case of discrete of and continuous spectrum respectively.*

The proof of this theorem is in [9].

Theorem 3 ***(Birman - Schwinger's Estimate).*** *Suppose* $q \in R$*. Then the number of discrete eigenvalues of Shrödinger operator satisfies the estimate*

$$N(q) \leq \frac{1}{(4\pi)^2} \int_{R^3}\int_{R^3} \frac{q(x)q(y)}{|x-y|^2} dxdy.$$

The proof of this theorem is in [14], p.114.

Definition 2. *([8], p.118)*

$$T_{\pm}(k,k^{'}) = \frac{1}{(2\pi)^3} \int_{R^3} \varphi_{\pm}(x,k^{'}) e^{\mp i(k,x)} q(x) dx.$$

$T_{\pm}(.,.)$ *is called T-matrix. Let us take into consideration a series for* $T_{\pm}$:

$$T_{\pm}(k,k^{'}) = \sum_{n=0}^{\infty} T_{n_{\pm}}(k,k^{'}),$$

where

$$T_{0_{\pm}}(k,k^{'}) = \frac{1}{(2\pi)^3} \int_{R^3} e^{i(k^{'} \mp k,x)} q(x) dx,$$

$$T_{n_{\pm}}(k,k^{'}) = \frac{1}{(2\pi)^3} \frac{(-1)^n}{(4\pi)^n} \int_{R^{3(n+1)}} e^{\mp i(k,x_0)} \times \\ \times q(x_0) \frac{e^{\pm i|k^{'}||x_0-x_1|}}{|x_0-x_1|} q(x_1) \ldots q(x_{n-1}) \times \\ \times \frac{e^{\pm i|k^{'}||x_{n-1}-x_n|}}{|x_{n-1}-x_n|} q(x_n) e^{i(k^{'},x_n)} dx_0 \ldots dx_n.$$

As well as in [8], p.120 we formulate.

Definition 3. *Series (4) is called Born's series.*

Theorem 4. *Let* $q \in L_1(R^3) \cap \boldsymbol{R}$ *. If* $PqP_{\boldsymbol{R}}^2 \leq 4\pi$*, then Born's series for* $T(k,k^{'})$ *converges as* $k, k^{'} \in R^3$.

The proof of the theorem is in [8], 121.

Definition 4. *Suppose* $q \in R$*; then the function* $A(k,\lambda)$*, denoted by the following equality*

$$A(k,l) = \frac{1}{(2\pi)^3} \int_{R^3} q(x) \times \\ \times \varphi_{+}(k,x) e^{-i(\lambda,x)} dx,$$

iscalled scattering amplitude

Corollary 1. *Scattering amplitude* $A(k,\lambda)$ *is equal to T-matrix*

$$A(k,l) = T_{+}(l,k) = \\ = \frac{1}{(2\pi)^3} \int_{R^3} q(x)\varphi_{+}(k,x) e^{-i(\lambda,x)} dx.$$

The proof follows from definition 4.

It is a well-known fact [1] that the solutions $\varphi_{+}(k,x)$ and $\varphi_{-}(k,x)$ of equation (1) are linearly dependent

$$\varphi_{+} = S\varphi_{-} \qquad (3)$$

where S is a scattering operator with the nucleus $S(k,\lambda)$ of the form

$$S(k,\lambda) = \int_{R^3} \varphi_{+}(k,x)\varphi_{+}^{*}(\lambda,x) dx.$$

Theorem 5 ***(Conservation law of Impulse and Energy).*** *Assume that* $q \in \boldsymbol{R}$*, then*

$$SS^{*} = I, S^{*}S = I,$$

were I *- isanunit operator.*

The proof is in [1].

Let us use the following definitions

$$\tilde{q}(k) = \int_{R^3} q(x) e^{i(k,x)} dx,$$

$$\tilde{q}(k-\lambda) = \int_{R^3} q(x) e^{i(k-\lambda,x)} dx,$$

$$\tilde{q}_{\text{mv}}(k) = \int_{R^3} \tilde{q}(k-\lambda))\delta(|k|^2 - |\lambda|^2)d\lambda,$$

$$A_{\text{mv}}(k) = \int_{R^3} A(k,l))\delta(|k|^2 - |l|^2)dl,$$

$$\int f(k,l)de_k = \int_{R^3} f(k,l)\delta(|k|^2 - |l|^2)dk,$$

$$\int f(k,l)de_\lambda = \int_{R^3} f(k,l)\delta(k^2 - |l|^2)dl,$$

where $k,\lambda \in R^3$ and $e_k = \frac{k}{|k|}$, $e_\lambda = \frac{\lambda}{|\lambda|}$.

3 Estimate of Amplitude Maximum

Let us consider the problem of estimating the maximum of amplitude, i.e., $\max_{k\in R^3}|A(k,k)|$. Let us estimate the n term of Born's series $|T_n(k,k)|$.

Lemma 1. $|T_n(k,k)|$ *satisfies the inequality*

$$|T_{n+1}(k,k)| \le \frac{1}{(2\pi)^3}\frac{1}{(4\pi)^{n+1}} \times$$
$$\times \frac{\gamma^n}{(2\pi)^{2(n+1)}} \int_{R^3} \frac{|\tilde{q}(k)|^2}{|k|^2} dk,$$

$$\gamma = C\delta||q|| + 4\pi M\tilde{q}\delta, C\delta = 2\frac{\sqrt{\pi}}{\sqrt{\delta}},$$

where δ *-is a small value,* C *is a positive number,* $M\tilde{q} = \max_{k\in R^3}|\tilde{q}|$.

Theorem 6. *Suppose that* $\gamma < 16\pi^3$ *, then* $\max_{k\in R^3}|A(k,k)|$ *satisfies the following estimate*

$$\max_{k\in R^3}|A(k,k)| \le \frac{1}{(2\pi)^3}\frac{1}{16\pi^3 - \gamma} \int_{R^3} \frac{|\tilde{q}(k)|^2}{|k|^2} dk,$$

where $\gamma = C\delta||q|| + 4\pi M\tilde{q}\delta$ *,* δ *is a small value,* $C\delta = 2\sqrt{\frac{\pi}{\delta}}$*,* $M\delta = \max_{k\in R^3}|\tilde{q}|$.

4 Representation of Functions by its Spherical Averages

Let us consider the problem of defining a function by its spherical average. This problem emerged in the course of our calculation and we shall consider it hereinafter.

Let us consider the following integral equation

$$\int_{R^3} \tilde{q}(t)\delta(|t-k|^2 - |k|^2)dt = f(2k),$$

where $k,t \in R^3$, δ is Dirac's delta function,

$f \in W_2^2(R^3)$, $|k|^2 = \sum_{i=1}^3 k_i^2$, $(k,t) = \sum_{i=1}^3 k_i t_i$.

Let us formulate the basic result.

Theorem 7. *Suppose that* $f \in W_2^2(R^3)$*, then*

$$(2\pi)^2\tilde{q}(r,\xi,\eta) =$$
$$= -\frac{1}{r}\frac{\partial^2}{\partial r^2}\int_0^\pi\int_0^{2\pi} \Big(f(\frac{2r}{(e_k,e_s)}, e_k) +$$
$$+ f(\frac{2r}{(e_k,e_s)}, -e_k)\Big) \times$$
$$\times \frac{r^2}{(e_k,e_s)^2} \sin\theta \; d\theta \; d\varphi,$$

where

$$f(\frac{2r}{(e_k,e_s)}, e_k) = \tilde{q}(\frac{2r}{(e_k,e_s)}, e_k),$$
$$\sin\theta \; d\theta \; d\varphi = de_k,$$
$$\sin\xi \; d\xi d\eta = de_s, \qquad r = |t|.$$

Theorem 8. *Fourier transformation of the function* q *satisfies the following estimate*

$$|\tilde{q}|_{L_1} \le \frac{1}{4}\left|z\frac{\partial \tilde{q}_{mv}}{\partial z^2}\right|_{L_1} + 2\left|\frac{\partial \tilde{q}_{mv}}{\partial z^2}\right|_{L_1} + \left|\frac{\tilde{q}_{mv}}{z}\right|_{L_1},$$

5 Correlation of Amplitude and Wave Functions

We take the relationship for φ_+, φ_- from (6)

$$\varphi_+(k,x) = \varphi_-(k,x) -$$
$$-2\pi i \int_{R^3} \delta(|k|^2 - |l|^2) \times$$
$$\times A(k,\lambda)\varphi_-(\lambda,x)d\lambda. \qquad (4)$$

