Stability Analysis for Neo-Hookean Machine-Tools Foundations

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Abstract: - It is well known that the vibrations are a major cause for the instability in the mechanical systems and major source of noises. In this paper we propose a simple system with two degrees of freedom based on a non-linear elastic element and the hypothesis for the coefficients of the elastic force.. For this system, it is proved in the present paper that the motion is stable, but not asymptotically stable. A comparison between the non-linear case and the linear case is performed, and for the both cases the eigenpulsations are also determined. All theoretical results are validated by numerical simulation. Finally, we considered the general case.

Key-Words: - neo-Hookean, motion, stability, numerical validation.

1 Introduction

The system purposed for the study is described in figure 1, *a*. It consists of the masses m_1 and m_2 linked one to another by the linear spring of stiffness k. The mass m_1 can be considered to be the foundation of the machine-tool, and the mass m_2 the machine-tool itself. The mass m_2 is linked to the ground by the non-linear spring 1 for which the elastic force writes

$$F = k_1 x - \frac{\varepsilon_1}{x^2}, \quad (1)$$

where x is the elongation of the spring.

The fundamental working hypothesis is that

$$\varepsilon_1 > 0. \quad (2)$$

The system has two degrees of freedom, that is the displacements z_1 and z_2 of the two masses in the vertical direction.

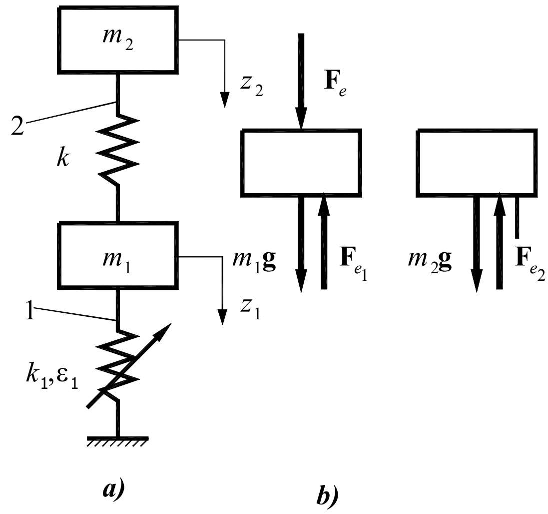


Fig. 1. The mathematical model.

2 The equations of motion

Isolating the two masses (fig. 1, *b*), one obtains the differential equations of the motion

$$m_1 \ddot{z}_1 = -k_1 z_1 + \frac{\varepsilon_1}{z_1^2} + k(z_2 - z_1) + m_1 g,$$

$$m_2 \ddot{z}_2 = m_2 g - k(z_2 - z_1), \quad (3)$$

where g is the gravitational acceleration.

Denoting

$$\xi_1 = z_1,\ \xi_2 = z_2,\ \xi_3 = \dot{z}_1,\ \xi_4 = \dot{z}_2, \quad (4)$$

the relations (3) transform in a system of four first order non-linear differential equations,

$$\dot{\xi}_1 = \xi_3,\ \dot{\xi}_2 = \xi_4,$$

$$\dot{\xi}_3 = \frac{1}{m_1}\left[-k_1\xi_1 + \frac{\varepsilon_1}{\xi_1^2} + k(\xi_2 - \xi_1) + m_1 g\right],$$

$$\dot{\xi}_4 = \frac{1}{m_2}\left[m_2 g - k(\xi_2 - \xi_1)\right]. \quad (5)$$

3 The equilibrium positions

These positions found at the intersections of the nullclines, so that one obtains the system

$$\xi_3 = 0,\ \xi_4 = 0,\ -k_1\xi_1 + \frac{\varepsilon_1}{\xi_1^2} + k(\xi_2 - \xi_1) + m_1 g = 0,$$

$$m_2 g - k(\xi_2 - \xi_1) = 0. \quad (6)$$

Summing the last two relations (6), it results

$$-k_1\xi_1+\frac{\varepsilon_1}{\xi_1^2}+(m_1+m_2)g=0\,, \qquad (7)$$

wherefrom

$$k_1\xi_1^3-(m_1+m_2)g\xi_1^2-\varepsilon_1=0\,. \qquad (8)$$

In the sequence of the coefficients of powers of ξ_1 in the relation (8) there exists only one variation of sign such that applying the Descartes theorem one deduces that the equation (8) has only one positive real root. Making now $\xi_1 \mapsto -\xi_1$, one obtains the equation

$$k_1\xi_1^3+(m_1+m_2)g\xi_1^2+\varepsilon_1=0 \qquad (9)$$

for which there exists no variation of sigh in the sequence of the coefficients so that the Descartes theorem assures us that we have no negative real root for the equation (8). In conclusion, the equation (8) has exactly one positive real root, name it $\overline{\xi}_1$.

The last relation (6) becomes now a linear equation in the unknown ξ_2 and therefore it has only one solution,

$$\overline{\xi}_2=\frac{m_2g}{k}+\overline{\xi}_1\,. \qquad (10)$$

We proved in this way that the system has only one equilibrium position $\left(\overline{\xi}_1,\overline{\xi}_2,0,0\right)$.

Let us denote by $f(\xi_1)$ the function $f:\mathbb{R}\to\mathbb{R}$,

$$f(\xi_1)=k_1\xi_1^3-(m_1+m_2)g\xi_1^2-\varepsilon_1 \qquad (11)$$

for which the derivative is

$$f'(\xi_1)=3k_1\xi_1^2-2(m_1+m_2)g\xi_1\,. \qquad (12)$$

The equation $f'(\xi_1)=0$ has the solutions

$$\xi_1^{(1)}=0\,,\ \xi_1^{(2)}=\frac{2(m_1+m_2)g}{3k_1}\,, \qquad (13)$$

$\xi_1^{(1)}$ being a point of maximum, and $\xi_1^{(2)}$ a point of minimum. In addition,

$$f(0)=-\varepsilon_1<0\,. \qquad (14)$$

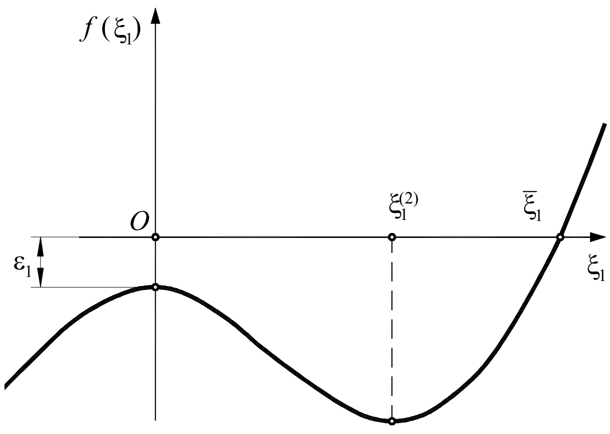


Fig. 2. The graphic of the function $f(\xi_1)$.

