2.4 The stability of the equilibrium

Let us consider the system (4) of the moving equations and let us rewrite it as

$$\frac{d\xi_i}{dt} = f_i(\xi_1, \xi_2, \xi_3, \xi_4);\ i = \overline{1,4}, \tag{16}$$

where

$$f_1(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_3;\ f_2(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_4;$$
$$f_3(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{1}{m_1}\left[-k\xi_1 + k_1(\xi_2 - \xi_1) - \frac{k_2}{(\xi_2 - \xi_1)^2} + m_1 g\right];$$
$$f_4(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{1}{m_2} \times \left[-k(\xi_2 - \xi_1) + \frac{k_2}{(\xi_2 - \xi_1)^2} + m_2 g\right]. \tag{17}$$

Let us denote by

$$j_{kl} = \frac{\partial f_k}{\partial \xi_l};\ k = \overline{1,4};\ l = \overline{1,4}. \tag{18}$$

We have

$$j_{11} = 0;\ j_{12} = 0;\ j_{13} = 1;\ j_{14} = 0, \tag{19}$$

$$j_{21} = 0;\ j_{22} = 0;\ j_{23} = 0;\ j_{24} = 1, \tag{20}$$

$$j_{31} = \frac{1}{m_1} - \left[-k - k_1 - \frac{2k_2}{(\xi_2 - \xi_1)^3}\right];$$
$$j_{32} = \frac{1}{m_1}\left[\frac{2k_2}{(\xi_2 - \xi_1)^3} + k_1\right];\ j_{33} = 0;\ j_{34} = 0, \tag{21}$$

$$j_{41} = \frac{1}{m_2}\left[k_1 + \frac{2k_2}{(\xi_2 - \xi_1)^3}\right];$$
$$j_{42} = \frac{1}{m_2}\left[-\frac{2k_2}{(\xi_2 - \xi_1)^3} - k_1\right];\ j_{43} = 0;$$
$$j_{44} = 0. \tag{22}$$

The characteristic equation

$$\det([\mathbf{J}] - \varphi[\mathbf{I}]) = 0, \tag{23}$$

in which $[\mathbf{J}]$ is the Jacobi matrix,

$$[\mathbf{J}] = [j_{kl}]_{\substack{k=\overline{1,4}\\ l=\overline{1,4}}}, \tag{24}$$

and $[\mathbf{I}]$ is the fourth order unity matrix, offers us

$$\begin{vmatrix} -\varphi & 0 & 1 & 0 \\ 0 & -\varphi & 0 & 1 \\ j_{31} & j_{32} & -\varphi & 0 \\ j_{41} & j_{42} & 0 & -\varphi \end{vmatrix} = 0, \tag{25}$$

$$-\varphi\begin{vmatrix} -\varphi & 0 & 1 \\ j_{32} & -\varphi & 0 \\ j_{42} & 0 & -\varphi \end{vmatrix} + \begin{vmatrix} 0 & -\varphi & 1 \\ j_{31} & j_{32} & 0 \\ j_{41} & j_{42} & -\varphi \end{vmatrix} = 0, \tag{26}$$

$$\varphi^4 - (j_{42} + j_{31})\varphi^2 + j_{31}j_{42} - j_{41}j_{32} = 0. \tag{27}$$

The discriminate of this bi-square equation is

$$\Delta = (j_{42} + j_{31})^2 - 4(j_{31}j_{42} - j_{41}j_{32}) = (j_{42} - j_{31})^2 + 4j_{41}j_{32} > 0, \tag{28}$$

because both j_{41} and j_{32} are strict positive expressions.

On the other hand,

$$j_{42} < 0;\ j_{31} < 0 \tag{29}$$

and

$$\sqrt{\Delta} < -(j_{42} + j_{31}). \tag{30}$$

Indeed, the relation (30), keeping into account that $\sqrt{\Delta} > 0$ and $-(j_{42} + j_{31}) > 0$, reads in the equivalent form

$$\Delta < (j_{42} + j_{31})^2. \tag{31}$$

It results successively

$$(j_{42} + j_{31})^2 - 4(j_{31}j_{42} - j_{41}j_{32}) < (j_{42} + j_{31})^2, \tag{32}$$

$$j_{31}j_{42} - j_{41}j_{32} > 0, \tag{33}$$

$$\frac{1}{m_1 m_2}\left[-k - k_1 - \frac{2k_2}{(\xi_2 - \xi_1)^3}\right] \times \left[-\frac{2k_2}{(\xi_2 - \xi_1)^3} + k_1\right] > 0, \tag{34}$$

$$\frac{1}{m_1 m_2} k\left[\frac{2k_2}{(\xi_2 - \xi_1)^3} + k_1\right] > 0, \tag{35}$$

the last relation being obviously true from the above discussion.

The roots of the bi-squared equation read

$$\varphi_{1,2}^2 = \frac{j_{42} + j_{31} \pm \sqrt{\Delta}}{2} \tag{36}$$

and one observes that

$$\varphi_1^2 < 0;\ \varphi_2^2 < 0, \tag{37}$$

that is, the roots of the characteristic equation are pure imaginary,

$$\varphi_1 = i\sqrt{-\frac{j_{42} + j_{31} + \sqrt{\Delta}}{2}};$$
$$\varphi_2 = -i\sqrt{-\frac{j_{42} + j_{31} + \sqrt{\Delta}}{2}};$$
$$\varphi_3 = i\sqrt{-\frac{j_{42} + j_{31} - \sqrt{\Delta}}{2}};$$
$$\varphi_4 = -i\sqrt{-\frac{j_{42} + j_{31} - \sqrt{\Delta}}{2}}. \tag{38}$$

Therefore, the equilibrium is simply stable.

Let us consider the equilibrium position $(\xi_1, \xi_2, 0, 0)$ and a deviation (u_1, u_2, u_3, u_4) sufficiently small in its norm.

Keeping into account that $(\xi_1, \xi_2, 0, 0)$ is a solution of the differential equations system (4), it results the system in deviations

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} = u_3, \tag{39a}$$

$$\frac{\mathrm{d}u_2}{\mathrm{d}t} = u_4, \tag{39b}$$

$$\frac{\mathrm{d}u_3}{\mathrm{d}t} = \frac{1}{m_1}\left[-k(\xi_1 + u_1) + k_1 \times (\xi_2 - \xi_1 + u_2 - u_1) - \frac{k_2}{(\xi_2 - \xi_1 + u_2 - u_1)^2} + m_1 g\right] - \frac{1}{m_1} \times \left[-k\xi_1 + k_1(\xi_2 - \xi_1) + \frac{k_2}{(\xi_2 - \xi_1)^2} + m_1 g\right], \tag{39c}$$

$$\frac{\mathrm{d}u_4}{\mathrm{d}t} = \frac{1}{m_2}\left[-k_1(\xi_2 - \xi_1 + u_2 - u_1) + \frac{k_2}{(\xi_2 - \xi_1 + u_2 - u_1)^2} + m_2 g\right] - \frac{1}{m_2}\left[-k_1(\xi_2 - \xi_1) + \frac{k_2}{(\xi_2 - \xi_1)^2} + m_2 g\right]. \tag{39d}$$

We can make the approximation

$$\frac{1}{(\xi_2 - \xi_1 + u_2 - u_1)^2} \approx \frac{1}{(\xi_2 - \xi_1)^2} - \frac{2(u_2 - u_1)}{(\xi_2 - \xi_1)^3}, \tag{40}$$

such that the system (39) becomes $\left(k^* = k_1 + \frac{2k_2}{(\xi_2 - \xi_1)^3}\right)$