Let us denote new functions and operators we will use further

$$\varphi_0(\sqrt{z}e_k,x) = e^{i(\sqrt{z}e_k,x)},$$
$$\Phi_0(\sqrt{z}e_k,x) = \varphi_0(\sqrt{z}e_k,x) + \varphi_0(-\sqrt{z}e_k,x),$$
$$\Phi_+(\sqrt{z}e_k,x) = \varphi_+(\sqrt{z}e_k,x) - e^{i(\sqrt{z}e_k,x)} +$$
$$+\varphi_+(-\sqrt{z}e_k,x) - e^{-i(\sqrt{z}e_k,x)},$$
$$\Phi_-(\sqrt{z}e_k,x) = \varphi_-(\sqrt{z}e_k,x) - e^{i(\sqrt{z}e_k,x)} +$$
$$+\varphi_-(-\sqrt{z}e_k,x) - e^{-i(\sqrt{z}e_k,x)},$$
$$D_1 f = -2\pi i \int_{R^3} A(k,\lambda)\delta(z-l)f(\lambda,x)d\lambda,$$
$$D_2 f = -2\pi i \int_{R^3} A(-k,\lambda)\delta(z-l)f(\lambda,x)d\lambda,$$
$$D_3 f = D_1 f + D_2 f,$$

where $z = |k|^2$, $l = |\lambda|^2$, $\pm k = \pm\sqrt{z}e_k$. Let us introduce the operators $T_\pm$, T for the function $f \in W_2^1(R)$ by the formulas

$$T_+ f = \frac{1}{\pi i}\lim_{Imz\to 0}\int_{-\infty}^{\infty}\frac{f(\sqrt{s})}{s-z}ds,$$

where $Imz > 0$,

$$T_- f = \frac{1}{\pi i}\lim_{Imz\to 0}\int_{-\infty}^{\infty}\frac{f(\sqrt{s})}{s-z}ds,$$

where $Imz < 0$,

$$Tf = \frac{1}{2}(T_+ + T_-)f.$$

Use (4) and the symbols $e_r = \frac{k}{|k|}$ to come to Riemann' problem of finding a function Φ_+, which is analytic by the variable z in the top half plane, and the function Φ_-, which is analytical on the variable z in the bottom half plane by the specified jump of discontinuity f onto the positive semi axis.

For the jump the discontinuity of an analytical function, we have the following equations

$$f = \Phi_+ - \Phi_-,$$
$$f = D_3[\Phi_-] - D_3[\varphi_-],$$

where $\varphi_- = \varphi_-(-\lambda, x)$.

Theorem 9. *Suppose that* $q \in \boldsymbol{R}$,

$$\varphi_\pm|_{x=0,z=0} = 0;$$

then the functions

$$\Psi_1 = \Phi_\pm(\sqrt{z}e_k, x)|_{x=0} - \Phi_0(\sqrt{z}e_k, x)|_{x=0},$$
$$\Psi_2 = T_\pm f|_{x=0}$$

are coincided according to the class of analytical functions, coincide with bounded derivatives all over the complex plane with a slit along the positive semi axis.

Lemma 2. *There exists* $0 < |\varepsilon| < \infty$ *such that it satisfies the following condition* $\varphi_+|_{x=0,z=0} = 0$ *holds for the potential of the form* $v = \varepsilon q$, *where* $q \in \boldsymbol{R}$.

Now, we can formulate Riemann's problem. Find the analytic function $\Phi_\pm$ that satisfies (5), (6) and its solution is set by the following theorem.

Theorem 10. *Assume that* $q \in \boldsymbol{R}$,

$$\varphi_\pm|_{x=0,z=0} = 0,$$

then

$$\Phi_\pm = T_\pm f + \Phi_0,$$
$$f = D_3[f[T_- f + \Phi_0]] - D_3\varphi_-,$$

where $\varphi_- = \varphi_-(-\lambda, x)$.

Lemma 3. *Suppose that* $q \in \boldsymbol{R}$, $\varphi_\pm|_{x=0,z=0} = 0$; *then*

$$\Delta_x T_\pm[f]|_{x=0} = T_\pm \Delta_x[f]|_{x=0}.$$

Theorem 11. *Suppose that* $q \in \boldsymbol{R}$,

$$\varphi_\pm|_{x=0,z=0} = 0, \; q(0) \neq 0,$$

then

$$q(0)f|_{x=0} = D_3 T_-[qf|_{x=0} - $$
$$-D_3[q\varphi_-]|_{x=0} + D_3 \int_0^\infty f ds|_{x=0}.$$

6 Auxiliary Propositions

For wave functions let us use integral representations following from Lippman-Schwinger's theorem

$$\varphi_\pm(k, x) = e^{i(k,x)} +$$
$$+\frac{1}{4\pi}\int_{R^3} \frac{e^{\pm i\sqrt{z}|x-y|}}{|x-y|} q(y)\varphi_\pm(k, y)dy,$$
$$\varphi_\pm(-k, x) = e^{-i(k,x)} +$$
$$+\frac{1}{4\pi}\int_{R^3} \frac{e^{\mp i\sqrt{z}|x-y|}}{|x-y|} q(y)\varphi_\pm(-k, y)dy.$$

Lemma 4. *Suppose that* $q \in \boldsymbol{R}$,

$$\varphi_\pm|_{x=0,z=0} = 0;$$

then

$$A(k, k') = c_0\tilde{q}(k - k') +$$
$$+\frac{c_0}{4\pi}\int_{R^3}\int_{R^3} e^{-i(k',x)} q(x) \frac{e^{i\sqrt{z}|x-y|}}{|x-y|} \times$$
$$\times q(y)e^{i(k,y)} dy dx + A_3(k, k'),$$
$$A(-k, k') = c_0\tilde{q}(-k - k') +$$
$$+\frac{c_0}{4\pi}\int_{R^3}\int_{R^3} e^{-i(k',x)} q(x) \frac{e^{-i\sqrt{z}|x-y|}}{|x-y|} \times$$
$$\times q(y)e^{-i(k,y)} dy dx + A_3(-k, k'),$$

where $c_0 = \frac{1}{(2\pi)^2}$, *and* $A_3(k, k'), A_3(-k, k')$ *are terms of order higher than 2 with regards to* q.

Theorem 12 *(Parseval).* *The functions*

$$f, g \in L_2(R^3)$$

satisfy the equation

$$(f, g) = c_0(\tilde{f}, \tilde{g}^*),$$

where $(\cdot,\cdot)$ *is a scalar product and* $c_0 = \frac{1}{(2\pi)^3}$.

The Proof is in work [12].

Lemma 5. *Suppose that* $q \in \boldsymbol{R}$, $\varphi_\pm|_{x=0,z=0} = 0$, *then*

$$A(k, k') = c_0\tilde{q}(k - k') -$$
$$-c_0^2 \int_{R^3} \frac{\tilde{q}(k + p)\tilde{q}(p - k')}{|p|^2 - z - i0} dp +$$
$$+A_3(k, k'),$$
$$A(-k, k') = c_0\tilde{q}(-k - k') -$$
$$-c_0^2 \int_{R^3} \frac{\tilde{q}(-k + p)\tilde{q}(p - k')}{|p|^2 - z - i0} dp +$$
$$+A_3(-k, k').$$