Graphically, the situation is presented in figure 2. It follows from here that

$$\overline{\xi}_1>\frac{2(m_1+m_2)g}{3k_1}\,. \qquad (15)$$

4 The stability of the equilibrium

Let us denote by f_k, $k=\overline{1,4}$ the right-hand terms of the relations (5) and by j_{kl} the partial derivatives

$$j_{kl}=\frac{\partial f_k}{\partial \xi_l}\,,\ k=\overline{1,4}\,,\ l=\overline{1,4}\,. \qquad (16)$$

We have

$$j_{11}=0\,,\ j_{12}=0\,,\ j_{13}=1\,,\ j_{14}=0\,, \qquad (17)$$

$$j_{21}=0\,,\ j_{22}=0\,,\ j_{23}=0\,,\ j_{24}=1\,, \qquad (18)$$

$$j_{31}=\frac{1}{m_1}\left(-k_1-k-\frac{2\varepsilon_1}{\overline{\xi}_1^3}\right),0\ j_{32}=\frac{k}{m_1}\,,\ j_{33}=0\,,$$

$$j_{34}=0\,, \qquad (19)$$

$$j_{41}=\frac{k}{m_2}\,,\ j_{42}=-\frac{k}{m_2}\,,\ j_{43}=0\,,\ j_{44}=0\,. \qquad (20)$$

The characteristic equation

$$\det(\mathbf{J}-\lambda\mathbf{I})=0\,, \qquad (21)$$

in which $\mathbf{J}$ is the Jacobi matrix, $\mathbf{J}=\left[j_{kl}\right]_{\substack{k=\overline{1,4}\\ l=\overline{1,4}}}$, and $\mathbf{I}$ is the fourth order unity matrix, reads

$$\begin{vmatrix}-\lambda & 0 & 1 & 0\\ 0 & -\lambda & 0 & 1\\ j_{31} & j_{32} & -\lambda & 0\\ j_{41} & j_{42} & 0 & -\lambda\end{vmatrix}=0\,. \qquad (22)$$

Multiplying the third column by λ and adding it to the first column, the fourth column by λ and adding it to the second column, results the equation

$$\lambda^4-(j_{31}+j_{42})\lambda^2+j_{31}j_{42}-j_{32}j_{41}=0\,. \qquad (23)$$

From the Routh–Hurwitz criterion we deduce that the equation has not all the roots with negative real part and therefore the equilibrium can not be asymptotically stable.

On the other hand, the roots of the equation (23) are

$$\lambda_1^2=\frac{j_{31}+j_{42}\pm\sqrt{(j_{31}-j_{42})^2+4j_{32}j_{41}}}{2}\,. \qquad (24)$$

Keeping into account the expressions (17)-(20), we have

$$j_{31}+j_{42}=-\frac{k_1+k+\frac{2\varepsilon_1}{\overline{\xi}_1^3}}{m_1}-\frac{k}{m_2}<0\,, \qquad (25)$$

$$4j_{32}j_{41} = \frac{4k^2}{m_1 m_2} > 0, \quad (26)$$

$$(j_{31} - j_{42})^2 + 4j_{32}j_{41} > 0. \quad (27)$$

More,

$$|j_{31} + j_{42}| > \sqrt{(j_{31} - j_{42})^2 + 4j_{32}j_{41}} \quad (28)$$

ecause it is equivalent to

$$j_{31}j_{42} > j_{32}j_{41}, \quad (29)$$

that is

$$\frac{k_1 k}{m_1 m_2} + \frac{k^2}{m_1 m_2} + \frac{2\varepsilon_1}{\xi_1^3}\frac{k}{m_2} > \frac{k^2}{m_1 m_2}. \quad (30)$$

The previous relations assure us that $\lambda_1^2 < 0$, $\lambda_2^2 < 0$ so that the roots of the characteristic equation (23) are all pure imaginary. The equilibrium position $(\overline{\xi}_1, \overline{\xi}_2, 0, 0)$ is simply stable.

5 The stability of the motion

Let $(\xi_1, \xi_2, \xi_3, \xi_4)$ a solution of the system (5) and (u_1, u_2, u_3, u_4) a deviation sufficiently small in its norm. We can write

$$\dot{\xi}_1 + \dot{u}_1 = \xi_3 + u_3, \ \dot{\xi}_2 + \dot{u}_2 = \xi_4 + u_4,$$

$$\dot{\xi}_3 + \dot{u}_3 = \frac{1}{m_1}\Bigg[-k_1(\xi_1 + u_1) + \frac{\varepsilon_1}{(\xi_1 + u_1)^2},$$
$$+ k(\xi_2 - \xi_1 + u_2 - u_1) + m_1 g\Bigg]$$

$$\dot{\xi}_4 + \dot{u}_4 = \frac{1}{m_2}[m_2 g - k(\xi_2 - \xi_1 + u_2 - u_1)]. \quad (31)$$

Since $u_1 << \xi_1$ we can approximate

$$\frac{\varepsilon_1}{(\xi_1 + u_1)^2} \approx \frac{\varepsilon_1}{\xi_1^2} - \frac{2\varepsilon_1}{\xi_1^3}u_1. \quad (32)$$

Keeping into account that $(\xi_1, \xi_2, \xi_3, \xi_4)$ is a solution of the system (5), from the relations (31) and (32) we obtain the system in deviations

$$\dot{u}_1 = u_3, \ \dot{u}_2 = u_4,$$

$$\dot{u}_3 = \frac{1}{m_1}\left[-k_1 u_1 - \frac{2\varepsilon_1 u_1}{\xi_1^3} + k(u_2 - u_1)\right],$$

$$\dot{u}_4 = \frac{1}{m_2}[-k(u_2 - u_1)], \quad (33)$$

wherefrom

$$m_1\ddot{u}_1 = -k_1 u_1 - \frac{2\varepsilon_1 u_1}{\xi_1^3} + k(u_2 - u_1),$$

$$m_2\ddot{u}_2 = -ku_2 + ku_1. \quad (34)$$

From the second relation (34) we find

$$u_1 = \frac{m_2\ddot{u}_2}{k} + u_2, \ \ddot{u}_1 = \frac{m_2 u_2^{(iv)}}{k} + \ddot{u}_2, \quad (35)$$

the first relation (34) offering now

$$\frac{m_1 m_2}{k}u_2^{(iv)} + \left[\frac{m_2}{k}\left(k_1 + k + \frac{2\varepsilon_1}{\xi_1^3}\right) + m_1\right]\ddot{u}_2 + ku_2 = 0. \quad (36)$$

The characteristic equation reads now

$$\frac{m_1 m_2}{k}r^4 + \left[\frac{m_2}{k}\left(k_1 + k + \frac{2\varepsilon_1}{\xi_1^3}\right) + m_1\right]r^2 + k = 0. \quad (37)$$

The discriminate of this equation is

$$\Delta = \left[\frac{m_2}{k}\left(k_1 + k + \frac{2\varepsilon_1}{\xi_1^3}\right) + m_1\right]^2 - 4m_1 m_2$$
$$= (m_2 - m_1)^2 + \left[\frac{m_2}{k}(k_1 + k)\right]^2 \quad (38)$$
$$+ 2(m_2 + m_1)\frac{m_2}{k}\left(k_1 + \frac{2\varepsilon_1}{\xi_1^3}\right) > 0$$

and, in addition,

$$\Delta < \left[\frac{m_2}{k}\left(k_1 + k + \frac{2\varepsilon_1}{\xi_1^3}\right) + m_1\right]^2. \quad (39)$$

Keeping into account that

$$a = \frac{m_2}{k}\left(k_1 + k + \frac{2\varepsilon_1}{\xi_1^3}\right) + m_1 > 0 \quad (40)$$

it immediately results that $r_1^2 < 0$, $r_2^2 < 0$ so that the roots of the characteristic equation (37) are all pure imaginary, the motion being stable, but not asymptotically stable.