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} = u_3;\ \frac{\mathrm{d}u_2}{\mathrm{d}t} = u_4;$$

$$\frac{\mathrm{d}u_3}{\mathrm{d}t} = \frac{1}{m_1}\left[-ku_1 + k^*(u_2 - u_1)\right];$$

$$\frac{\mathrm{d}u_4}{\mathrm{d}t} = \frac{1}{m_2}\left[-k^*(u_2 - u_1)\right]. \tag{41}$$

The Jacobi matrix of the system (41) has the expression

$$[\mathbf{J}] = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k - k^*}{m_1} & \frac{k^*}{m_1} & 0 & 0 \\ \frac{k^*}{m_2} & -\frac{k^*}{m_2} & 0 & 0 \end{vmatrix}, \tag{42}$$

the characteristic equation reading

$$\begin{vmatrix} -\varphi & 0 & 1 & 0 \\ 0 & -\varphi & 0 & 1 \\ -\frac{k + k^*}{m_1} & \frac{k^*}{m_1} & -\varphi & 0 \\ \frac{k^*}{m_2} & -\frac{k^*}{m_2} & 0 & -\varphi \end{vmatrix} = 0, \tag{43}$$

$$-\varphi\begin{vmatrix} -\varphi & 0 & 1 \\ \frac{k^*}{m_1} & -\varphi & 0 \\ -\frac{k^*}{m_2} & 0 & -\varphi \end{vmatrix} + \begin{vmatrix} 0 & -\varphi & 1 \\ -\frac{k + k^*}{m_1} & \frac{k^*}{m_1} & 0 \\ \frac{k^*}{m_2} & -\frac{k^*}{m_2} & -\varphi \end{vmatrix} = 0, \tag{44}$$

$$\varphi^4 - \left(\frac{k^*}{m_2} + \frac{k + k^*}{m_1}\right)\varphi^2 + \frac{k^*}{m_2}\frac{k + k^*}{m_1} = 0. \tag{45}$$

The roots of the characteristic equation (45) are

$$\varphi_1^2 = -\frac{k^*}{m_2};\ \varphi_2^2 = -\frac{k + k^*}{m_1}, \tag{46}$$

wherefrom

$$\varphi_1 = i\sqrt{\frac{k^*}{m_2}};\ \varphi_2 = -i\sqrt{\frac{k^*}{m_2}};\ \varphi_3 = i\sqrt{\frac{k + k^*}{m_1}};$$

$$\varphi_4 = -i\sqrt{\frac{k + k^*}{m_2}}. \tag{47}$$

THEOREM 1. *If* $k_2 > 0$, *then the system has only one equilibrium position which is stable and not asymptotically stable.*

Proof: Since the roots (47) of the characteristic equation (45) are pure imaginary, it follows that the system (41) has the solution

$$u_1 = C_1\cos(p_1 t) + C_2\cos(p_2 t);$$

$$u_2 = D_1\cos(p_1 t) + D_2\cos(p_2 t), \tag{48}$$

where C_1, C_2, D_1, D_2 are constants of integration which result from the initial conditions, and p_1, p_2 have the expressions

$$p_1 = \sqrt{\frac{k^*}{m_2}};\ p_2 = \sqrt{\frac{k + k^*}{m_1}}. \tag{49}$$

The solutions (48) are bounded and it follows that the equilibrium is stable but not asymptotically stable.

2.5 The small oscillations around the position of stable equilibrium

Let us return to the system (4) and let us make the approximation

$$\frac{1}{(\xi_2-\xi_1+u_2-u_1)^2} \approx \frac{1}{(\xi_2-\xi_1)^2}-\frac{2(u_2-u_1)}{(\xi_2-\xi_1)^3} \quad (50)$$

obtained developing the function $g(x)=\frac{1}{x^2}$ about a point x_0 and retaining only the first term of the development.

The system (39) takes now the form

$$\frac{\mathrm{d}u_1}{\mathrm{d}t}=u_3;\ \frac{\mathrm{d}u_2}{\mathrm{d}t}=u_4;$$
$$\frac{\mathrm{d}u_3}{\mathrm{d}t}=\frac{1}{m_1}\left\{\left[-k-k_1-\frac{2k_2}{(\xi_2-\xi_1)^3}\right]u_1 +\left[k_1+\frac{2k_2}{(\xi_2-\xi_1)^3}\right]u_2\right\};$$
$$\frac{\mathrm{d}u_4}{\mathrm{d}t}=\frac{1}{m_2}\left\{\left[k_1+\frac{2k_2}{(\xi_2-\xi_1)^3}\right]u_1 +\left[-k_1-\frac{2k_2}{(\xi_2-\xi_1)^3}\right]u_2\right\}. \quad (51)$$

Let us denote

$$a_1=-k-k_1-\frac{2k_2}{(\xi_2-\xi_1)^3};\ b_1=k_1+\frac{2k_2}{(\xi_2-\xi_1)^3};$$
$$a_2=k_1+\frac{2k_2}{(\xi_2-\xi_1)^3};\ b_2=-k_1-\frac{2k_2}{(\xi_2-\xi_1)^3}, \quad (52)$$

the system (51) reading

$$m_1\ddot{u}_1=a_1u_1+b_1u_2;\ m_2\ddot{u}_2=a_2u_1+b_2u_2. \quad (53)$$

For the system (53) we shall look for solutions in the form

$$u_1=\alpha\cos(\omega t);\ u_2=\beta\cos(\omega t);\ \alpha^2+\beta^2\neq 0. \quad (54)$$

It follows

$$\alpha\omega^2 m_1=a_1\alpha+b_1\beta;\ -\beta\omega^2 m_2=a_2\alpha+b_2\beta. \quad (55)$$

The system (55) has non-zero solution if its determinant is equal to 0, that is

$$\begin{vmatrix} a_1+m_1\omega^2 & b_1 \\ a_2 & b_2+m_2g\omega^2 \end{vmatrix}=0, \quad (56)$$

wherefrom

$$(a_1+m_1\omega^2)(b_2+m_2\omega)^2-a_2b_1=0. \quad (57)$$

Let us denote by a the expression

$$a=k_1+\frac{2k_2}{(\xi_2-\xi_1)^3}, \quad (58)$$

such that

$$a_1=-a-k;\ b_1=a;\ a_2=a;\ b_2=-a. \quad (59)$$

The equation (57) reads now

$$(-a-k+m_1\omega^2)(-a+m_2\omega^2)-a^2=0 \quad (60)$$

or, equivalently,

$$m_1m_2\omega^4+\omega^2(-am_1-am_2-km_2)+ak=0. \quad (61)$$

It results

$$\omega_{1,2}^2=\frac{a(m_1+m_2)+km_2}{2m_1m_2} \pm\frac{\sqrt{[a(m_1+m_2)-km_2]^2+4m_1m_2a^2}}{2m_1m_2}. \quad (62)$$

2.6 Comparison with the linear case

The linear case is defined by

$$k_2=0. \quad (63)$$

In this situation, the parameter a takes its minimum value, which is

$$a=k_1. \quad (64)$$

Let us consider the expression (62), which gives the eigenpulsations and let ω_1^2 be the value corresponding to the sign – and ω_2^2 the value corresponding to the sign +.

If a increases, then ω_2^2 will increase, such that we can always write

$$\omega_2>(\omega_2)_l, \quad (65)$$

where the index l signifies linear.

On the other hand, the increasing of a implies both the increasing of the expression $a(m_1+m_2)$ and the expression under the radical, the increasing of the expression $a(m_1+m_2)$ being greater than that of the expression under the radical. It therefore results

$$\omega_1>(\omega_1)_l. \quad (66)$$

Finally, we found that the use of the neo–Hookean element leads to the displacement of the fundamental pulsation in an increasing sense.