Corollary 2. *Suppose that* $q \in \boldsymbol{R}$,

$$\varphi_\pm|_{x=0,z=0} = 0,$$

then

$$A_{\text{mv}}(k) = c_0\tilde{q}_{\text{mv}}(k) -$$
$$-c_0^2 \frac{\sqrt{z}}{2}\int_0^\pi \int_0^{2*\pi} \int_{R^3} \frac{\tilde{q}(k + p)\tilde{q}(p - k')}{|p|^2 - z - i0} dp de_{k'} +$$
$$+A_{3\text{mv}}(k),$$

where

$$A_{3\text{mv}}(k) = \int_{R^3} A_3(k, k')\delta(z - |k'|^2)dk'.$$

And

$$A_{\text{mv}}(-k) = c_0\tilde{q}_{\text{mv}}(-k) -$$
$$-c_0^2\frac{\sqrt{z}}{2}\int_0^{\pi}\int_0^{2*\pi}\int_{R^3}\frac{\tilde{q}(-k+p)\tilde{q}(p-k')}{|p|^2-z-i0}dp\,de_{k'} +$$
$$+A_{3\text{mv}}(-k),$$

where

$$A_{3\text{mv}}(-k) = \int_{R^3} A_3(-k,k')\delta(z-|k'|^2)dk'.$$

Lemma 6. *Suppose that* $q \in R$ *and* $x = 0$*, then*

$$\varphi_\pm(k,0) = 1 + \frac{1}{4\pi}\int_{R^3}\frac{e^{\pm i\sqrt{z}|y|}}{|y|}q(y)e^{i(k,y)}dy +$$
$$+\frac{1}{(4\pi)^2}\int_{R^3}\int_{R^3}\frac{e^{\pm i\sqrt{z}|y|}}{|y|}q(y)\frac{e^{\pm i\sqrt{z}|y-t|}}{|y-t|}\times$$
$$\times q(t)e^{i(k,t)}dtdy + \varphi_\pm^{(3)}(k,0),$$

where $\varphi_\pm^{(3)}(k,0)$ *are terms of order higher than 2 with regards to* q*., i.e.,*

$$\varphi_\pm^{(3)}(k,x) = \frac{1}{(4\pi)^3}\int_{R^3}\int_{R^3}\int_{R^3}\frac{e^{\pm i\sqrt{z}|x-y|}}{|x-y|}q(y)\times$$
$$\times\frac{e^{\pm i\sqrt{z}|y-t|}}{|y-t|}q(t)\frac{e^{\pm i\sqrt{z}|t-s|}}{|t-s|}q(s)\varphi_\pm(k,s)dsdtdy.$$

And

$$\varphi_\pm(-k,0) = 1 + \frac{1}{4\pi}\int_{R^3}\frac{e^{\mp i\sqrt{z}|y|}}{|y|}q(y)e^{-i(k,y)}dy +$$
$$+\frac{1}{(4\pi)^2}\int_{R^3}\int_{R^3}\frac{e^{\mp i\sqrt{z}|y|}}{|y|}q(y)\frac{e^{\mp i\sqrt{z}|y-t|}}{|y-t|}q(t)\times$$
$$\times e^{-i(k,t)}dtdy + \varphi_\pm^{(3)}(-k,0),$$

where $\varphi_\pm^{(3)}(-k,0)$ *are terms of order higher than 2 with regards to* q*., i.e.,*

$$\varphi_\pm^{(3)}(-k,x) = \frac{1}{(4\pi)^3}\int_{R^3}\int_{R^3}\int_{R^3}\frac{e^{\mp i\sqrt{z}|x-y|}}{|x-y|}q(y)\times$$
$$\times\frac{e^{\mp i\sqrt{z}|y-t|}}{|y-t|}q(t)\frac{e^{\mp i\sqrt{z}|t-s|}}{|t-s|}q(s)\varphi_\pm(-k,s)dsdtdy.$$

Lemma 7. *Suppose that* $q \in \boldsymbol{R}$*,* $\varphi_\pm|_{x=0,z=0} = 0$*, then*

$$\varphi_\pm(k,0) = 1 - c_0\int_{R^3}\frac{\tilde{q}(k+p)}{|p|^2-z\mp i0}dp +$$
$$+c_0^2\int_{R^3}\int_{R^3}\frac{\tilde{q}(k+p)\tilde{q}(p+p_1)}{(|p|^2-z\mp i0)(|p_1|^2-z\mp i0)}dp_1dp +$$
$$+\varphi_\pm^{(3)}(k,0), \quad (7)$$

$$\varphi_\pm(-k,0) = 1 - c_0\int_{R^3}\frac{\tilde{q}(-k+p)}{|p|^2-z\mp i0}dp +$$
$$+c_0^2\int_{R^3}\int_{R^3}\frac{\tilde{q}(-k+p)\tilde{q}(p+p_1)}{(|p|^2-z\mp i0)(|p_1|^2-z\mp i0)}dp_1dp +$$
$$+\varphi_\pm^{(3)}(-k,0). \quad (8)$$

Lemma 8. *Suppose that* $q \in R$*,* $x = 0$*; then*

$$F(k,0) = -\pi ic_0\sqrt{z}\int_0^{\pi}\int_0^{2\pi}\tilde{q}(k-\sqrt{z}e_p)de_p +$$
$$+\pi ic_0^2\sqrt{z}\int_0^{\pi}\int_0^{2\pi}V.p.\int_{R^3}\frac{\tilde{q}(k-\sqrt{z}e_p)}{|p_1|^2-z}\times$$
$$\times\tilde{q}(-\sqrt{z}e_p-p_1)dp_1de_p +$$
$$+\pi ic_0^2\sqrt{z}V.p.\int_{R^3}\int_0^{\pi}\int_0^{2\pi}\frac{\tilde{q}(k-p)}{|p|^2-z}\times$$
$$\times\tilde{q}(-p-\sqrt{z}e_{p_1})de_{p_1}dp +$$
$$+\varphi_+^{(3)}(k,0) - \varphi_-^{(3)}(k,0).$$

And

$$F(-k,0) = -\pi ic_0\sqrt{z}\int_0^{\pi}\int_0^{2\pi}\tilde{q}(-k-\sqrt{z}e_p)de_p +$$
$$+\pi ic_0^2\sqrt{z}\int_0^{\pi}\int_0^{2\pi}V.p.\int_{R^3}\frac{\tilde{q}(-k-\sqrt{z}e_p)}{|p_1|^2-z}\times$$
$$\times\tilde{q}(-\sqrt{z}e_p-p_1)dp_1de_p +$$
$$+\pi ic_0^2\sqrt{z}\,V.p.\int_{R^3}\int_0^{\pi}\int_0^{2\pi}\frac{\tilde{q}(-k-p)}{|p|^2-z}\times$$
$$\times\tilde{q}(-p-\sqrt{z}e_{p_1})de_{p_1}dp +$$
$$+\varphi_+^{(3)}(-k,0) - \varphi_-^{(3)}(-k,0).$$

7 Two Representations of Scattering Amplitude

Lemma 9. *Suppose that* $f \in W_2^1(R)$*, then*

$$T_\pm f = \mp f + Tf.$$

Lemma 10. *Suppose that* $q \in \boldsymbol{R}$*,* $\varphi_\pm|_{x=0,z=0} = 0$*, then*

$$f(k,0) = F(k,0) + F(-k,0).$$

Lemma 11. *Suppose that* $q \in \boldsymbol{R}$*,* $\varphi_\pm|_{x=0,z=0} = 0$*, then*

$$A_{\text{mv}}(k) + A_{\text{mv}}(-k) = c_0(\tilde{q}_{\text{mv}}(k) + \tilde{q}_{\text{mv}}(-k)) +$$
$$+\pi ic_0^2\sqrt{z}\int_0^{\pi}\int_0^{2\pi}(\tilde{q}(k-\sqrt{z}e_\lambda) + \tilde{q}(-k-\sqrt{z}e_\lambda))\times$$
$$\times\tilde{q}_{\text{mv}}(\sqrt{z}e_\lambda)de_\lambda +$$
$$+\pi ic_0^2\frac{\sqrt{z}}{2}\int_0^{\pi}\int_0^{2\pi}(\tilde{q}(k-\sqrt{z}e_\lambda) + \tilde{q}(-k-\sqrt{z}e_\lambda))\times$$

$$\times \tilde{q}_{\rm mv}(-\sqrt{z}e_\lambda)de_\lambda -$$
$$-\pi i c_0^2\sqrt{z}\int_0^{\pi}\int_0^{2\pi} (\tilde{q}(k-\sqrt{z}e_\lambda)+\tilde{q}(-k-\sqrt{z}e_\lambda))\times$$
$$\times (T[\tilde{q}_{\rm mv}](\sqrt{z}e_\lambda)+T[\tilde{q}_{\rm mv}](-\sqrt{z}e_\lambda))de_\lambda -$$
$$-c_0^2\sqrt{z}\int_0^{\pi}\int_0^{2\pi} (\tilde{q}(k-\sqrt{z}e_\lambda)+\tilde{q}(-k-\sqrt{z}e_\lambda))\times$$
$$\times V.p.\int_{R^3}\frac{\tilde{q}(-\sqrt{z}e_\lambda-p)}{|p|^2-z}dpde_\lambda +$$
$$+c_0^2\frac{\sqrt{z}}{2}V.p.\int_{R^3}\int_0^{\pi}\int_0^{2\pi}\frac{\tilde{q}(k-\lambda)+\tilde{q}(-k-\lambda)}{l-z}\times$$
$$\times \tilde{q}(-l-\sqrt{z}e_p)de_pd\lambda - 2\pi i(F^{(3)}(k,0)+$$
$$+F^{(3)}(-k,0)+Q_3(k,0)+Q^{(3)}(k,0)),$$