The solution of the equation (36) is

$$u_2 = C_1\cos\left(\sqrt{\frac{\frac{a - \sqrt{\Delta}}{2k}}{m_1 m_2}}t + \varphi_1\right) + C_2\cos\left(\sqrt{\frac{\frac{a + \sqrt{\Delta}}{2k}}{m_1 m_2}}t + \varphi_2\right). \quad (41)$$

By twice derivation of the expression (41), we obtain

$$\ddot{u}_2 = -\frac{a-\sqrt{\Delta}}{\frac{2k}{m_1 m_2}} C_1 \cos\left(\sqrt{\frac{a-\sqrt{\Delta}}{\frac{2k}{m_1 m_2}}}t + \varphi_1\right) - \frac{a+\sqrt{\Delta}}{\frac{2k}{m_1 m_2}} C_2 \cos\left(\sqrt{\frac{a+\sqrt{\Delta}}{\frac{2k}{m_1 m_2}}}t + \varphi_2\right) \quad (42)$$

and from the first relation (35) it results

$$u_1 = \left(-\frac{m_2}{k}\frac{a-\sqrt{\Delta}}{\frac{2k}{m_1 m_2}} + 1\right) C_1 \cos\left(\sqrt{\frac{a-\sqrt{\Delta}}{\frac{2k}{m_1 m_2}}}t + \varphi_1\right) + \left(-\frac{m_2}{k}\frac{a+\sqrt{\Delta}}{\frac{2k}{m_1 m_2}} + 1\right) C_2 \cos\left(\sqrt{\frac{a+\sqrt{\Delta}}{\frac{2k}{m_1 m_2}}}t + \varphi_2\right) \quad (43)$$

Everywhere C_1, C_2, φ_1 and φ_2 are constants of integration, which result from the initial conditions $u_1(0) = u_{10}$, $u_2(0) = u_{20}$, $\dot{u}_1(0) = \dot{u}_{10}$, $\dot{u}_2(0) = \dot{u}_{20}$. The expressions (41) and (43) approximate the solution of the system in deviations (31).

6 The small oscillations around the equilibrium position

These can be obtained as a particular case of the previous paragraph for

$$\xi_1 = \overline{\xi}_1 . \quad (44)$$

Result the eigenpulsations

$$\omega_1 = \left\{\left\{\frac{m_2}{k}\left(k_1 + k + \frac{2\varepsilon_1}{\overline{\xi}_1^3}\right) + m_1 - \left\{\left[\frac{m_2}{k}\left(k_1 + k + \frac{2\varepsilon_1}{\overline{\xi}_1^3}\right) + m_1\right]^2 - 4m_1 m_2\right\}^{\frac{1}{2}}\right\} \Big/ \left(\frac{2k}{m_1 m_2}\right)\right\}^{\frac{1}{2}},$$

$$\omega_2 = \left\{\left\{\frac{m_2}{k}\left(k_1 + k + \frac{2\varepsilon_1}{\overline{\xi}_1^3}\right) + m_1 + \left\{\left[\frac{m_2}{k}\left(k_1 + k + \frac{2\varepsilon_1}{\overline{\xi}_1^3}\right) + m_1\right]^2 - 4m_1 m_2\right\}^{\frac{1}{2}}\right\} \Big/ \left(\frac{2k}{m_1 m_2}\right)\right\}^{\frac{1}{2}}. \quad (45)$$

7 Comparison with the linear case

The linear case is obtained for $\varepsilon_1 = 0$.

The equation (7) writes

$$-k_1\xi_1 + (m_1 + m_2)g = 0 , \quad (46)$$

with the solution

$$\overline{\xi}_1^{(l)} = \frac{m_1 + m_2}{k_1} g . \quad (47)$$

One observes that $\overline{\xi}_1^{(l)} < \overline{\xi}_1$ for which holds true the relation (15).

The relation (10) offers

$$\overline{\xi}_2^{(l)} = \frac{m_2 g}{k} + \frac{m_1 + m_2}{k_1} g \quad (48)$$

and therefore $\overline{\xi}_2^{(l)} < \overline{\xi}_2$, too.

The equilibrium remains again simply stable because the relations (25)-(30) still hold true.

The motion is again simply stable and we have in addition

$$\Delta^{(l)} = \left(m_2 + m_1 + m_2\frac{k_1}{k}\right)^2 - 4m_1 m_2 = (m_2 - m_1)^2 + \left(m_2\frac{k_1}{k}\right)^2 + 2(m_2 + m_1)m_2\frac{k_1}{k}, \quad (49)$$

$$a^{(l)} = m_2 + m_1 + \frac{m_2 k_1}{k}, \quad (50)$$

with

$$\Delta^{(l)} > 0 ,\quad \Delta^{(l)} < \Delta ,\ a^{(l)} > 0 ,\ a^{(l)} < a . \quad (51)$$

The eigenpulsations are

$$\omega_1^{(l)} = \Bigg\{ m_2 + m_1 + \frac{m_2 k_1}{k_2} - \left[\left(m_2 + m_1 + \frac{m_2 k_1}{k_2} \right)^2 , - 4 m_1 m_2 \right]^{\frac{1}{2}} \Bigg/ \left(\frac{2k}{m_1 m_2} \right) \Bigg\}^{\frac{1}{2}}$$

$$\omega_2^{(l)} = \Bigg\{ m_2 + m_1 + \frac{m_2 k_1}{k_2} + \left[\left(m_2 + m_1 + \frac{m_2 k_1}{k_2} \right)^2 . - 4 m_1 m_2 \right]^{\frac{1}{2}} \Bigg/ \left(\frac{2k}{m_1 m_2} \right) \Bigg\}^{\frac{1}{2}} \quad (52)$$

8 Numerical application

Let us consider the case

$$m_1 = 2000\,\text{kg},\ m_2 = 1000\,\text{kg},\ g = 10\,\text{m/s}^2,$$
$$k_1 = 10^6\ \text{N/m},\ \varepsilon_1 = 700\ \text{Nm}^2,$$
$$k = 10^5\ \text{N/m}. \quad (53)$$

The equation (8) leads us to

$$10^6 \xi_1^3 - 3000 \times 10 \xi_1^2 - 700 = 0 \quad (54)$$

with the solution

$$\xi_1 = 0.1\,\text{m}. \quad (55)$$

The relation (10) offers us

$$\xi_2 = \frac{1000 \times 10}{10^5} + 0.1 = 0.2\ \text{m}. \quad (56)$$

The expression (15) assures us that

$$0.1\,\text{m} > \frac{2 \times (2000 + 1000) \times 10}{3 \times 10^6} = 0.02\ \text{m}. \quad (57)$$

The equations (17)-(20) lead us to

$$j_{31} = \frac{1}{2000} \times \left(-10^6 - 10^5 - \frac{2 \times 700}{0.1^3} \right) = -1250,$$
$$j_{32} = \frac{10^5}{2000} = 50,\ j_{41} = \frac{10^5}{1000} = 100,$$
$$j_{42} = -\frac{10^5}{2000} = -50, \quad (58)$$

the roots of the characteristic equation (23) being given by (24)

$$\lambda_1^2 = -45.848,\ \lambda_2^2 = -1254.152. \quad (59)$$

The parameters a and Δ are

$$a = 27000, \quad (60)$$

$$\Delta = 721000000. \quad (61)$$

The eigenpulsations read

$$\omega_1 = \sqrt{\frac{27000 - \sqrt{721000000}}{\frac{2 \times 10^5}{2000 \times 1000}}} = 38.543\,\text{s}^{-1},$$
$$\omega_2 = \sqrt{\frac{27000 + \sqrt{721000000}}{\frac{2 \times 10^5}{2000 \times 1000}}} = 733.835\,\text{s}^{-1}. \quad (62)$$

In the linear case we have

$$\xi_1^{(l)} = \frac{2000 + 1000}{10^6} \times 10 = 0.03\,\text{m},$$
$$\xi_2^{(l)} = \frac{1000 \times 10}{10^5} + \frac{2000 + 1000}{10^6} \times 10 = 0.13\,\text{m}, \quad (63)$$

$$a^{(l)} = 1000 + 2000 + \frac{1000 \times 10^6}{10^5} = 13000, \quad (64)$$

$$\Delta^{(l)} = \left(1000 + 2000 + 1000 \times \frac{10^6}{10^5} \right)^2, - 4 \times 2000 \times 1000 = 161000000 \quad (65)$$

$$\omega_1^{(l)} = \sqrt{\frac{13000 - \sqrt{161000000}}{\frac{2 \times 10^5}{2000 \times 1000}}} = 55.805\,\text{s}^{-1},$$
$$\omega_2^{(l)} = \sqrt{\frac{13000 + \sqrt{161000000}}{\frac{2 \times 10^5}{2000 \times 1000}}} = 506.839\,\text{s}^{-1}. \quad (66)$$

One observes that

$$\omega_1 < \omega_1^{(l)},\ \omega_2 > \omega_2^{(l)}, \quad (67)$$

so that the non-linearity has as effect the increasing of the domain of pulsations where the resonance doesn't appear.