In addition, the increasings of the two fundamental pulsations ω_1 and ω_2 are not equal, in the sense that ω_2 increases more than ω_1. This thing is mathematically written by the relation

$$\omega_2-\omega_1>(\omega_2)_l-(\omega_1)_l. \quad (67)$$

In this way, the safety domain where the resonance doesn't appear increases and it goes to superior value comparing to those in the linear case.

2.7 Numerical application

Let us consider a numerical case defined by

$$k = 4 \cdot 10^5 \left[\mathrm{N/m}\right];\ k_1 = 5 \cdot 10^4 \left[\mathrm{N/m}\right];$$
$$k_2 = 5\left[\mathrm{Nm^2}\right];\ m_1 = 25\left[\mathrm{kg}\right];\ m_2 = 350\left[\mathrm{kg}\right];$$
$$g = 9.8065\left[\mathrm{m/s^2}\right]. \qquad (68)$$

The static elongations (the equilibrium position) are given by

$$\xi_1 = 0.00919\left[\mathrm{m}\right];\ \xi_2 = 0.0923\left[\mathrm{m}\right]. \qquad (69)$$

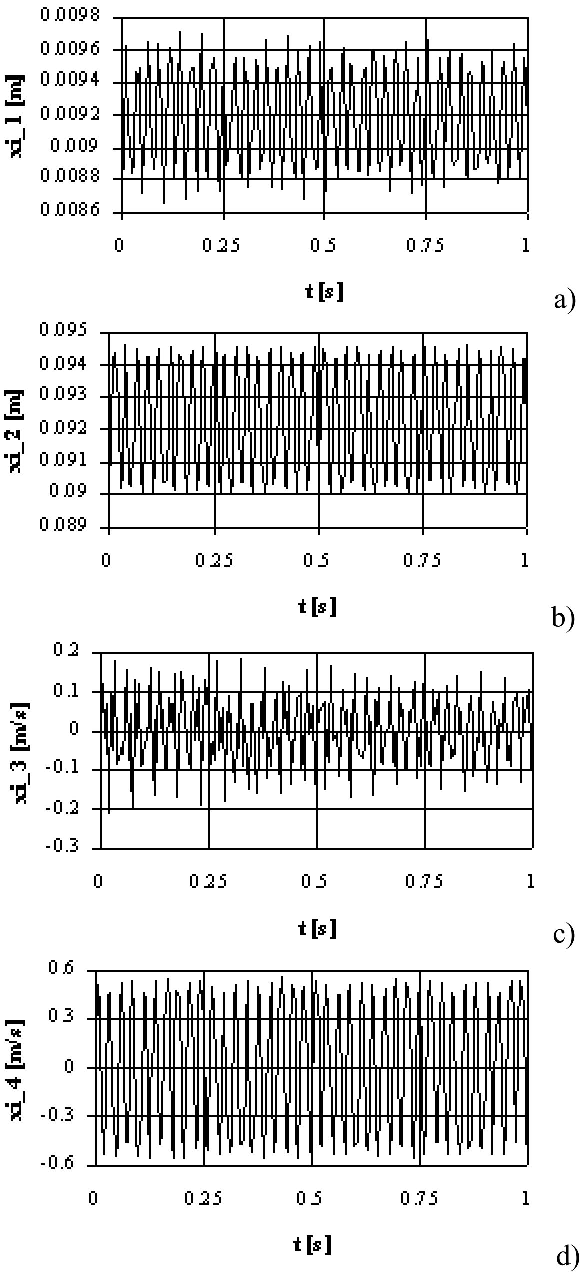

Fig. 4. The graphics of variation for different characteristic variables; a) $\xi_1 = \xi_1(t)$ for $0 \le t \le 1[\mathrm{s}]$; b) $\xi_2 = \xi_2(t)$ for $0 \le t \le 1[\mathrm{s}]$; c) $\xi_3 = \xi_3(t)$ for $0 \le t \le 1[\mathrm{s}]$; d) $\xi_4 = \xi_4(t)$ for $0 \le t \le 1[\mathrm{s}]$.

The initial values are

$$\xi_1^0 = 0.009\left[\mathrm{m}\right];\ \xi_2^0 = 0.09\left[\mathrm{m}\right];\ \xi_3^0 = 0\left[\mathrm{m/s}\right];$$
$$\xi_4^0 = 0\left[\mathrm{m/s}\right], \qquad (70)$$

and the parameter a from the expression (58) is

$$a = 67419.68\left[\mathrm{N/m}\right]. \qquad (71)$$

In figure 4 were drawn the graphics of variation for different characteristic variables obtained by numerical simulation.

One can obviously see the quasi-periodic character of the presented variations.

The formula (62) offers us the eigenpulsations in the non-linear case

$$\omega_1 = 12.83\left[\mathrm{s^{-1}}\right];\qquad \omega_2 = 136.84\left[\mathrm{s^{-1}}\right]. \qquad (72)$$

In the linear case one obtains the values

$$\left(\omega_1\right)_l = 11.26\left[\mathrm{s^{-1}}\right];\qquad \omega_2 = 134.22\left[\mathrm{s^{-1}}\right]. \qquad (73)$$

The breadth of the non-dangerous eigenpulsations is in the non-linear case

$$\Delta\omega = \omega_2 - \omega_1 = 124.01\left[\mathrm{s^{-1}}\right], \qquad (74)$$

and in the linear case

$$\left(\Delta\omega\right)_l = \left(\omega_2\right)_l - \left(\omega_1\right)_l = 122.96\left[\mathrm{s^{-1}}\right]. \qquad (75)$$

The results are in good agreement with the above theoretical considerations.

2.8 The stability of the motion

Let us consider that ξ_i, $i = \overline{1,4}$ is a solution of the system (4) of the moving equations and let $\left(u_1, u_2, u_3, u_4\right)$ be a deviation sufficiently small in its norm.

The system in deviations has the expression

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} = u_3;\ \frac{\mathrm{d}u_2}{\mathrm{d}t} = u_4;$$
$$\frac{\mathrm{d}u_3}{\mathrm{d}t} = \frac{1}{m_1}\left[-ku_1 + k_1\left(u_2 - u_1\right)\right.$$
$$\left. - \frac{k_2}{\left(\xi_2 - \xi_1 + u_2 - u_1\right)^2} + \frac{k_2}{\left(\xi_2 - \xi_1\right)^2}\right];$$
$$\frac{\mathrm{d}u_4}{\mathrm{d}t} = \frac{1}{m_2}\left[-k_1\left(u_2 - u_1\right)\right.$$
$$\left. - \frac{k_2}{\left(\xi_2 - \xi_1 + u_2 - u_1\right)} + \frac{k_2}{\left(\xi_2 - \xi_1\right)^2}\right]. \qquad (76)$$

Let us denote by $g\left(u_1, u_2, u_3, u_4\right)$ the functions in the right-hand terms of the expressions (76) and by j_{kl} the partial derivatives

$$j_{kl} = \frac{\partial g_k}{\partial u_l};\ k = \overline{1,4};\ l = \overline{1,4}. \qquad (77)$$