where $Q_3(k,0)$, $Q^{(3)}(k,0)$ *are defined by formulas*

$$Q_3(k,0)=-4\pi^2c_0^2\int_{R^3}(A_2(k,\lambda)+A_2(-k,\lambda))\times$$
$$\times\delta(z-l)(\tilde{q}_{\rm mv}(\lambda)+\tilde{q}_{\rm mv}(-\lambda))d\lambda +$$
$$+2\pi i c_0\int_{R^3}(A_2(k,\lambda)+A_2(-k,\lambda))\delta(z-l)\times$$
$$\times f_2(l,0)dl+4\pi^2c_0^2\int_{R^3}(A_2(k,\lambda)+A_2(-k,\lambda))\times$$
$$\times\delta(z-l)(T[\tilde{q}_{\rm mv}](\lambda)+T[\tilde{q}_{\rm mv}](-\lambda))d\lambda -$$
$$-2\pi i c_0\int_{R^3}(A_2(k,l)+A_2(-k,l))\times$$
$$\times\delta(z-l)T[f_2](\lambda,0)d\lambda. \qquad (9)$$

$$Q^{(3)}(k,0)=2\pi i c_0^2\int_{R^3}(\tilde{q}(k-\lambda)+\tilde{q}(-k-\lambda))\times$$
$$\times\delta(z-l)\varphi_-^{(2)}(-\lambda,0)d\lambda +$$
$$+2\pi i c_0^2\int_{R^3}(A_2(k,\lambda)+A_2(-k,\lambda))\times$$
$$\times\delta(z-l)(\int_{R^3}\frac{\tilde{q}(-\lambda-p)}{|p|^2-l+i0}dp+$$
$$+\varphi_-^{(2)}(-l,0))d\lambda. \qquad (10)$$

correspondingly,

$$F^{(3)}(k,0)=\varphi_+^{(3)}(k,0)-\varphi_-^{(3)}(k,0),$$
$$F^{(3)}(-k,0)=\varphi_+^{(3)}(-k,0)-\varphi_-^{(3)}(-k,0),$$

and $\varphi_\pm^{(3)}(\pm k,0)$ *are terms of order 3 and higher w.r.t.* $\tilde{q}$ *in the representations (7), (8).*

Lemma 12. *Suppose that* $q\in\boldsymbol{R}$, $\varphi_\pm|_{x=0,z=0}=0$, *then*

$$A_{\rm mv}(k)+A_{\rm mv}(-k)=$$
$$=-\frac{i\sqrt{z}}{4\pi q(0)}\int_0^{\pi}\int_0^{2\pi}(A(k,\sqrt{z}e_\lambda)+A(-k,\sqrt{z}e_\lambda))\times$$
$$\times\int_0^{\infty}f(se_\lambda,0)dsde_\lambda.$$

8 Nonlinear Representation of Potential

Let us proceed to the construction of potential nonlinear representation.

Lemma 13. *Assume that* $q\in\boldsymbol{R}$, $\varphi_\pm|_{x=0,z=0}=0$; *then*

$$\tilde{q}_{\rm mv}(k)+\tilde{q}_{\rm mv}(-k)=$$
$$=-\pi i c_0\sqrt{z}\int_0^{\pi}\int_0^{2\pi}(\tilde{q}(k-\sqrt{z}e_\lambda)+\tilde{q}(-k-\sqrt{z}e_\lambda))\times$$
$$\times\tilde{q}_{\rm mv}(\sqrt{z}e_\lambda)de_\lambda -$$
$$-\pi i c_0\frac{\sqrt{z}}{2}\int_0^{\pi}\int_0^{2\pi}(\tilde{q}(k-\sqrt{z}e_\lambda)+\tilde{q}(-k-\sqrt{z}e_\lambda))\times$$
$$\times\tilde{q}_{\rm mv}(-\sqrt{z}e_\lambda)de_\lambda +$$
$$+\pi i c_0\sqrt{z}\int_0^{\pi}\int_0^{2\pi}(\tilde{q}(k-\sqrt{z}e_\lambda)+\tilde{q}(-k-\sqrt{z}e_\lambda))\times$$
$$\times(T[\tilde{q}_{\rm mv}](\sqrt{z}e_\lambda)+T[\tilde{q}_{\rm mv}](-\sqrt{z}e_\lambda))de_\lambda -$$
$$-c_0\sqrt{z}\int_0^{\pi}\int_0^{2\pi}(\tilde{q}(k-\sqrt{z}e_\lambda)+\tilde{q}(-k-\sqrt{z}e_\lambda))\times$$
$$\times V.p.\int_{R^3}\frac{\tilde{q}(-\sqrt{z}e_\lambda-p)}{|p|^2-z}dpde_\lambda -$$
$$-c_0\frac{\sqrt{z}}{2}V.p.\int_{R^3}\int_0^{\pi}\int_0^{2\pi}\frac{(\tilde{q}(k-\lambda)+\tilde{q}(-k-\lambda))}{l-z}\times$$
$$\times\tilde{q}(-l-\sqrt{z}e_p)de_pd\lambda -$$
$$-\frac{i\sqrt{z}}{4\pi c_0q(0)}\int_0^{\pi}\int_0^{2\pi}(A(k,\sqrt{z}e_\lambda)+A(-k,\sqrt{z}e_\lambda))\times$$
$$\times\int_0^{\infty}f(se_\lambda,0)dsde_\lambda+\frac{2\pi i}{c_0}(F^{(3)}(k,0)+$$
$$+F^{(3)}(-k,0)+Q_3(k,0)+Q^{(3)}(k,0)),$$

where $Q_3(k,0)$, $Q^{(3)}(k,0)$ *are defined by formulas (9), (10) accordingly,*

$$F^{(3)}(k,0)=\varphi_+^{(3)}(k,0)-\varphi_-^{(3)}(k,0),$$
$$F^{(3)}(-k,0)=\varphi_+^{(3)}(-k,0)-\varphi_-^{(3)}(-k,0),$$

and $\varphi_\pm^{(3)}(\pm k,0)$ *are term of order 3 and higher w.r.t.* $\tilde{q}$ *in representations (7), (8).*

Lemma 14. *Suppose that* $q\in\boldsymbol{R}$, $\varphi_\pm|_{x=0,z=0}=0$, *then*

$$V.p.\int_{R^3}\int_0^{\pi}\int_0^{2\pi}\frac{(\tilde q(k-\lambda)+\tilde q(-k-\lambda))}{l-z}\times$$
$$\times\tilde q(-l-\sqrt{z}e_p)de_p dl=$$
$$=\pi i\int_0^{\pi}\int_0^{2\pi}(\tilde q(k-\sqrt{z}e_\lambda)+\tilde q(-k-\sqrt{z}e_\lambda))\times$$
$$\times\tilde q_{\text{mv}}(-\sqrt{z}e_\lambda)de_\lambda.$$

Lemma 15. *Let* $\tilde q\in W_2^1(R)$ *and* $q\in R$, *then*

$$\int_0^{\pi}\int_0^{2\pi}(\tilde q(k-\sqrt{z}e_\lambda)+\tilde q(-k-\sqrt{z}e_\lambda))\times$$
$$\times(T[\tilde q_{\text{mv}}](\sqrt{z}e_\lambda)+T[\tilde q_{\text{mv}}](-\sqrt{z}e_\lambda))de_\lambda=$$
$$=\int_0^{\pi}\int_0^{2\pi}(\tilde q(k-\sqrt{z}e_\lambda)+\tilde q(-k-\sqrt{z}e_\lambda))\times$$
$$\times(\tilde q_{\text{mv}}(\sqrt{z}e_\lambda)+\tilde q_{\text{mv}}(-\sqrt{z}e_\lambda))de_\lambda,$$
$$\int_0^{\pi}\int_0^{2\pi}(\tilde q(k-\sqrt{z}e_\lambda)+\tilde q(-k-\sqrt{z}e_\lambda))\times$$
$$\times V.p.\int_{R^3}\frac{\tilde q(-\sqrt{z}e_\lambda-p)}{|p|^2-z}dp de_\lambda=$$
$$=\pi i\int_0^{\pi}\int_0^{2\pi}(\tilde q(k-\sqrt{z}e_\lambda)+\tilde q(-k-\sqrt{z}e_\lambda))\times$$
$$\times\tilde q_{\text{mv}}(-\sqrt{z}e_\lambda)de_\lambda.$$

Theorem 14. *Let* $q\in\boldsymbol{R}$, $\varphi_\pm|_{x=0,z=0}=0$, *then*

$$\tilde q_{\text{mv}}(k)+\tilde q_{\text{mv}}(-k)=$$
$$=-\pi i c_0\sqrt{z}\int_0^{\pi}\int_0^{2\pi}(\tilde q(k-\sqrt{z}e_\lambda)+\tilde q(-k-\sqrt{z}e_\lambda))\times$$
$$\times\tilde q_{\text{mv}}(-\sqrt{z}e_\lambda)de_\lambda+\mu(k),$$
$$\mu(k)=\frac{2\pi i}{c_0}(F^{(3)}(k,0)+F^{(3)}(-k,0)+$$
$$+Q_3(k,0)+Q^{(3)}(k,0)),$$

where $c_0=4\pi$.