9 The general case

We shall recall the equation (8). If

$$\varepsilon_1 < 0, \quad (68)$$

then in the sequence of the coefficients we have two variations of sign and, according to the Descartes theorem, the equation (8) can have zero or two positive roots. Making $\xi_1 \mapsto -\xi_1$ and recalling the equation (9) we have only one variation of sign in the sequence of its coefficients. Applying again the Descartes theorem, it follows that the equation (8) has exactly one negative root.

The graphic of the function $f : \mathbb{R} \to \mathbb{R}$ defined by the relation (11) is captured in the figure 3 and in the figure 4.

The expression of $f\left(\xi_1^{(2)}\right)$ is given by the relation (13) and we obtain

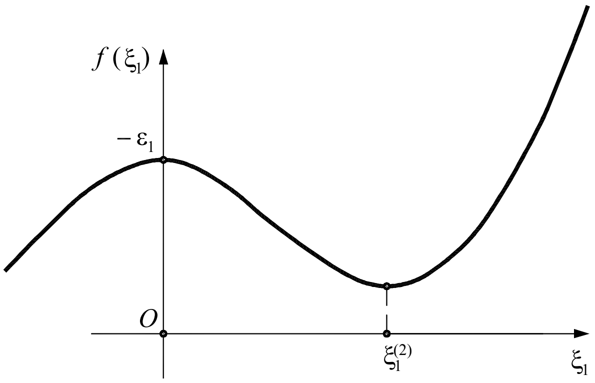


Fig. 3. The graphic of the function $f(\xi_1)$ in the case $f\left(\xi_1^{(2)}\right) > 0$ and $\varepsilon_1 < 0$.

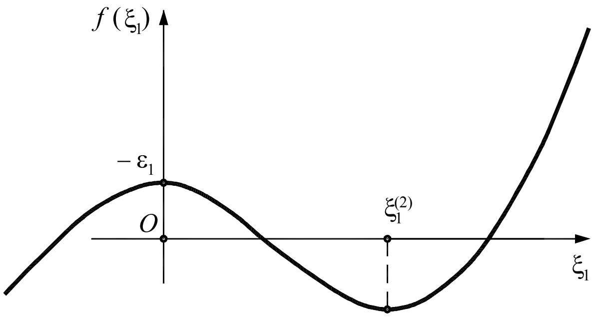


Fig. 4. The graphic of the function $f(\xi_1)$ in the case $f\left(\xi_1^{(2)}\right) < 0$ and $\varepsilon_1 < 0$.

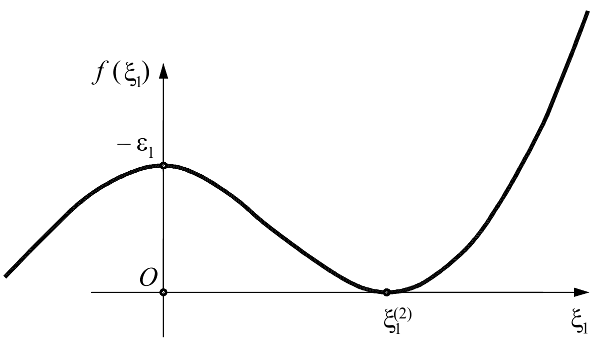


Fig. 5. The graphic of the function $f(\xi_1)$ in the case $f\left(\xi_1^{(2)}\right) = 0$ and $\varepsilon_1 < 0$.

$$f\left(\xi_1^{(2)}\right) = -\frac{4(m_1 + m_2)^3 g^3}{27k_1^2} - \varepsilon_1 . \quad (69)$$

The condition

$$f\left(\xi_1^{(2)}\right) > 0 \quad (70)$$

implies

$$\varepsilon_1 < -\frac{4(m_1 + m_2)^3 g^3}{27k_1^2} . \quad (71)$$

In conclusion, if the relation (71) is fulfilled, then the equation (8) has only one negative root and the situation is captured in the figure 3. if the relation (71) is not satisfied, then the equation (8) has three real roots, two of them being positive and one negative, the situation being described by the figure 4.

There exists a particular case when

$$\varepsilon_1 = -\frac{4(m_1 + m_2)^3 g^3}{27k_1^2} , \quad (72)$$

and $f'\left(\xi_1^{(2)}\right) = 0$, $f\left(\xi_1^{(2)}\right) = 0$. In this situation the equation (8) has three real roots, one negative and two positive, the two positive roots being equal. Graphically this case is drawn in the figure 5.

Further on, we shall analyze the stability of the equilibrium positions.

Again, the characteristic equation is given by the relation (23), where

$$j_{31} = -\frac{k_1 + k}{m_1} - \frac{2\varepsilon_1}{m_1 \xi_1^3} , \quad (73)$$

$$j_{32} = \frac{k}{m_1} , \quad (74)$$

$$j_{41} = \frac{k}{m_2} , \quad (75)$$

$$j_{42} = -\frac{k}{m_2} . \quad (76)$$

We have

$$j_{31} + j_{42} = -\frac{k_1 + k}{m_1} - 2\frac{\varepsilon_1}{m_1 \xi_1^3} - \frac{k}{m_2} , \quad (77)$$

$$j_{31} j_{42} - j_{32} j_{41} = \frac{k}{m_1 m_2}\left(k_1 + \frac{2\varepsilon_1}{\xi_1^3}\right) . \quad (78)$$

The discriminate of the equation (23) is given by

$$\Delta = (j_{31} - j_{42})^2 + 4 j_{32} j_{41} \quad (79)$$

and, keeping into account that

$$(j_{31} - j_{42})^2 > 0 , \quad (80)$$

$$j_{32} j_{41} = \frac{k^2}{m_1 m_2} > 0 , \quad (81)$$

it results that $\Delta > 0$ and therefore the characteristic equation (23) has always two real roots λ_1^2 and λ_2^2.

For the stability is necessary and sufficient that both λ_1^2 and λ_2^2 to be negative. This implies

$$j_{31} + j_{42} < 0 \quad (82)$$

and

$$j_{31} j_{42} - j_{32} j_{41} > 0 . \quad (83)$$

The condition (82)leads us to

$$\frac{2\varepsilon_1}{m_1\xi_1^3} > -\frac{k_1 + k}{m_1} - \frac{k}{m_2}. \quad (84)$$

The condition (83) implies

$$k_1 + \frac{2\varepsilon_1}{\xi_1^3} > 0. \quad (85)$$

Let us denote by $\overline{\xi}_1$, $\overline{\overline{\xi}}_1$ and $\overline{\overline{\overline{\xi}}}_1$ the possible three real roots of the equation (8), where $\overline{\xi}_1 < 0$, $\overline{\overline{\xi}}_1 > 0$, $\overline{\overline{\overline{\xi}}}_1 > 0$, $\overline{\overline{\xi}}_1 \le \overline{\overline{\overline{\xi}}}_1$.