We have

$$j_{11} = 0;\ j_{12} = 0;\ j_{13} = 1;\ j_{14} = 0, \qquad (78)$$

$$j_{21} = 0;\ j_{22} = 0;\ j_{23} = 0;\ j_{24} = 1, \quad (79)$$

$$j_{31} = \frac{1}{m_1}\left[-k - k_1 - \frac{2k_2}{(\xi_2 - \xi_1 + u_2 - u_1)^3}\right];$$

$$j_{32} = \frac{1}{m_1}\left[k_1 + \frac{2k_2}{(\xi_2 - \xi_1 + u_2 - u_1)^3}\right];\ j_{33} = 0;$$

$$j_{34} = 0, \quad (80)$$

$$j_{41} = \frac{1}{m_2}\left[k_1 + \frac{2k_2}{(\xi_2 - \xi_1 + u_2 - u_1)^3}\right];$$

$$j_{42} = \frac{1}{m_2}\left[-k_1 - \frac{2k_2}{(\xi_2 - \xi_1 + u_2 - u_1)^3}\right];\ j_{43} = 0;$$

$$j_{44} = 0. \quad (81)$$

We denote by a the expression

$$a = k_1 + \frac{2k_2}{(\xi_2 - \xi_1 + u_2 - u_1)^3}, \quad (82)$$

such that we can write

$$j_{31} = \frac{-k - a}{m_1};\ j_{32} = \frac{a}{m_1};\ j_{41} = \frac{a}{m_2};$$

$$j_{42} = \frac{-a}{m_2}. \quad (83)$$

The characteristic equation

$$\det([\mathbf{J}] - \varphi[\mathbf{I}]) = 0, \quad (84)$$

where $[\mathbf{J}]$ is the Jacobi matrix,

$$[\mathbf{J}] = [j_{kl}]_{\substack{k=\overline{1,4}\\ l=\overline{1,4}}}, \quad (85)$$

and $[\mathbf{I}]$ is the fourth order unity matrix, takes the form

$$\begin{vmatrix} -\varphi & 0 & 1 & 0 \\ 0 & -\varphi & 0 & 1 \\ -\frac{a+k}{m_1} & \frac{a}{m_1} & -\varphi & 0 \\ \frac{a}{m_2} & -\frac{a}{m_2} & 0 & -\varphi \end{vmatrix} = 0, \quad (86)$$

$$-\varphi\begin{vmatrix} -\varphi & 0 & 1 \\ \frac{a}{m_1} & -\varphi & 0 \\ -\frac{a}{m_2} & 0 & -\varphi \end{vmatrix} + \begin{vmatrix} 0 & -\varphi & 1 \\ -\frac{a+k}{m_1} & \frac{a}{m_1} & 0 \\ \frac{a}{m_2} & -\frac{a}{m_2} & -\varphi \end{vmatrix} = 0, \quad (87)$$

$$\varphi^4 + \left(\frac{a}{m_2} + \frac{a+k}{m_1}\right)\varphi^2 + \frac{ak}{m_1 m_2} = 0. \quad (88)$$

The discriminate of this equation is

$$\begin{aligned}\Delta &= \left(\frac{a}{m_2} + \frac{a+k}{m_1}\right)^2 - \frac{4ak}{m_1 m_2} \\ &= \frac{a^2}{m_2^2} + \frac{(a+k)^2}{m_1^2} + \frac{2a(a+k)}{m_1 m_2} - \frac{4ak}{m_1 m_2} \\ &= \frac{a^2}{m_2^2} + \frac{(a+k)^2}{m_1^2} + \frac{2a(a-k)}{m_1 m_2} + \frac{4ak}{m_1^2} \\ &= \left(\frac{a}{m_2} + \frac{a-k}{m_1}\right)^2 + \frac{4ak}{m_1^2} > 0.\end{aligned} \quad (89)$$

The roots of the characteristic equation (88) read

$$\varphi_{1,2}^2 = \frac{1}{2}\left[-\left(\frac{a}{m_2} + \frac{a+k}{m_1}\right) \pm \pm\sqrt{\left(\frac{a}{m_2} + \frac{a+k}{m_1}\right)^2 - \frac{4ak}{m_1 m_2}}\right] \quad (90)$$

and one can easy see that they are pure imaginary,

$$\varphi_1 = \mathrm{i} \times \sqrt{\frac{\frac{a}{m_2} + \frac{a+k}{m_1} + \sqrt{\left(\frac{a}{m_2} + \frac{a+k}{m_1}\right)^2 - \frac{4ak}{m_1 m_2}}}{2}}, \quad (91a)$$

$$\varphi_2 = -\mathrm{i} \times \sqrt{\frac{\frac{a}{m_2} + \frac{a+k}{m_1} + \sqrt{\left(\frac{a}{m_2} + \frac{a+k}{m_1}\right)^2 - \frac{4ak}{m_1 m_2}}}{2}}, \quad (91b)$$

$$\varphi_3 = \mathrm{i} \times \sqrt{\frac{\frac{a}{m_2} + \frac{a+k}{m_1} - \sqrt{\left(\frac{a}{m_2} + \frac{a+k}{m_1}\right)^2 - \frac{4ak}{m_1 m_2}}}{2}}, \quad (91c)$$

$$\varphi_4 = -\mathrm{i} \times \sqrt{\frac{\frac{a}{m_2} + \frac{a+k}{m_1} - \sqrt{\left(\frac{a}{m_2} + \frac{a+k}{m_1}\right)^2 - \frac{4ak}{m_1 m_2}}}{2}}. \quad (91d)$$

Result the solutions of the system (81) in the form

$$u_1 = C_1\cos(p_1 t) + C_2\cos(p_2 t);$$

$$u_2 = D_1\cos(p_1 t) + D_2\cos(p_2 t), \quad (92)$$

in which C_1, C_2, D_1, D_2 are constants of integration which are obtained from the initial conditions, and p_1, p_2 read as

$$p_1 = \sqrt{\frac{\frac{a}{m_2} + \frac{a+k}{m_1} + \sqrt{\left(\frac{a}{m_2} + \frac{a+k}{m_1}\right)^2 - \frac{4ak}{m_1 m_2}}}{2}};$$

$$p_2 = \sqrt{\frac{\frac{a}{m_2} + \frac{a+k}{m_1} - \sqrt{\left(\frac{a}{m_2} + \frac{a+k}{m_1}\right)^2 - \frac{4ak}{m_1 m_2}}}{2}}. \quad (93)$$

THEOREM 2. *In the case* $k_2 > 0$ *the motion is always stable and not asymptotically stable.*

Proof: It is obvious from the above discussions.

2.9 Numerical application

As numerical application let us consider the case defined by the relations (68), the initial conditions being

$$\xi_1^0 = 0.012[\text{m}];\ \xi_2^0 = 0.022[\text{m}];\ \xi_3^0 = 0[\text{m/s}];$$
$$\xi_4^0 = 0[\text{m/s}], \quad (94)$$

and the deviations

$$u_1^0 = -0.001[\text{m}];\ u_2^0 = -0.001[\text{m}];\ u_3^0 = 0.01[\text{m/s}];$$
$$u_4^0 = 0.00001[\text{m/s}]. \quad (95)$$

In the figure 5 we presented the representative diagrams for the unperturbed motion; in the figure 6 for the perturbed motion, and in figure 7 the deviations of the perturbed motion with respect to the unperturbed motion.

2.10 Discussion

Until now we considered the case $k_2 > 0$. The Case $k_2 = 0$ is not an interesting one because we arrive to the linear case.

Let us now consider $k_2 < 0$.

One obtains the same value for ξ_1 given by the formula (7) and the same roots of the derivative $f'(\xi_2)$ given by the expressions (12).