Theorem 15. *Suppose* $q\in\boldsymbol{R}$, $\varphi_\pm|_{x=0,z=0}=0$; *then*

$$\mu(k)=\sqrt{z}\int_0^{\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi}(\tilde q(-k-\sqrt{z}e_\lambda)+$$
$$+\tilde q(k-\sqrt{z}e_\lambda))\tilde q(\sqrt{z}e_\lambda-\sqrt{z}e_s)\times$$
$$\times\mu_0(\sqrt{z}e_s)de_\lambda de_s,$$

where $|\mu_0|<C|q_{\text{mv}}|$

9 The Cauchy Problem for Navier-Stokes' Equations

Let us apply the obtained results to estimate the solutions of Cauchy problem for Navier-Stokes' set of equations

$$q_t-\nu\Delta q+\sum_{k=1}^{3}q_k q_{x_k}=$$
$$=-\nabla p+F_0(x,t),\ div q=0,\qquad(11)$$
$$q|_{t=0}=q_0(x)\qquad(12)$$

in the domain of $Q_T=R^3\times(0,T)$. With respect to q_0, assume

$$div\ q_0=0.\qquad(13)$$

Problem (11), (12), (13) has at least one weak solution (q,p) in the so-called Leray-Hopf class, see [4].

Let us mention the known statements proved in [13].

Theorem 16. *Suppose that*

$$q_0\in W_2^1(R^3),\quad f\in L_2(Q_T);$$

then there exists a unique weak solution of problem (11), (12), (13), in Q_{T_1}, $T_1\in[0,T]$, *that satisfies*

$$q_t,\ q_{xx},\nabla p\in L_2(Q_T).$$

Note that T_1 depends on q_0, f.

Lemma 16. *If* $q_0\in W_2^1(R^3)$, $f\in L_2(Q_T)$, *then*

$$\sup_{0\le t\le T}||q||^2_{L_2(R^3)}+\int_0^t||q_x||^2_{L_2(R^3)}d\tau\le$$
$$\le||q_0||^2_{L_2(R^3)}+||F_0||_{L_2(Q_T)}$$

Our goal is to prove the global unicity weak solution of (11), (12), (13) irrespective of initial velocity and power smallness conditions.

Therefore let us obtain uniform estimates.

Statement 1. *Weak solution of problem (11), (12), (13), from Theorem 16 satisfies the following equation*

$$\tilde q(z(e_k-e_\lambda),t)=\tilde q_0(z(e_k-e_\lambda))+$$
$$+\int_0^t e^{-\nu z^2|e_k-e_\lambda|(t-\tau)}([\widetilde{(q,\nabla)q}]+\tilde F)\times$$
$$\times(z(e_k-e_\lambda),\tau)d\tau,$$

where $F=-\nabla p+F_0$.

Lemma 17. *The solution of the problem (11), (12), (13) from Theorem 16, satisfies the following equation*

$$\tilde p=\sum_{i,j}\frac{k_i k_j}{|k|^2}\widetilde{q_\iota q_\jmath}+i\sum_i\frac{k_i}{|k|^2}\tilde F_\iota$$

and the following estimates

$$||p||_{L_2(R^3)}\le 3||q_x||^{\frac{3}{2}}_{L_2(R^3)}||q||^{\frac{1}{2}}_{L_2(R^3)},$$
$$\left|\frac{\partial\tilde p}{\partial k}\right|\le\frac{|\tilde q^2|}{|k|}+\frac{|\tilde F|}{|k|^2}+\frac{1}{|k|}\left|\frac{\partial\tilde F}{\partial k}\right|+3\left|\frac{\partial\tilde q^2}{\partial|k|}\right|;$$

Lemma 18. *Weak solution of problem (11), (12), (13), from Theorem 16 satisfies the following inequalities*

$$\sup_{0\le t\le T}[\int_{R^3}|x|^2|q(x,t)|^2dx+$$

$$+\int_0^t \int_{R^3} |x|^2 |q_x(x,\tau)|^2 dx d\tau] \le const,$$

$$\sup_{0\le t\le T} [\int_{R^3} |x|^4 |q(x,t)|^2 dx +$$

$$+\int_0^t \int_{R^3} |x|^4 |q_x(x,\tau)|^2 dx d\tau] \le const,$$

or

$$\sup_{0\le t\le T} [\left\|\frac{\partial \tilde{q}}{\partial z}\right\|_{L_2(R^3)} +$$

$$+\int_0^t \int_{R^3} z^2 |\widetilde{q_k}(k,\tau)|^2 dk d\tau] \le const,$$

$$\sup_{0\le t\le T} [\left\|\frac{\partial^2 \tilde{q}}{\partial z^2}\right\|_{L_2(R^3)} +$$

$$+\int_0^t \int_{R^3} z^2 |\widetilde{q_{kk}}(k,\tau)|^2 dk d\tau] \le const.$$

Lemma 19. *Weak solution of problem (11), (12), (13), from Theorem 16, satisfies the following inequalities*

$$\max_k |\tilde{q}| \le \max_k |\tilde{q}_0| +$$

$$+\frac{T}{2} \sup_{0\le t\le T} ||q||^2_{L_2(R^3)} + \int_0^t ||q_x||^2_{L_2(R^3)} d\tau,$$

$$\max_k \left|\frac{\partial \tilde{q}}{\partial z}\right| \le \max_k \left|\frac{\partial \tilde{q}_0}{\partial z}\right| +$$

$$+\frac{T}{2} \sup_{0\le t\le T} \left\|\frac{\partial \tilde{q}}{\partial z}\right\|_{L_2(R^3)} + \int_0^t \int_{R^3} z^2 |\widetilde{q_k}(k,\tau)|^2 dk d\tau,$$

$$\max_k \left|\frac{\partial^2 \tilde{q}}{\partial z^2}\right| \le \max_k \left|\frac{\partial^2 \tilde{q}_0}{\partial z^2}\right| +$$

$$+\frac{T}{2} \sup_{0\le t\le T} \left\|\frac{\partial^2 \tilde{q}}{\partial z^2}\right\|_{L_2(R^3)} + \int_0^t \int_{R^3} z^2 |\widetilde{q_{kk}}(k,\tau)|^2 dk d\tau.$$

Lemma 20. *Weak solution of problem (11), (12), (13), from Theorem 16 satisfies the following inequalities*

$$|\tilde{q}_{\mathrm{mv}}(z,t)| \le zM_1, \qquad \left|\frac{\partial \tilde{q}_{\mathrm{mv}}(z,t)}{\partial z}\right| \le zM_2,$$

$$\left|\frac{\partial^2 \tilde{q}_{\mathrm{mv}}(z,t)}{\partial z^2}\right| \le zM_3,$$

where M_1, M_2, M_3 *are limited.*

Lemma 21. *Weak solution of problem (11), (12), (13), from Theorem 16 satisfies the following inequalities* $C_i \le const,\ (i = \overline{0,2,4})$, *where*

$$C_0 = \int_0^t |\tilde{F}_1|^2 d\tau, \qquad F_1 = (q,\nabla)q + F,$$

$$C_2 = \int_0^t \left|\frac{\partial \tilde{F}_1}{\partial z}\right|^2 d\tau, \qquad C_4 = \int_0^t \left|\frac{\partial^2 \tilde{F}_1}{\partial z^2}\right|^2 d\tau.$$