For the root $\overline{\xi}_1$ we have

$$\frac{2\varepsilon_1}{\xi_1^3} > 0 \quad (86)$$

and the conditions (84) and (85) are fulfilled; therefore $\overline{\xi}_1$ is always a stable equilibrium position.

It is easy to calculate that in the situation described the figure 5, the inequality (87) becomes

$$-\frac{4(m_1 + m_2)^3 g^3}{27k_1^2} > -\frac{8h_1(m_1 + m_2)^3 g^3}{27 \cdot 2k_1^2}, \quad (87)$$

which is obviously false; the equilibrium position $\overline{\overline{\xi}}_1 = \overline{\overline{\overline{\xi}}}_1$ is unstable.

For the case of three different real roots we have

$$\overline{\overline{\overline{\xi}}}_1 > \xi_1^{(2)} = \frac{2(m_1 + m_2)g}{3k_1}, \quad (88)$$

$$\frac{k_1\overline{\overline{\overline{\xi}}}_1^3}{2} > \frac{4(m_1 + m_2)^3 g^3}{27k_1^3}, \quad (89)$$

$$\varepsilon_1 < \frac{4(m_1 + m_2)^3}{27k_1^2} \quad (90)$$

and the condition (85) offers us (the condition (84) is useless now)

$$-\varepsilon_1 < \frac{4(m_1 + m_2)^3 g^3}{27k_1^2} < \frac{k_1\overline{\overline{\overline{\xi}}}_1^3}{2}. \quad (91)$$

It results that $\overline{\overline{\overline{\xi}}}_1$ is a stable equilibrium position.

Let us consider the root $\overline{\overline{\xi}}_1$. The equation (8) can be written as

$$\varepsilon_1 = k_1\overline{\overline{\xi}}_1^3 - (m_1 + m_2)g\overline{\overline{\xi}}_1^2. \quad (92)$$

The condition (85) lead us to (again the condition (84) is useless)

$$-\frac{k_1\overline{\overline{\xi}}_1^3}{2} < k_1\overline{\overline{\xi}}_1^3 - (m_1 + m_2)g\overline{\overline{\xi}}_1^2, \quad (93)$$

wherefrom

$$\frac{3k_1\overline{\overline{\xi}}_1^3}{2} > (m_1 + m_2)g\overline{\overline{\xi}}_1^2, \quad (94)$$

$$\overline{\overline{\xi}}_1 > \frac{2(m_1 + m_2)g}{3k_1} = \xi_1^{(2)}, \quad (95)$$

which is absurd. The equilibrium position $\overline{\overline{\xi}}_1$ is always unstable.

For the stability of the motion we obtain the same equation (37). Denoting now

$$a = \frac{\frac{m_2}{k}\left(k_1 + k + \frac{2\varepsilon_1}{\xi_1^3}\right) + m_1}{\frac{m_1 m_2}{k}}, \quad (96)$$

$$b = \frac{k}{\frac{m_1 m_2}{k}}, \quad (97)$$

it results the equation

$$r^4 + ar^2 + b = 0, \quad (98)$$

or

$$p^2 + ap + b = 0, \quad (99)$$

the notation being obvious.

The equation (99) has negative real roots if and only if

$$\Delta = a^2 - 4b > 0, \quad (100)$$

$$a > 0, \quad (101)$$

and

$$b > 0. \quad (102)$$

Recalling the relations (96) and (97), one can easily observe that the condition (102) is always true.

The only conditions for the stability of the motion are (100) and (101).

10 Numerical analysis

Let us consider the following three numerical cases (for the simplicity of the calculation, we shall take the gravitational acceleration $g = 10\ \text{m/s}^2$).

The first case is defined by

$$m_1 = 2000\ \text{kg},\ m_2 = 1000\ \text{kg},\ g = 10\ \text{m/s}^2,$$
$$k_1 = 10^6\ \text{N/m},\ \varepsilon_1 = 700\ \text{Nm}^2,\ k = 10^5\ \text{N/m} \quad (102)$$

and the initial conditions

$$\xi_1^0 = 0.11\ \text{m},\ \xi_2^0 = 0.19\ \text{m},\ \xi_3^0 = 0\ \text{m/s},$$
$$\xi_4^0 = 0\ \text{m/s}. \quad (103)$$

The second case is characterized by

$$m_1 = 2000\ \text{kg},\ m_2 = 1000\ \text{kg},\ k_1 = 4 \cdot 10^5\ \text{N/m},$$
$$k = 10^5\ \text{N/m},\ \varepsilon_1 = -700\ \text{Nm}^2 \quad (104)$$

and the initial conditions

$$\xi_1^0 = -0.11\ \text{m},\ \xi_2^0 = -0.01\ \text{m},\ \xi_3^0 = 0\ \text{m/s},$$
$$\xi_4^0 = 0\ \text{m/s}. \quad (105)$$

Finally, the third case is described by

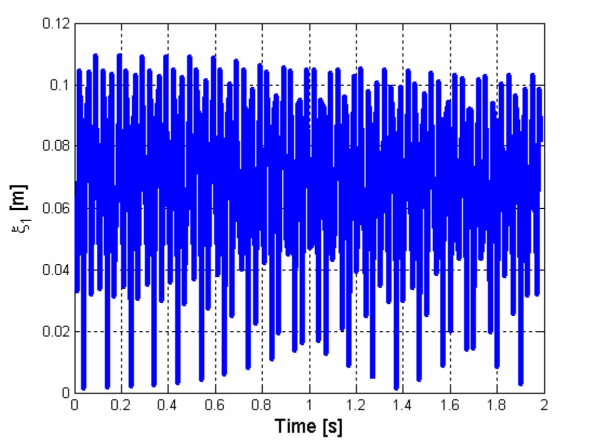
$m_1 = 2000\,\text{kg}$, $m_2 = 1000\,\text{kg}$, $k_1 = 4 \cdot 10^5\ \text{N/m}$,
$k = 10^5\ \text{N/m}$, $\varepsilon_1 = -20\,\text{Nm}^2$ (106)
and the following three variants of initial conditions
$\xi_1^0 = 0.063\,\text{m}$, $\xi_2^0 = 0.163$, $\xi_3^0 = 0\,\text{m/s}$,
$\xi_4^0 = 0\,\text{m/s}$, (107)
$\xi_1^0 = 0.037\,\text{m}$, $\xi_2^0 = 0.137$, $\xi_3^0 = 0\,\text{m/s}$,
$\xi_4^0 = 0\,\text{m/s}$, (108)
$\xi_1^0 = -0.023\,\text{m}$, $\xi_2^0 = 0.977$, $\xi_3^0 = 0\,\text{m/s}$,
$\xi_4^0 = 0\,\text{m/s}$, (109)

In the first situation there exists an unique equilibrium position
$\xi_1 = 0.1\,\text{m}$, (110)
$\xi_2 = 0.2\,\text{m}$. (111)

In the second case there is only one equilibrium position given by
$\xi_1 = -0.1\,\text{m}$, (112)
$\xi_2 = 0.9\,\text{m}$. (113)

Finally, in the third case we have three equilibrium positions
$\overline{\xi}_1 = -0.022629\,\text{m}$, (114)
$\overline{\xi}_2 = 0.077371\,\text{m}$, (115)
$\overline{\overline{\xi}}_1 = 0.03567\,\text{m}$, (116)
$\overline{\overline{\xi}}_2 = 0.13567\,\text{m}$, (117)
$\overline{\overline{\overline{\xi}}}_1 = 0.06198\,\text{m}$, (118)
$\overline{\overline{\overline{\xi}}}_2 = 0.16198\,\text{m}$. (119)

The results of the numerical simulations are presented in the figures 6, 7, 8, 9 and 10.