For $\xi_2^{(1)}$ we find

$$f\left(\xi_2^{(1)}\right) = -k_2 > 0, \quad (96)$$

and for $\xi_2^{(2)}$ we have

$$f\left(\xi_2^{(2)}\right) = -\frac{4m_2^3 g^3}{27k_1^2} - k_2. \quad (97)$$

If $f\left(\xi_2^{(2)}\right) > 0$, that is

$$k_2 < -\frac{4m_2^3 g^3}{27k_1^2}, \quad (98)$$

then we are in the situation drawn in the figure 8, the equation $f(\xi_2) = 0$ having only one root $\xi_2 < \xi_2^{(1)}$.

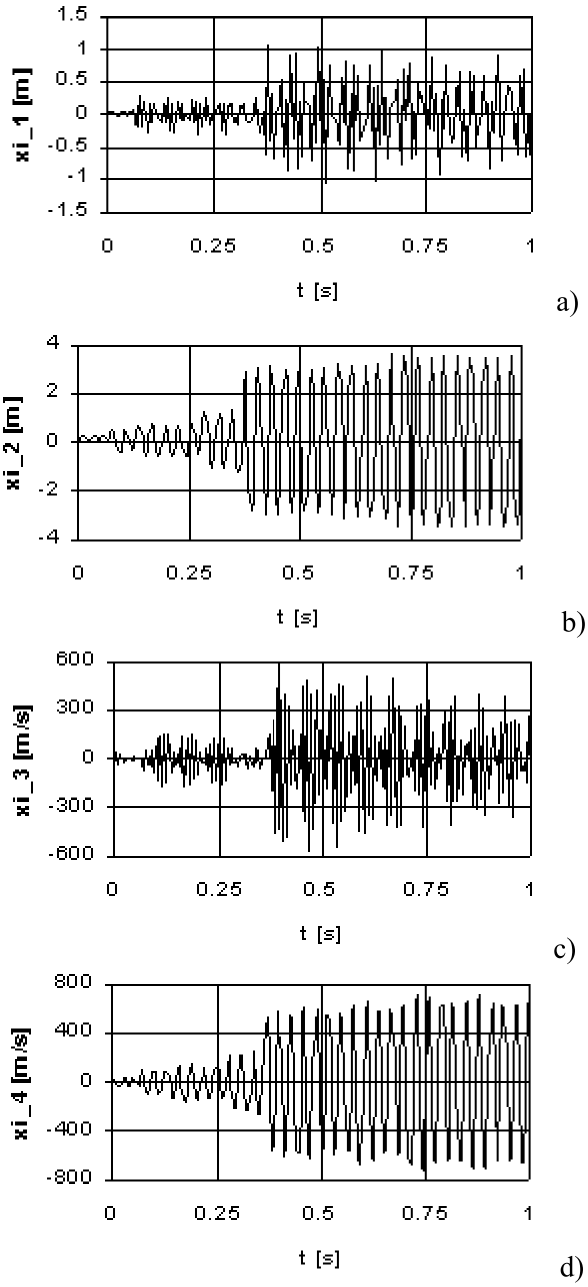

Fig. 5. The representative diagrams for the unperturbed motion; a) $\xi_1 = \xi_1(t)$ for $0 \le t \le 1[\text{s}]$; b) $\xi_2 = \xi_2(t)$ for $0 \le t \le 1[\text{s}]$; c) $\xi_3 = \xi_3(t)$ for $0 \le t \le 1[\text{s}]$; d) $\xi_4 = \xi_4(t)$ for $0 \le t \le 1[\text{s}]$.

If $f\left(\xi_2^{(2)}\right) < 0$, that is

$$k_2 > -\frac{4m_2^3 g^3}{27k_1^2}, \quad (99)$$

then we are in the situation presented in the figure 9, the equation $f(\xi_2) = 0$ having now three real roots situated in the intervals $\left(-\infty, \xi_2^{(1)}\right)$, $\left(\xi_2^{(1)}, \xi_2^{(2)}\right)$,

$\left(\xi_2^{(2)}, +\infty\right)$.

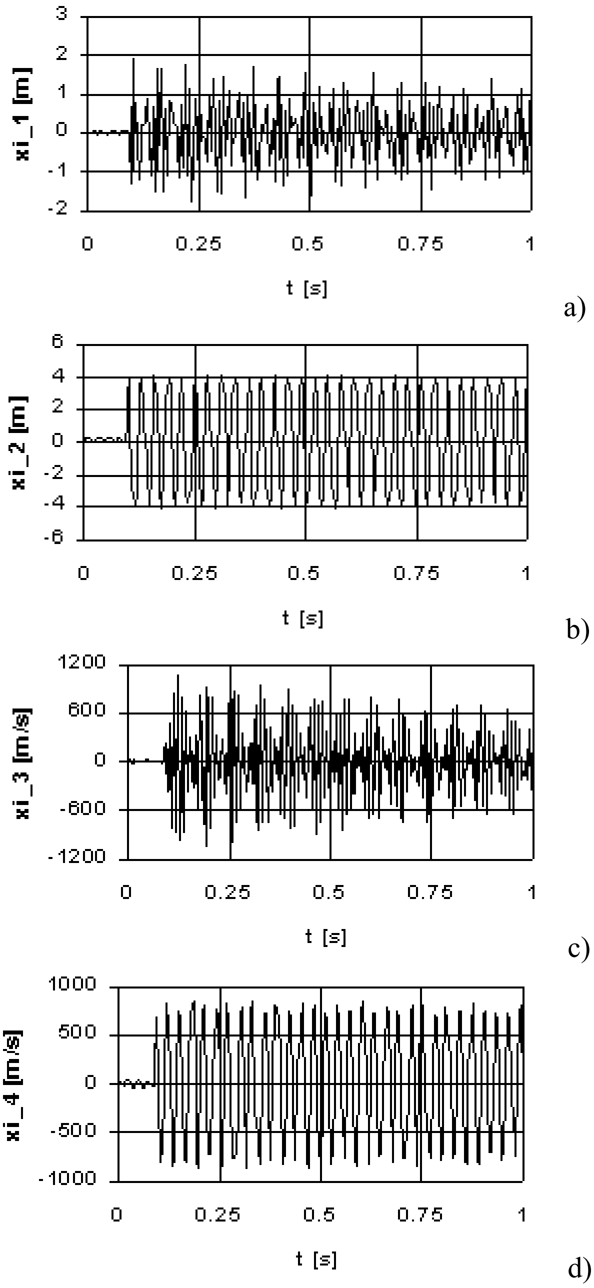

Fig. 6. The representative diagrams for the perturbed motion; a) $\xi_1 = \xi_1(t)$ for $0 \le t \le 1[\mathrm{s}]$; b) $\xi_2 = \xi_2(t)$ for $0 \le t \le 1[\mathrm{s}]$; c) $\xi_3 = \xi_3(t)$ for $0 \le t \le 1[\mathrm{s}]$; d) $\xi_4 = \xi_4(t)$ for $0 \le t \le 1[\mathrm{s}]$.

No matter in what situation we are, the components of the Jacobi matrix have the same expressions given by the formulas (19), (21) and (22), and the characteristic equation reads in the same form (27).