Lemma 22. *Suppose that* $q \in R$, $\max_k |\tilde{q}| < \infty$, *then*

$$\int_{R^3} \int_{R^3} \frac{q(x)q(y)}{|x-y|^2} dx dy \le C(|q|_{L_2} + \max_k |\tilde{q}|)^2.$$

Lemma 23. *Weak solution of problem (11), (12), (13), from Theorem 16 satisfies the following inequalities*

$$|\tilde{q}(z(e_k - e_\lambda),t)| \le |\tilde{q}_0(z(e_k - e_\lambda))| +$$

$$+\left(\frac{1}{2\nu}\right)^{\frac{1}{2}} \frac{C_0^{\frac{1}{2}}}{z|e_k - e_\lambda|},$$

where

$$C_0 = \int_0^t |\tilde{F}_1|^2 d\tau, F_1 = (q,\nabla)q + F.$$

Now, we have the uniform estimates of Rolnik norms for the solution of problems (11), (12), (13). Our further and basic aim is to get the uniform estimates $|\tilde{q}_i|_{L_1(R^3)}$, a component of velocity components in the Cauchy problem for Navier-Stokes' equations. In order to achieve the aim, we use Theorem 8 it implies to get estimates of spherical average.

Lemma 24. *Weak solution of problem (11), (12), (13), from Theorem 16 satisfies the following inequalities*

$$|\tilde{q}_{\mathrm{mv}}|_{L_1(R^3)} \le \frac{C}{2}\left(A_0^{(1)} + \beta_1 |\tilde{q}_{\mathrm{mv}}|_{L_1(R^3)}\right) +$$

$$+|\mu|_{L_1(R^3)},$$

the function μ *is defined in Theorem 15,*

$$A_0^{(1)} = \int_{R^3} z \int_0^\pi \int_0^{2\pi} |\tilde{q}_0(z(e_k - e_\lambda))| \times$$

$$\times |\tilde{q}_{\mathrm{mv}}(ze_\lambda,t)| de_\lambda dk, \beta_1 = \left(\frac{1}{\nu}\right)^{\frac{1}{2}} 8\pi C_0^{\frac{1}{2}},$$

and C_0 *is defined in Lemma 23.*

Theorem 17. *Weak solution of problem (11), (12), (13), from Theorem 16 satisfies the following inequalities*

$$\left|\frac{\tilde{q}_{\mathrm{mv}}}{z}\right|_{L_1(R^3)} \le \frac{C}{2}\left(A_0 + \beta_1 \left|\frac{\tilde{q}_{\mathrm{mv}}}{z}\right|_{L_1(R^3)}\right) + \left|\frac{\mu}{z}\right|_{L_1(R^3)},$$

where

$$A_0 = \int_{R^3} \int_0^\pi \int_0^{2\pi} |\tilde{q}_0(z(e_k - e_\lambda))||\tilde{q}_{\mathrm{mv}}(ze_\lambda,t)| de_\lambda dk$$

and β_1 *is defined in Lemma 24.*

Corollary 3. *Weak solution of problem (11), (12), (13), from Theorem 16 satisfies the following inequalities*

$$\left|\frac{\tilde{q}_{\mathrm{mv}}}{z}\right|_{L_1(R^3)} \le \left(\frac{C}{2}A_0 + \left|\frac{\mu}{z}\right|_{L_1(R^3)}\right)K,$$

where

$$K = \frac{\nu^{\frac{1}{2}}}{\nu^{\frac{1}{2}} - 4\pi C C_0^{\frac{1}{2}}}.$$

Let's consider the influence of the following large scale transformations in Navier-Stokes' equation on K

$$t' = tA, \qquad \nu' = \frac{\nu}{A}, \qquad v' = \frac{v}{A}, \qquad F_0' = \frac{F_0}{A^2}.$$

Statement 2. *Let*

$$A = \frac{4}{\nu^{\frac{1}{3}}(CC_0 + 1)^{\frac{2}{3}}},$$

then $K \le \frac{8}{7}$.

Lemma 25. *Weak solution of problem (11), (12), (13), from Theorem 16 satisfies the following inequalities*

$$\left|\frac{\partial \tilde{q}(z(e_k - e_\lambda), t)}{\partial z}\right| \le \left|\frac{\partial \tilde{q}_0(z(e_k - e_\lambda))}{\partial z}\right| +$$
$$+4\alpha\left(\frac{1}{\nu}\right)^{\frac{1}{2}} \frac{C_0^{\frac{1}{2}}}{z^2|e_k - e_\lambda|} + \left(\frac{1}{2\nu}\right)^{\frac{1}{2}} \frac{C_2^{\frac{1}{2}}}{z|e_k - e_\lambda|},$$

where

$$C_2 = \int_0^t \left|\frac{\partial \tilde{F}_1}{\partial z}\right|^2 d\tau.$$

Theorem 18. *Weak solution of problem (11), (12), (13), from Theorem 16 satisfies the following inequalities*

$$\left|\frac{\partial \tilde{q}_{\mathrm{mv}}}{\partial z}\right|_{L_1(R^3)} \le \frac{C}{2}(A_0 + A_1 + A_2 +$$
$$+\beta_3|\tilde{q}_{\mathrm{mv}}|_{L_1(R^3)} + (\beta_1 + \beta_2)\left|\frac{\tilde{q}_{\mathrm{mv}}}{z}\right|_{L_1(R^3)} +$$
$$+\beta_1\left|\frac{\partial \tilde{q}_{\mathrm{mv}}}{\partial z}\right|_{L_1(R^3)}) + \left|\frac{\partial \mu}{\partial z}\right|_{L_1(R^3)},$$

where

$$A_1 = \int_{R^3} z \int_0^\pi \int_0^{2\pi} \left|\frac{\partial \tilde{q}_0(z(e_k - e_\lambda))}{\partial z}\right| \times$$
$$\times |\tilde{q}_{\mathrm{mv}}(ze_\lambda, t)| de_\lambda dk,$$

$$A_2 = \int_{R^3} z \int_0^\pi \int_0^{2\pi} |\tilde{q}_0(z(e_k - e_\lambda))| \times$$
$$\times \left|\frac{\partial \tilde{q}_{\mathrm{mv}}(ze_\lambda, t)}{\partial z}\right| de_\lambda dk,$$

$$\beta_2 = \left(\frac{1}{\nu}\right)^{\frac{1}{2}} 2^{\frac{11}{2}}\pi\alpha C_0^{\frac{1}{2}}, \beta_3 = \left(\frac{1}{\nu}\right)^{\frac{1}{2}} 8\pi C_2^{\frac{1}{2}},$$

and C_2 *is defined in Lemma 25,* $C = const.$

Lemma 26. *Weak solution of problem (11), (12), (13), from Theorem 16 satisfies the following inequalities*

$$\left|\frac{\partial^2 \tilde{q}(z(e_k - e_\lambda), t)}{\partial z^2}\right| \le \left|\frac{\partial^2 \tilde{q}_0(z(e_k - e_\lambda))}{\partial z^2}\right| +$$
$$+\left(\frac{1}{\nu}\right)^{\frac{1}{2}} \frac{16\alpha C_0^{\frac{1}{2}}}{z^3|e_k - e_\lambda|} + \left(\frac{1}{\nu}\right)^{\frac{1}{2}} \frac{8\alpha C_2^{\frac{1}{2}}}{z^2|e_k - e_\lambda|} +$$
$$+\left(\frac{1}{2\nu}\right)^{\frac{1}{2}} \frac{C_4^{\frac{1}{2}}}{z|e_k - e_\lambda|},$$

where

$$\sup_t |t^m e^{-t}| < \alpha,$$

as $m > 0$,

$$C_4 = \int_0^t \left|\frac{\partial^2 \tilde{F}_1}{\partial z^2}\right|^2 d\tau.$$