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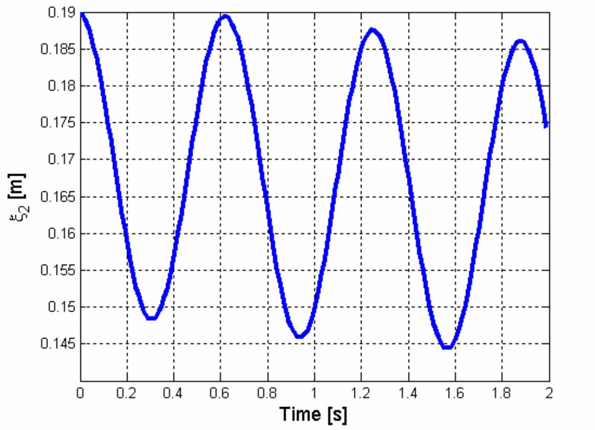

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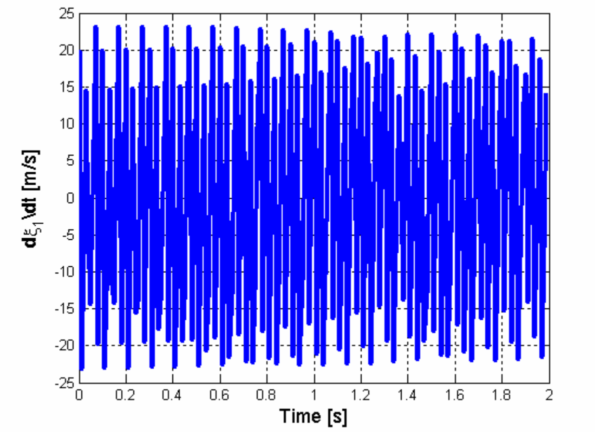

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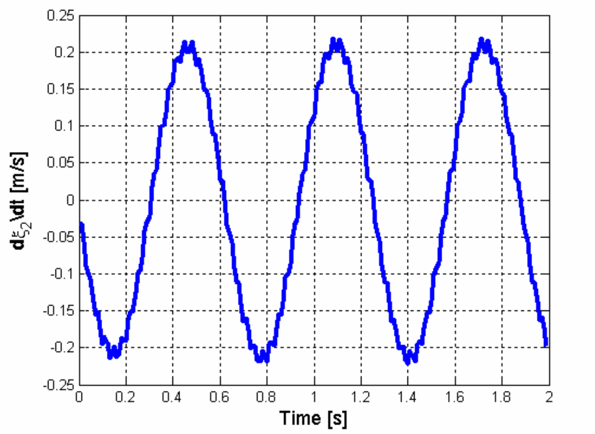
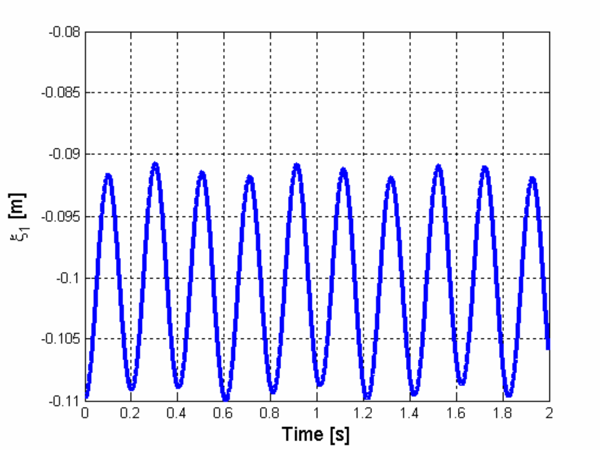


d)

Fig. 6. Time history in the first case.



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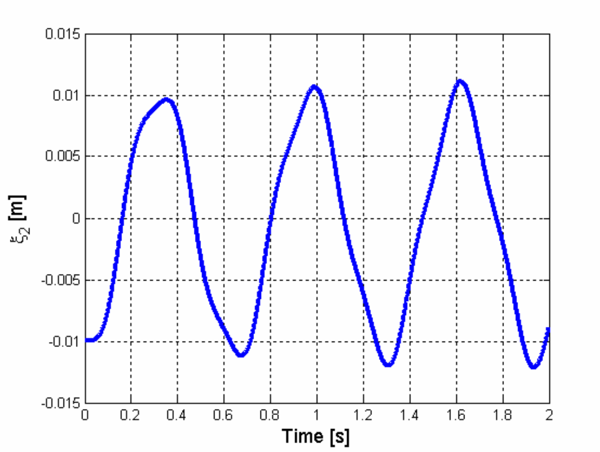

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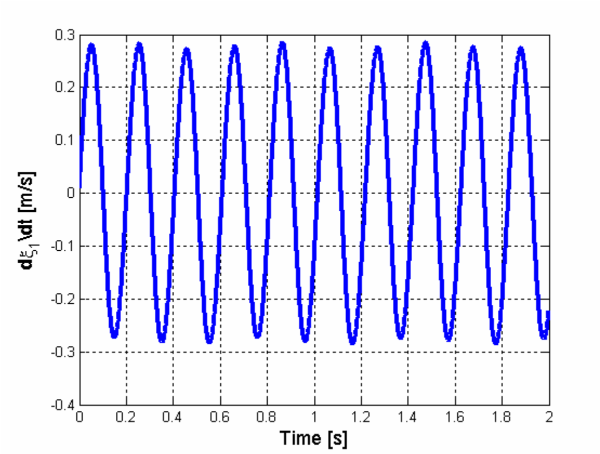

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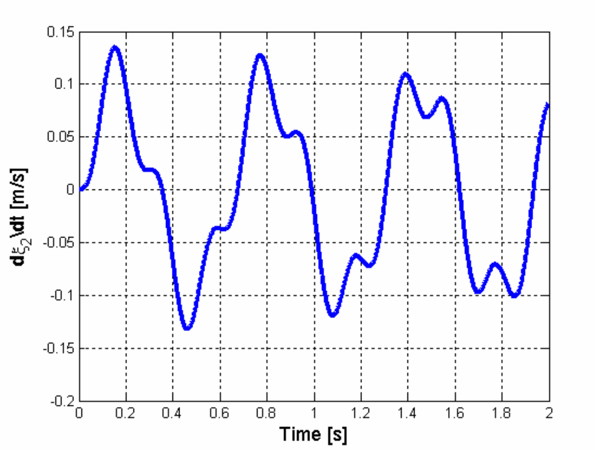
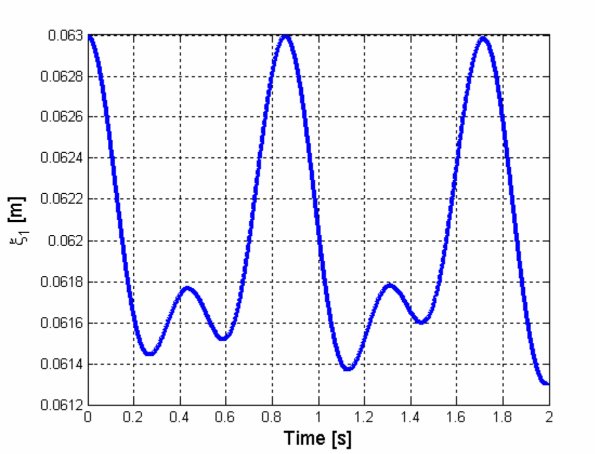


d)

Fig. 7. Time history in the second case.