Let us consider for the beginning the situation described in the figure 8. In this case $\xi_2 - \xi_1 < 0$, $k_2 < 0$ and it results

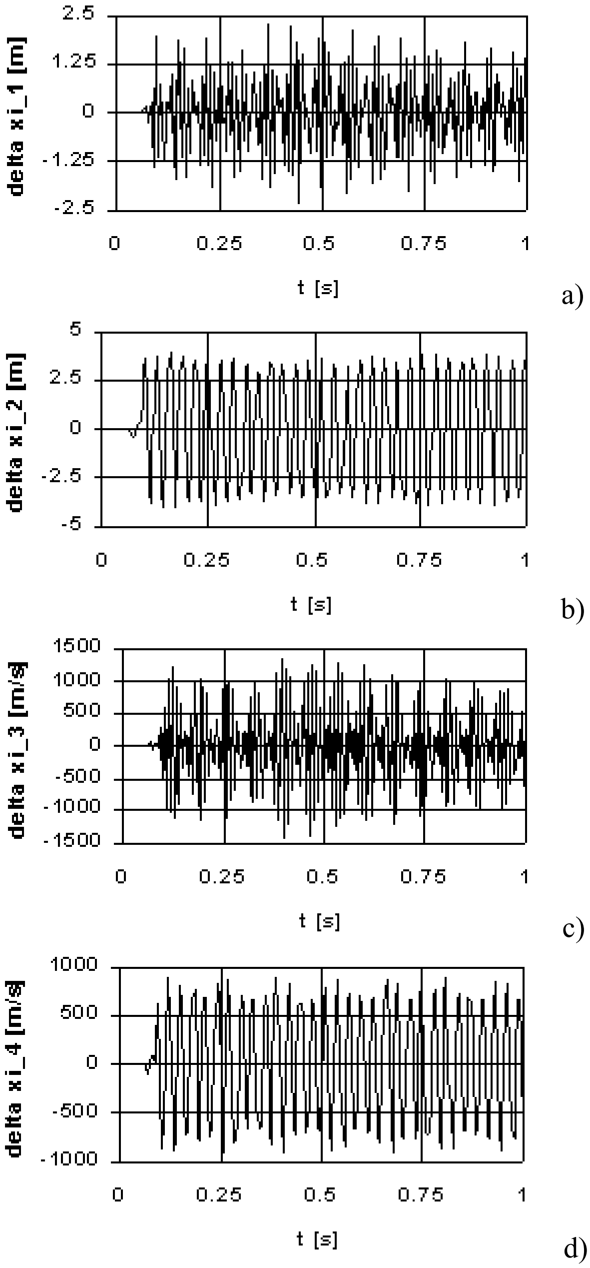

Fig. 7. The deviations of the perturbed motion with respect to the unperturbed motion; a) $\Delta\xi_1 = \Delta\xi_1(t)$ for $0 \le t \le 1[\mathrm{s}]$; b) $\Delta\xi_2 = \Delta\xi_2(t)$ for $0 \le t \le 1[\mathrm{s}]$; c) $\Delta\xi_3 = \Delta\xi_3(t)$ for $0 \le t \le 1[\mathrm{s}]$; d) $\Delta\xi_4 = \Delta\xi_4(t)$ for $0 \le t \le 1[\mathrm{s}]$.

$$j_{31} < 0;\ j_{32} > 0;\ j_{41} < 0;\ j_{42} < 0. \quad (100)$$

The discriminate of the characteristic equation is given by the formula (28) and it is also positive, the expression (30) remaining true.

In addition, the roots of the characteristic equation are pure imaginary, the equilibrium being simply stable.

The system in deviations has again the form (41), the roots of the characteristic equation being pure imaginary and given by the formulas (47).

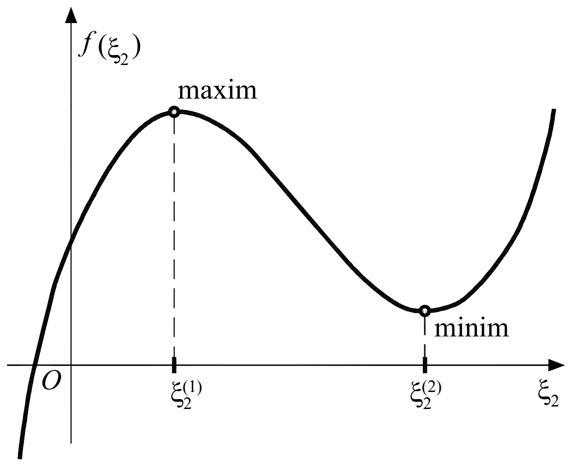

Fig. 8. The case $f\left(\xi_2^{(2)}\right) > 0$.

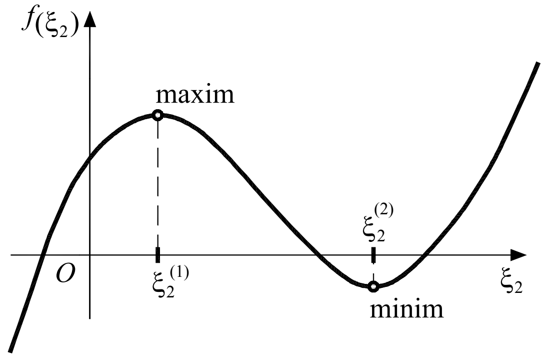

Fig. 9. The case $f\left(\xi_2^{(2)}\right) < 0$.

THEOREM 3. *In the case* $k_2 < 0$, $k_2 < -\dfrac{4m_2^3g^3}{27k_1^2}$, *the equilibrium is always stable and not asymptotically stable.*

Proof: It is identical to that of the theorem 1.

We shall go now to the case described in the figure 9, that is $k_2 < 0$, $k_2 > -\dfrac{4m_2^3g^3}{27k_1^2}$.

Obviously, if $\xi_2 \in \left(-\infty, \xi_2^{(1)}\right)$, the discussion doesn't change being the same as in the previous case.

THEOREM 4. *In the case* $k_2 < 0$, $k_2 > -\dfrac{4m_2^3g^3}{27k_1^2}$, *the equilibrium position* $\xi_2 \in \left(-\infty, \xi_2^{(1)}\right)$ *is always stable and not asymptotically stable.*

Proof: It is identical to that of the theorem 1.

Let us now consider the situations for which $\xi_2 \in \left(\xi_2^{(1)}, \xi_2^{(2)}\right)$ or $\xi_2 \in \left(\xi_2^{(2)}, +\infty\right)$.

In this case, the inequality

$$\sqrt{\Delta} = \left|j_{42} + j_{31}\right| \tag{101}$$

is equivalent, according to formula (35), with

$$\frac{2k_2}{\left(\xi_2 - \xi_1\right)^3} + k_1 > 0, \tag{102}$$

or, equivalently,

$$k_2 < -\frac{k_1\left(\xi_2 - \xi_1\right)^3}{2}. \tag{103}$$

Let be the equilibrium position $\xi_2 \in \left(\xi_2^{(2)}, \infty\right)$. From the expression (12) one deduces

$$\xi_2 - \xi_1 > \xi_2^{(2)} - \xi_1 = \frac{2m_2g}{3k_1}. \tag{104}$$

Keeping into account the formula (104) and the fact that $k_2 < 0$, the relation (102) leads us to

$$\frac{2k_2}{\left(\xi_2 - \xi_1\right)^3} + k_1 > \frac{2k_2}{\left(\dfrac{2m_2g}{3k_1}\right)^3} + k_1 = \frac{27k_1^3k_2}{4m_2^3g^3} + k_1 = k_1\left(1 + \frac{27k_1^2k_2}{4m_2^3g^3}\right). \tag{105}$$

But, since $k_2 > -\dfrac{4m_2^3g^3}{27k_1^2}$, the relation (105) offers us

$$\frac{2k_2}{\left(\xi_2 - \xi_1\right)^3} + k_1 > k_1\left(1 - \frac{27k_1^2}{4m_2^3g^3}\frac{4m_2^3g^3}{27k_1^2}\right) = 0. \tag{106}$$

Therefore we proved that for $\xi_2 > \xi_2^{(2)}$, the relation (102) is always true.