Theorem 19. *Weak solution of problem (11), (12), (13), from Theorem 16 satisfies the following estimate*

$$\left|z\frac{\partial^2 \tilde{q}_{\mathrm{mv}}}{\partial z^2}\right|_{L_1(R^3)} \le \frac{C}{2}(2(A_1 + A_2 + A_3) +$$
$$+A_4 + A_5 + (2\beta_2 + \beta_4)\left|\frac{\tilde{q}_{\mathrm{mv}}}{z}\right|_{L_1(R^3)} +$$
$$+(2\beta_3 + \beta_5)|\tilde{q}_{\mathrm{mv}}|_{L_1(R^3)} + \beta_6|z\tilde{q}_{\mathrm{mv}}|_{L_1(R^3)} +$$
$$+2(\beta_1 + \beta_2)\left|\frac{\partial \tilde{q}_{\mathrm{mv}}}{\partial z}\right|_{L_1(R^3)} + 2\beta_3\left|z\frac{\partial \tilde{q}_{\mathrm{mv}}}{\partial z}\right|_{L_1(R^3)} +$$
$$+\beta_1\left|z\frac{\partial^2 \tilde{q}_{\mathrm{mv}}}{\partial z^2}\right|_{L_1(R^3)}) + \left|z\frac{\partial^2 \mu}{\partial z^2}\right|_{L_1(R^3)},$$

where

$$A_3 = \int_{R^3} z^2 \int_0^\pi \int_0^{2\pi} \left|\frac{\partial \tilde{q}_0(z(e_k - e_\lambda))}{\partial z}\right| \times$$
$$\times \left|\frac{\partial \tilde{q}_{\mathrm{mv}}(ze_\lambda, t)}{\partial z}\right| de_\lambda dk,$$

$$A_4 = \int_{R^3} z^2 \int_0^\pi \int_0^{2\pi} \left|\frac{\partial^2 \tilde{q}_0(z(e_k - e_\lambda))}{\partial z^2}\right| \times$$
$$\times |\tilde{q}_{\mathrm{mv}}(ze_\lambda, t)| de_\lambda dk,$$

$$A_5 = \int_{R^3} z^2 \int_0^\pi \int_0^{2\pi} |\tilde{q}_0(z(e_k - e_\lambda))| \times$$
$$\times \left|\frac{\partial^2 \tilde{q}_{\mathrm{mv}}(ze_\lambda, t)}{\partial z^2}\right| de_\lambda dk,$$

$$\beta_4 = \left(\frac{1}{\nu}\right)^{\frac{1}{2}} 2^{\frac{15}{2}}\pi\alpha C_0^{\frac{1}{2}},$$

$$\beta_5 = \left(\frac{1}{\nu}\right)^{\frac{1}{2}} 2^{\frac{13}{2}}\pi\alpha C_2^{\frac{1}{2}},$$

$$\beta_6 = \left(\frac{1}{\nu}\right)^{\frac{1}{2}} 8\pi C_4^{\frac{1}{2}},$$

and C_4 *is defined in Lemma 26.*

Lemma 27. *Weak solution of problem (11), (12),* *(13), from Theorem 16 satisfies the following estimate*

$$\left|\frac{\tilde{q}_{\text{mv}}}{z}\right|_{L_1(R^3)} \le B_0 K,$$
$$|\tilde{q}_{\text{mv}}|_{L_1(R^3)} \le B_1 K,$$
$$|z\tilde{q}_{\text{mv}}|_{L_1(R^3)} \le B_2 K,$$

where

$$K = \frac{\nu^{\frac{1}{2}}}{\nu^{\frac{1}{2}} - 4\pi C C_0^{\frac{1}{2}}}, B_0 = \frac{C}{2}A_0 + \left|\frac{\mu}{z}\right|_{L_1(R^3)},$$
$$B_1 = \frac{C}{2}A_0^{(1)} + |\mu|_{L_1(R^3)},$$
$$B_2 = \frac{C}{2}A_0^{(2)} + |z\mu|_{L_1(R^3)},$$
$$A_0^{(2)} = \int_{R^3}\int_0^{\pi}\int_0^{2\pi} z^2 |\tilde{q}_0(z(e_k - e_\lambda))| \times$$
$$\times |\tilde{q}_{\text{mv}}(ze_\lambda, t)| de_\lambda dk.$$

Lemma 28. *Weak solution of problem (11), (12), (13), from Theorem 16 satisfies the following estimates*

$$\left|\frac{\partial \tilde{q}_{\text{mv}}}{\partial z}\right|_{L_1(R^3)} \le D_0 K^2 + D_1 K,$$
$$\left|z\frac{\partial \tilde{q}_{\text{mv}}}{\partial z}\right|_{L_1(R^3)} \le D_2 K^2 + D_3 K,$$

where

$$D_0 = \frac{C}{2}\left(\beta_3^{(0)} B_1 + (\beta_1^{(0)} + \beta_2^{(0)}) B_0\right),$$
$$D_1 = \frac{C}{2}(A_0 + A_1 + A_2) + \left|\frac{\partial \mu}{\partial z}\right|_{L_1(R^3)},$$
$$D_2 = \frac{C}{2}\left(\beta_3^{(0)} B_2 + (\beta_1^{(0)} + \beta_2^{(0)}) B_1\right),$$
$$D_3 = \frac{C}{2}\left(A_0^{(1)} + A_1^{(1)} + A_2^{(1)}\right) + \left|z\frac{\partial \mu}{\partial z}\right|_{L_1(R^3)},$$
$$A_1^{(1)} = \int_{R^3} z^2 \int_0^{\pi}\int_0^{2\pi} \left|\frac{\partial \tilde{q}_0(z(e_k - e_\lambda))}{\partial z}\right| \times$$
$$\times |\tilde{q}_{\text{mv}}(ze_\lambda, t)| de_\lambda dk,$$
$$A_2^{(1)} = \int_{R^3} z^2 \int_0^{\pi}\int_0^{2\pi} |\tilde{q}_0(z(e_k - e_\lambda))| \times$$
$$\times \left|\frac{\partial \tilde{q}_{\text{mv}}(ze_\lambda, t)}{\partial z}\right| de_\lambda dk,$$
$$\beta_1^{(0)} = \frac{8\pi C_0^{\frac{1}{2}}}{\nu^{\frac{1}{2}}}, \beta_2^{(0)} = \frac{2^{\frac{11}{2}}\pi\alpha C_0^{\frac{1}{2}}}{\nu^{\frac{1}{2}}},$$
$$\beta_3^{(0)} = \frac{8\pi C_2^{\frac{1}{2}}}{\nu^{\frac{1}{2}}},$$

Lemma 29. *The solution of the problem (11), (12), (13), from Theorem 16, satisfies the following estimate*

$$\left|z\frac{\partial^2 \tilde{q}_{\text{mv}}}{\partial z^2}\right|_{L_1(R^3)} \le P_0 K^3 + P_1 K^2 + P_2 K,$$

where

$$P_0 = C(\beta_3^{(0)} D_2 + (\beta_1^{(0)} + \beta_2^{(0)}) D_0),$$
$$P_1 = \frac{C}{2}\Big((2\beta_2^{(0)} + \beta_4^{(0)}) B_0 +$$
$$+ (2\beta_3^{(0)} + \beta_5^{(0)}) B_1 + \beta_6^{(0)} B_2 +$$
$$+ 2\beta_3^{(0)} D_3 + 2(\beta_1^{(0)} + \beta_2^{(0)}) D_1\Big),$$
$$P_2 = \frac{C}{2}(2(A_1 + A_2 + A_3) +$$
$$+ A_4 + A_5) + \left|z\frac{\partial^2 \mu}{\partial z^2}\right|_{L_1(R^3)},$$
$$\beta_4^{(0)} = \frac{2^{\frac{15}{2}}\pi\alpha C_0^{\frac{1}{2}}}{\nu^{\frac{1}{2}}}, \beta_5^{(0)} = \frac{2^{\frac{13}{2}}\pi\alpha C_2^{\frac{1}{2}}}{\nu^{\frac{1}{2}}},$$
$$\beta_6^{(0)} = \frac{8\pi C_4^{\frac{1}{2}}}{\nu^{\frac{1}{2}}}.$$