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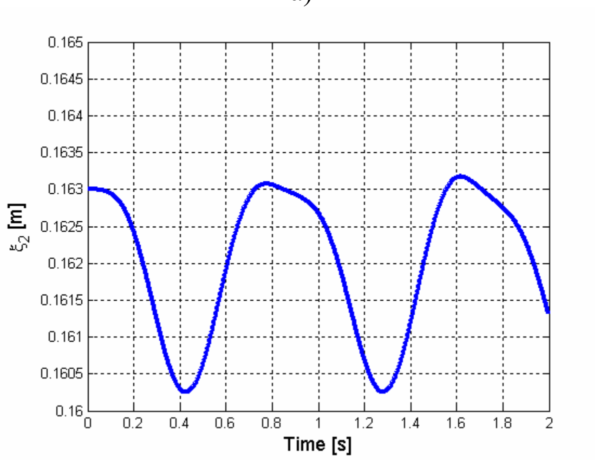

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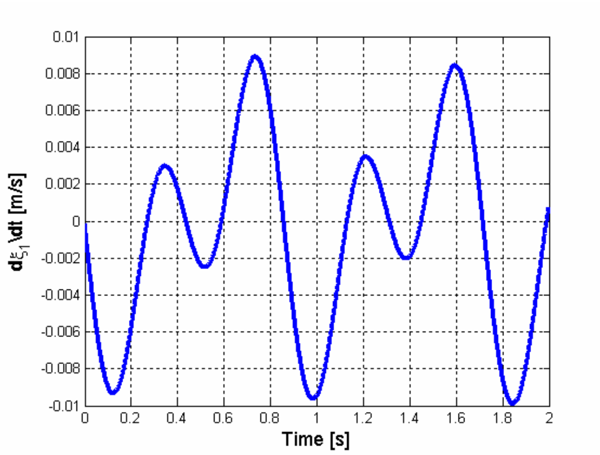

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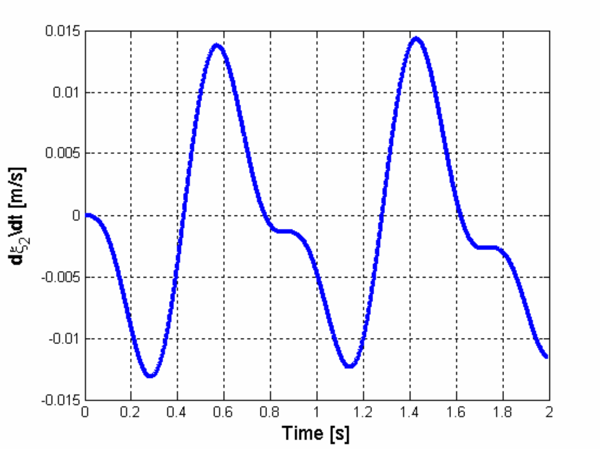
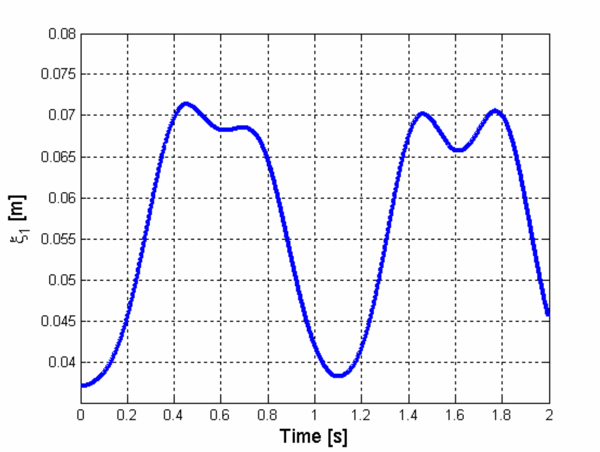


d)

Fig. 8. Time history in the third case, the initial conditions being given by (107).



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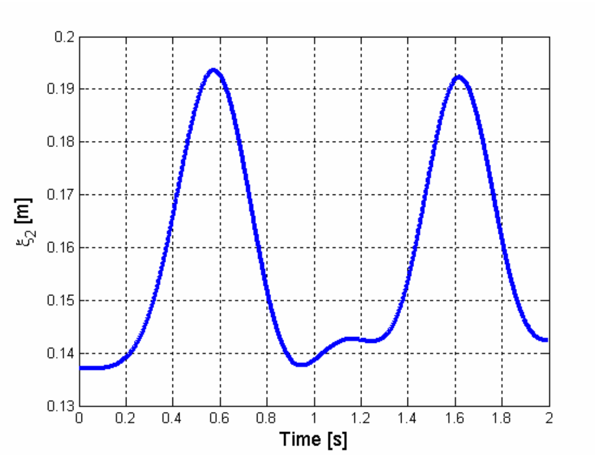

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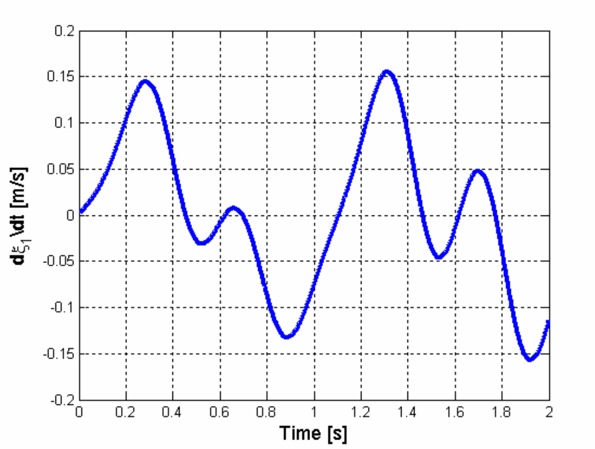

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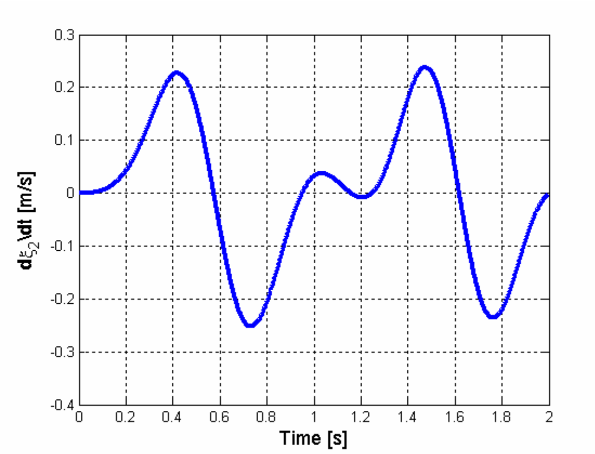
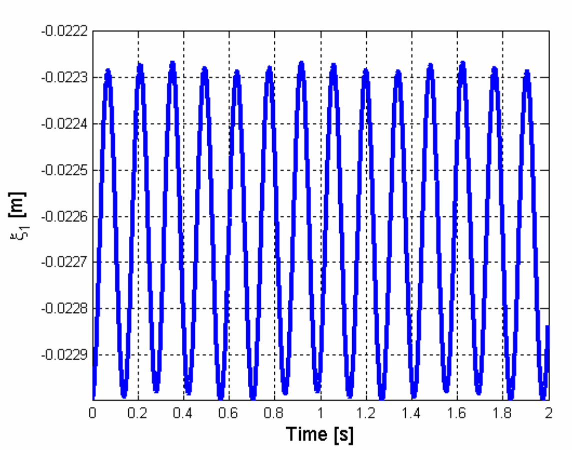
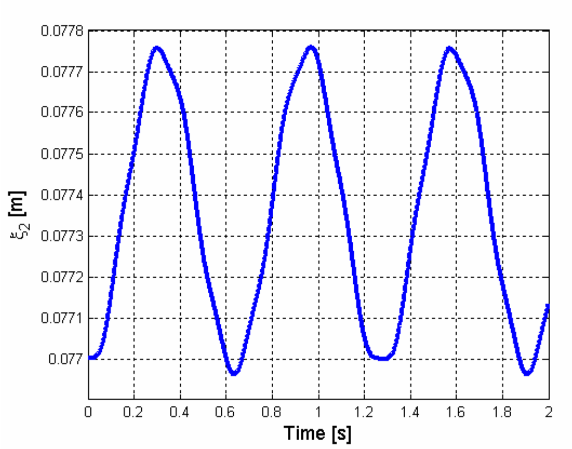


d)