On the other hand, the condition

$$j_{42} + j_{31} < 0 \tag{107}$$

leads us to

$$\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\left[-\frac{2k_2}{\left(\xi_2 - \xi_1\right)^3} - k_1\right] - \frac{k}{m_1} < 0. \tag{108}$$

One observes that if the formula (102) is true, then the formula (108) is also true. Therefore, we have to verify only the condition (102).

THEOREM 5. *In the case* $k_2 < 0$, $k_2 > -\dfrac{4m_2^3g^3}{27k_1^2}$, *the equilibrium position* $\xi_2 > \xi_2^{(2)}$ *is always stable and not asymptotically stable.*

Proof: It is obvious from the above discussion.

Let us consider now the root $\xi_2 \in \left(\xi_2^{(1)}, \xi_2^{(2)}\right)$.

THEOREM 6. *In the case* $k_2 < 0$, $k_2 > -\dfrac{4m_2^3g^3}{27k_1^2}$, *the equilibrium position* $\xi_2 \in \left(\xi_2^{(1)}, \xi_2^{(2)}\right)$ *is always unstable.*

Proof: From the second formula (5) we deduce

$$\frac{k_2}{(\xi_2-\xi_1)^2}=k_1(\xi_2-\xi_1)-m_2g\,, \tag{109}$$

wherefrom

$$\frac{2k_2}{(\xi_2-\xi_1)^3}+k_1=3k_1-\frac{2m_2g}{\xi_2-\xi_1}. \tag{110}$$

But $\xi_2\in\left(\xi_2^{(1)},\xi_2^{(2)}\right)$ and from the expression (13) we have

$$\xi_2-\xi_1<\frac{2m_2g}{3k_1}. \tag{111}$$

The relation (110) offers now

$$\frac{2k_2}{(\xi_2-\xi_1)^3}+k_1<3k_1-\frac{2m_2g}{\frac{2m_2g}{3k_1}}=0. \tag{112}$$

It results that the equilibrium is unstable and the theorem is proved.

3 Neo–Hookean Suspension for a Half of an Automobile

3.1 Mathematical model

We shall now present the study of the motion for four degrees of freedom system that models a half of an automobile. The model is presented in figure 10. This model consists of the masses m_1 and m_2, which mark the wheels of the automobile, masses linked to the ground by linear elastic springs of stiffness k_1 and k_2, respectively. By wheels is attached the chassis marked by the bar AB of mass M. The linking of the chassis is made by the non-linear neo–Hookean elastic elements by elastic stiffness d_1, e_1, respectively d_2, e_2. The elastic force that appears in such element is given by

$$F=d_iz_i-\frac{e_i}{z_i^2}, \tag{113}$$

where $i=\overline{1,2}$, z_i marks the elongation of the respective element, and $d_i>0$, $e_1>0$, $i=\overline{1,2}$.

The four degrees of freedom of the system were selected as follows: q_1, q_2 the elongations of the linear springs, q_3 the displacement in the vertical direction of the gravity centre G of the chassis and q_4 the rotation of the chassis with respect to the horizontal.

We assume that there are known the dimensions L_1 and L_2 that define the position of the gravity centre G of the chassis with respect to the two wheels and J the moment of the inertia with respect to a horizontal axis that passes through its gravity centre.

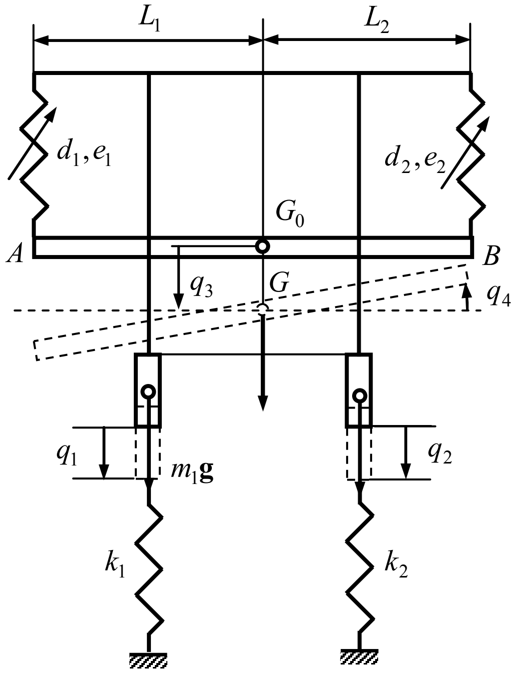

Fig. 10. The mathematical model.

3.2 The equations of motion

The kinetic energy of the system has the expression

$$T=\frac{1}{2}m_1\dot q_1^2+\frac{1}{2}m_2\dot q_2^2+\frac{1}{2}M\dot q_3^2+\frac{1}{2}J\dot q_4^2. \tag{114}$$

The forces, which appear in the system, derive from a potential, hence the potential energy reads

$$\begin{aligned} V&=\frac{1}{2}k_1q_1^2-m_1gq_1+\frac{1}{2}k_2q_2^2\\ &-m_2gq_2+\frac{1}{2}d_1(L_1q_4-q_1+q_3)^2\\ &+\frac{e_1}{L_1q_4-q_1+q_3}+\frac{d_2}{2}(q_3-L_2q_4-q_2)^2\\ &+\frac{e_2}{q_3-L_2q_4-q_2}-Mgq_3\,, \end{aligned} \tag{115}$$

g being the gravitational acceleration.

We successively calculate

$$\frac{\partial T}{\partial\dot q_1}=m_1\dot q_1;\ \frac{\partial T}{\partial\dot q_2}=m_2\dot q_2;\ \frac{\partial T}{\partial\dot q_3}=M\dot q_3;$$
$$\frac{\partial T}{\partial\dot q_4}=J\dot q_4, \tag{116}$$

$$\frac{\mathrm d}{\mathrm dt}\left(\frac{\partial T}{\partial\dot q_1}\right)=m_1\ddot q_1;\ \frac{\mathrm d}{\mathrm dt}\left(\frac{\partial T}{\partial\dot q_2}\right)=m_2\ddot q_2;$$
$$\frac{\mathrm d}{\mathrm dt}\left(\frac{\partial T}{\partial\dot q_3}\right)=M\ddot q_3;\ \frac{\mathrm d}{\mathrm dt}\left(\frac{\partial T}{\partial\dot q_4}\right)=J\ddot q_4, \tag{117}$$

$$\frac{\partial T}{\partial q_1} = 0\,;\ \frac{\partial T}{\partial q_2} = 0\,;\ \frac{\partial T}{\partial q_3} = 0\,;\ \frac{\partial T}{\partial q_4} = 0\,, \tag{118}$$

$$\frac{\partial V}{\partial q_1} = k_1 q_1 - m_1 g - d_1(L_1 q_4 - q_1 + q_3) + \frac{e_1}{(L_1 q_4 - q_1 + q_3)^2}\,, \tag{119a}$$

$$\frac{\partial V}{\partial q_2} = k_2 q_2 - m_2 g - d_2(q_3 - L_2 q_4 - q_2) + \frac{e_2}{(q_3 - L_2 q_4 - q_2)^2}\,, \tag{119b}$$

$$\frac{\partial V}{\partial q_3} = d_1(L_1 q_4 - q_1 + q_3) - \frac{e_1}{(L_1 q_4 - q_1 + q_3)^2} + d_2(q_3 - L_2 q_4 - q_2) - \frac{e_2}{(q_3 - L_2 q_4 - q_2)^2} - Mg\,, \tag{119c}$$

$$\frac{\partial V}{\partial q_4} = L_1 d_1(L_1 q_4 - q_1 + q_3) - \frac{L_1 e_1}{(L_1 q_4 - q_1 + q_3)^2} - L_2 d_2(q_3 - L_2 q_4 - q_2) + \frac{L_2 e_2}{(q_3 - L_2 q_4 - q_2)^2}\,, \tag{119d}$$