Theorem 20. *The solution of the problem (11), (12), (13), from Theorem 16, satisfies the following estimate*

$$|\tilde{q}|_{L_1(R^3)} \le \left(\gamma_1 C_0 + \gamma_2 C_0^{\frac{1}{2}} C_2^{\frac{1}{2}} + \gamma_3 C_2\right) K^3 +$$
$$+ \left(\gamma_4 C_0^{\frac{1}{2}} + \gamma_5 C_2^{\frac{1}{2}} + \gamma_6 C_4^{\frac{1}{2}}\right) K^2 +$$
$$+ \left(\gamma_7 C_0^{\frac{1}{2}} + \gamma_8 C_2^{\frac{1}{2}} + \gamma_9\right) K,$$

where

$$K = \frac{\nu^{\frac{1}{2}}}{\nu^{\frac{1}{2}} - 4\pi C C_0^{\frac{1}{2}}}, C_0 = \int_0^t |\tilde{F}_1|^2 d\tau,$$
$$F_1 = (q, \nabla) q + F,$$
$$C_2 = \int_0^t \left|\frac{\partial \tilde{F}_1}{\partial z}\right|^2 d\tau, C_4 = \int_0^t \left|\frac{\partial^2 \tilde{F}_1}{\partial z^2}\right|^2 d\tau,$$
$$\gamma_1 = \frac{C^2 2^3 \pi^2}{\nu}(1 + 2^{\frac{5}{2}}) B_0,$$
$$\gamma_2 = \frac{C^2 2^4 \pi^2}{\nu}(1 + 2^{\frac{5}{2}}) B_1,$$
$$\gamma_3 = \frac{C^2 2^3 \pi^2}{\nu} B_2,$$
$$\gamma_4 = \frac{C 2^3 \pi}{\nu^{\frac{1}{2}}}((1 + 2^{\frac{9}{2}}) B_0 + (1 + 2^{\frac{5}{2}}) D_1),$$
$$\gamma_5 = \frac{C 2^3 \pi}{\nu^{\frac{1}{2}}}((1 + 2^{\frac{3}{2}}) B_1 + D_3),$$

$$\gamma_6 = \frac{C2^3\pi}{\nu^{\frac{1}{2}}},$$
$$\gamma_7 = \frac{C2^2\pi}{\nu^{\frac{1}{2}}}(1+2^{\frac{5}{2}})B_0, \gamma_8 = \frac{C2^2\pi}{\nu^{\frac{1}{2}}}B_1,$$
$$\gamma_9 = \frac{C}{2}(D_1+P_2),$$
$$B_0 = \frac{C}{2}A_0 + \left|\frac{\mu}{z}\right|_{L_1(R^3)},$$
$$B_1 = \frac{C}{2}A_0^{(1)} + |\mu|_{L_1(R^3)},$$
$$B_2 = \frac{C}{2}A_0^{(2)} + |z\mu|_{L_1(R^3)},$$
$$D_1 = \frac{C}{2}(A_0+A_1+A_2) + \left|\frac{\partial\mu}{\partial z}\right|_{L_1(R^3)},$$
$$D_3 = \frac{C}{2}\left(A_0^{(1)}+A_1^{(1)}+A_2^{(1)}\right) + \left|z\frac{\partial\mu}{\partial z}\right|_{L_1(R^3)},$$
$$P_2 = \frac{C}{2}(2(A_1+A_2+A_3)+$$
$$+A_4+A_5) + \left|z\frac{\partial^2\mu}{\partial z^2}\right|_{L_1(R^3)},$$
$$\frac{C}{2} = \frac{9\pi}{4(2\pi)^3},$$

the function μ is defined in Theorem 15.

Lemma 30. *The function μ, defined in Theorem 15, satisfies the following estimates*

$$|\mu|_{L_1(R^3)} \le const, \quad |z\mu|_{L_1(R^3)} \le const,$$
$$\left|\frac{\partial\mu}{\partial z}\right|_{L_1(R^3)} \le const,$$
$$\left|z\frac{\partial\mu}{\partial z}\right|_{L_1(R^3)} \le const, \quad \left|z\frac{\partial^2\mu}{\partial z^2}\right|_{L_1(R^3)} \le const.$$

Lemma 31. *Weak solution of problem (11), (12), (13), from Theorem 16 satisfies the following estimates*

$$A_0 \le 2M_1\int_{R^3}(|\tilde{q}_0(ze_k)|)_{\mathrm{mv}}dk,$$
$$A_0^{(1)} \le 2M_1\int_{R^3}z(|\tilde{q}_0(ze_k)|)_{\mathrm{mv}}dk,$$
$$A_0^{(2)} \le 2M_1\int_{R^3}z^2(|\tilde{q}_0(ze_k)|)_{\mathrm{mv}}dk,$$
$$A_1 \le 2M_1\int_{R^3}z(\left|\frac{\partial\tilde{q}_0(ze_k)}{\partial z}\right|)_{\mathrm{mv}}dk,$$
$$A_1^{(1)} \le 2M_1\int_{R^3}z^2(\left|\frac{\partial\tilde{q}_0(ze_k)}{\partial z}\right|)_{\mathrm{mv}}dk,$$
$$A_2 \le 2M_2\int_{R^3}z(|\tilde{q}_0(ze_k)|)_{\mathrm{mv}}dk,$$
$$A_2^{(1)} \le 2M_2\int_{R^3}z^2(|\tilde{q}_0(ze_k)|)_{\mathrm{mv}}dk,$$
$$A_3 \le 2M_2\int_{R^3}z^2(\left|\frac{\partial\tilde{q}_0(ze_k)}{\partial z}\right|)_{\mathrm{mv}}dk,$$
$$A_4 \le 2M_1\int_{R^3}z^2(\left|\frac{\partial^2\tilde{q}_0(ze_k)}{\partial z^2}\right|)_{\mathrm{mv}}dk,$$
$$A_5 \le 2M_3\int_{R^3}z^2(|\tilde{q}_0(ze_k)|)_{\mathrm{mv}}dk.$$

Theorem 21. *Suppose that*

$$q_0 \in W_2^1(R^3), F_0 \in L_2(Q_T),$$
$$\tilde{F}_0 \in L_1(Q_T), \frac{\partial\tilde{F}_0}{\partial z} \in L_1(Q_T),$$
$$\frac{\partial^2\tilde{F}_0}{\partial z^2} \in L_1(Q_T), \tilde{q}_0 \in L_1(R^3),$$
$$I_j = \int_{R^3}z^{j-1}(|\tilde{q}_0(ze_k)|)_{\mathrm{mv}}dk \le const,$$
$$(j=\overline{1,3}),$$
$$I_j = \int_{R^3}z^{j-3}(\left|\frac{\partial\tilde{q}_0(ze_k)}{\partial z}\right|)_{\mathrm{mv}}dk \le const,$$
$$(j=\overline{4,5}),$$
$$I_6 = \int_{R^3}z^2(\left|\frac{\partial^2\tilde{q}_0(ze_k)}{\partial z^2}\right|)_{\mathrm{mv}}dk \le const.$$

Then there exists a unique weak solution of (11), (12), (13), satisfying the following inequalities

$$\max_t\sum_{i=1}^3|\tilde{q}_i|_{L_1(R^3)} \le const,$$

where $const$ depends only on the theorem conditions.

Note. In the estimate for $\tilde{q}$ the condition $q(0) > 1$ is used. This conditioncan be obviated if we use smooth and bounded function w and make all the estimates for $q_1 = q + w$ such that $q_1(0) > 1$ is satisfied. Using the function w, we also choose the constant A concordant with the constant ε from Lemma 3.

Theorem 21 proves the global solvability and unicity of the Cauchy problem for Navier-Stokes' equation.

10 Conclusion

In Introduction we mentioned the authors whose scientific researches we consider appropriate to call the prehistory of this work. The list of these authors may be considerably extended if we enumerate all the predecessors diachronically or by the significance of their contribution into this research. Actually we intended to obtain evident results which were directly and indirectly indicated by these authors in their scientific works. We do not concentrate on the solution to the multi-dimensional problem of quantum scattering theory although it follows from some certain statements proved in this work. In fact, the problem of overdetermination in the multi-dimensional inverse problem of quantum scattering theory is obviated since a potential can be defined by amplitude averaging when the amplitude is a function of three variables. In the classic case of the multi-dimensional inverse problem of quantum scattering theory the potential requires restoring with respect to the amplitude that depends on five variables. This obviously leads to the problem of overdetermination. Further detalization could have distracted us from the general research line of the work consisting in application of energy and momentum conservation laws in terms of wave functions to the theory of nonlinear equations. This very method we use in solving the problem of the century, the problem of solvability of the Cauchy problem for Navier-Stokes' equations of viscous incompressible fluid. Let us also note the importance of the fact that the laws of momentum and energy conservation in terms of wave functions are conservation laws in the microworld; but in the classic methods of studying nonlinear equations scientists usually use the priori estimates reflecting the conservation laws of macroscopic quantities. We did not focus attention either on obtaining exact estimates dependent on viscosity, lest the calculations be complicated. However, the pilot analysis shows the possibility of applying these estimates to the problem of limiting viscocity transition tending to zero.

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