Fig. 9. Time history in the third case, the initial conditions being given by (108).



a)



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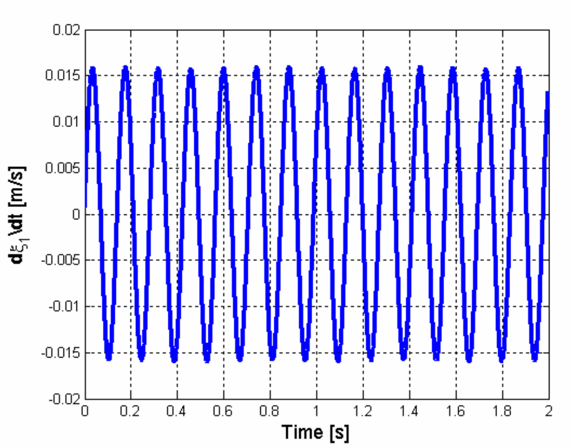

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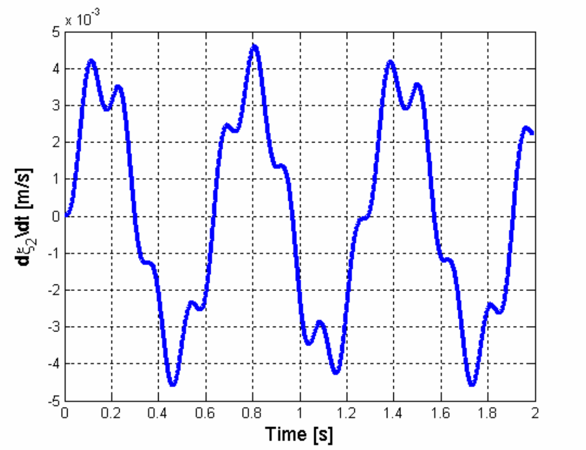

d)

Fig. 10. Time history in the third case, the initial conditions being given by (109).

11 Conclusions

In this paper we presented a study concerning the influence of the non-linear neo-Hookean elements on the stability of the system machine-tool-foundation. If $\varepsilon_1 > 0$, then we proved that both the equilibrium and the motion are simply stable and the neo-Hookean element increases the safety domain where the resonance doesn't appear. If $\varepsilon_1 < 0$, then the discussion is more difficult and it is presented in the ninth part of this paper where we also studied the stability of the equilibrium positions. Finally we made the numerical simulation for all the cases presented in the paper, the numerical results being in excellent agreement with the theory developed.

References:

[1] N.–D. Stănescu, L. Munteanu, V. Chiroiu, N. Pandrea, *Dynamical systems. Theory and applications, vol. 1*, The Publishing House of the Romanian Academy, Bucharest (Romania), 2007.

[2] N.–D. Stănescu, L. Munteanu, V. Chiroiu, N. Pandrea, *Dynamical systems. Theory and applications, vol. 2*, The Publishing House of the Romanian Academy, Bucharest (Romania), 2009 (in press).

[3] N. Pandrea, N.–D. Stănescu, N.–D., *Mechanics*, Didactical and Pedagogical Publishing House, Bucharest (Romania), 2002.

[4] N.–D. Stănescu, *Numerical methods*, Didactical and Pedagogical Publishing House, Bucharest (Romania), 2007.

[5] V. I. Arnold, *Dynamical Systems. Bifurcation Theory and Catastrophe Theory,* Springer–

Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, Hong Kong, Barcelona, Budapest, 1994.
[6] V. I. Arnold, *Catastrophe theory*, Springer–Verlag, Berlin, Heidelberg, New York, Tokyo, 1984.
[7] D. K. Arrowsmith, C. M. Place, *An Introduction to Dynamical Systems,* Cambridge University Press, 1994.
[8] H. W. Broer, B. Krauskopf, G. Vegter, (eds.) *Global Analysis of Dynamical Systems*, Institute of Physics Publishing, Bristol and Philadelphia, 2001.
[9] L. Carleson, T. W. Gamelin, *Complex Dynamics*, Springer–Verlag, New York, Heidelberg, London, Paris, 1993.
[10] J. Carr, *Applications of Centre Manifold Theory,* Springer–Verlag, New York, Heidelberg, Berlin, 1981.
[11] P. Collet, J. P. Eckmann, *Iterated maps on the interval as dynamical systems,* Birkhäuser, Boston, 1980.
[12] R. Curtu, *Introduction to the theory of the dynamical systems,* Infomarket Publishing House, Braşov, 2000 (in Romanian).
[13] R. L. Devaney, *An introduction to chaotic dynamical systems,* Addison-Wesley, New York, 1989.
[14] F. Dumortier, R. Roussarie, J. Sotomayor, H. Żoladek, *Bifurcations of Planar Vector Fields. Nilpotent Sigularities and Abelian Integrals,* Springer–Verlag, Berlin, Heidelberg, New York, 1991.
[15] A. Georgescu, M. Moroianu, I. Oprea, *The Theory of the Bifurcation,* University of Piteşti Publishing House, 1999 (in Romanian).
[16] J. Guckenheimer, *A strange attractor,* in The Hopf bifurcation and applications, ed. J. E. Marsden, McCracken, Springer, Berlin, 1976.
[17] J. Guckenheimer, P. J. Holmes, *Nonlinear oscillations, dynamical systems and bifurcations of vector fields,* Springer–Verlag, New York, 1983.
[18] J. K. Hale, H. Koçak, *Dynamics and Bifurcations,* Springer–Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, Hong Kong, Barcelona, Budapest, 1991.
[19] J. L. Kaplan, J. A. Yorke, *Functional differential equations and approximations of fixed points*. In H. O. Peiten, H. O. Walther (eds.), Springer, Berlin, 204, 1979.
[20] A. Katok, *Lyapunov exponents, entropy and periodic points for diffeomorfisms,* Publishing Mathematics, IHES, 1, 1980.
[21] Y. A. Kuznetsov, *Elements of Applied Bifurcation Theory,* Springer–Verlag, New York, Berlin, Heidelberg, London Paris, Tokyo, Hong Kong, Barcelona, Budapest, 1995.
[22] A. I. Lurie, *Analytical Mechanics*, Springer–Verlag, Berlin, Heidelberg, New York, Barcelona, Hong Kong, London, Milan, Paris, Tokyo, 2002.
[23] Ju. I. Neĭmark, N. A. Fufaev, *Dynamics of Nonholonomic Systems*, Translations of Mathematical Monographs, vol. 33, American Mathematical Society, Providence, Rhode Island, 1972.
[24] H. O. Peitgen, H. Jurgens, D. Saupe, *Chaos and fractals. New frontiers of science,* Springer Verlag, 1992.
[25] L. Perko, *Differential Equations and Dynamical Systems,* Springer–Verlag, New York, Berlin, Heidelberg, 1996.
[26] K. S. Sibirsky, *Introduction to topological dynamics*, Noordhoff International Publishing, Leyden, The Netherlands, 1975.
[27] P. P. Teodorescu, *Mechanical systems. Classic models*, 4 vol. Technical Publishing House, , Bucharest 1984, 1988, 1997, 2002.
[28] R. P. Voinea, I. V. Stroe, *Introduction to the theory of the dynamical systems,* Romanian Academy Publishing House, Bucharest, 2000.
[29] S. Wiggins., *Iintroduction to applied nonlinear dynamical systems and chaos,* Springer, New York, 1992.