such that the Lagrange equations read

$$m_1\ddot{q}_1 + k_1 q_1 - m_1 g - d_1(L_1 q_4 - q_1 + q_3) + \frac{e_1}{(L_1 q_4 - q_1 + q_3)^2} = 0\,, \tag{120a}$$

$$m_2\ddot{q}_2 + k_2 q_2 - m_2 g - d_2(q_3 - L_2 q_4 - q_2) + \frac{e_2}{(q_3 - L_2 q_4 - q_2)^2} = 0\,, \tag{120b}$$

$$M\ddot{q}_3 + d_1(L_1 q_4 - q_1 + q_3) - \frac{e_1}{(L_1 q_4 - q_1 + q_3)^2} + d_2(q_3 - L_2 q_4 - q_2) - \frac{e_2}{(q_3 - L_2 q_4 - q_2)^2} - Mg = 0\,, \tag{120c}$$

$$J\ddot{q}_4 + L_1 d_1(L_1 q_4 - q_1 + q_3) - \frac{L_1 e_1}{(L_1 q_4 - q_1 + q_3)^2} - L_2 d_2(q_3 - L_2 q_4 - q_2) + \frac{L_2 e_2}{(q_3 - L_2 q_4 - q_2)^2} = 0\,. \tag{120d}$$

Let us denote

$$\xi_1 = q_1\,;\ \xi_2 = q_2\,;\ \xi_3 = q_3\,;\ \xi_4 = q_4\,;\ \xi_5 = \dot{q}_1\,;\ \xi_6 = \dot{q}_2\,;\ \xi_7 = \dot{q}_3\,;\ \xi_8 = \dot{q}_4 \tag{121}$$

obtaining a system of eight first order non-linear differential equations

$$\frac{d\xi_1}{dt} = \xi_5\,;\ \frac{d\xi_2}{dt} = \xi_6\,;\ \frac{d\xi_3}{dt} = \xi_7\,;\ \frac{d\xi_4}{dt} = \xi_8\,;$$

$$\frac{d\xi_5}{dt} = \frac{1}{m_1}\left[-k_1\xi_1 + m_1 g + d_1(L_1\xi_4 - \xi_1 + \xi_3) - \frac{e_1}{(L_1\xi_4 - \xi_1 + \xi_3)^2}\right];$$

$$\frac{d\xi_6}{dt} = \frac{1}{m_2}\left[-k_2\xi_2 + m_2 g + d_2(\xi_3 - L_2\xi_4 - \xi_2) - \frac{e_2}{(\xi_3 - L_2\xi_4 - \xi_2)^2}\right];$$

$$\frac{d\xi_7}{dt} = \frac{1}{M}\left[-d_1(L_1\xi_4 - \xi_1 + \xi_3) + \frac{e_1}{(L_1\xi_4 - \xi_1 + \xi_3)^2} - d_2(\xi_3 - L_2\xi_4 - \xi_2) + \frac{e_2}{(\xi_3 - L_2\xi_4 - \xi_2)^2} + Mg\right],$$

$$\frac{d\xi_8}{dt} = \frac{1}{J}\left[-L_1 d_1(L_1\xi_4 - \xi_1 + \xi_3) + \frac{L_1 e_1}{(L_1\xi_4 - \xi_1 + \xi_3)^2} + L_2 d_2(\xi_3 - L_2\xi_4 - \xi_2) - \frac{L_2 e_2}{(\xi_3 - L_2\xi_4 - \xi_2)^2}\right]. \tag{122}$$

3.3 The equilibrium positions

These are obtained at the intersection of the nullclines, resulting the system

$$\xi_5 = 0\,;\ \xi_6 = 0\,;\ \xi_7 = 0\,;\ \xi_8 = 0\,, \tag{123}$$

$$-k_1\xi_1 + m_1 g + d_1(L_1\xi_4 - \xi_1 + \xi_3) - \frac{e_1}{(L_1\xi_4 - \xi_1 + \xi_3)^2} = 0\,, \tag{124a}$$

$$-k_2\xi_2 + m_2 g + d_2(\xi_3 - L_2\xi_4 - \xi_2) - \frac{e_2}{(\xi_3 - L_2\xi_4 - \xi_2)^2} = 0\,, \tag{124b}$$

$$-d_1(L_1\xi_4 - \xi_1 + \xi_3) + \frac{e_1}{(L_1\xi_4 - \xi_1 + \xi_3)^2} - d_2(\xi_3 - L_2\xi_4 - \xi_2) + \frac{e_2}{(\xi_3 - L_2\xi_4 - \xi_2)^2} + Mg = 0\,, \tag{124c}$$

$$-L_1d_1\left(L_1\xi_4-\xi_1+\xi_3\right)+\frac{L_1e_1}{\left(L_1\xi_4-\xi_1+\xi_3\right)^2}+L_2d_2\left(\xi_3-L_2\xi_4-\xi_2\right)-\frac{L_2e_2}{\left(\xi_3-L_2\xi_4-\xi_2\right)^2}=0. \quad (124d)$$

Adding the first three relations (124), one obtains the equation

$$-k_1\xi_1-k_2\xi_2+\left(m_1+m_2+M\right)g=0. \quad (125)$$

Multiplying the first relation (124) by L_1, the second relation (124) by $-L_2$ and summing the results at the last expression (124), we deduce

$$-L_1k_1\xi_1+L_2k_2\xi_2+\left(L_1m_1-L_2m_2\right)g=0. \quad (126)$$

The relations (125) and (126) form a linear system of two equations with two unknowns (ξ_1 and ξ_2)

$$k_1\xi_1+k_2\xi_2=\left(m_1+m_2+M\right)g;\quad L_1k_1\xi_1-L_2k_2\xi_2=\left(L_1m_1-L_2m_2\right)g, \quad (127)$$

the solution of this system being

$$\xi_1=\frac{\begin{vmatrix}\left(m_1+m_2+M\right)g & k_2\\ \left(L_1m_1-L_2m_2\right)g & -L_2k_2\end{vmatrix}}{\begin{vmatrix}k_1 & k_2\\ L_1k_1 & -L_2k_2\end{vmatrix}}=\frac{m_1\left(L_1+L_2\right)+L_2M}{\left(L_1+L_2\right)k_1}g, \quad (128a)$$

$$\xi_2=\frac{\begin{vmatrix}k_1 & \left(m_1+m_2+M\right)g\\ L_1k_1 & \left(L_1m_1-L_2m_2\right)g\end{vmatrix}}{\begin{vmatrix}k_1 & k_2\\ L_1k_1 & -L_2k_2\end{vmatrix}}=\frac{m_2\left(L_1+L_2\right)+L_1M}{\left(L_1+L_2\right)k_2}g. \quad (128b)$$

We multiply now the third equation (124) by $-L_1$ and we add it to the last equation (124) obtaining

$$\left(L_1+L_2\right)d_2\left(\xi_3-L_2\xi_4-\xi_2\right)-\frac{\left(L_1+L_2\right)e_2}{\left(\xi_3-L_2\xi_4-\xi_2\right)^2}-L_1Mg=0 \quad (129)$$

or, equivalently,

$$\left(\xi_3-L_2\xi_4-\xi_2\right)^3-\frac{L_1}{\left(L_1+L_2\right)d_2}\times Mg\left(\xi_3-L_2\xi_4-\xi_2\right)^2-\frac{e_2}{d_2}=0. \quad (130)$$

We multiply the third equation (124) by L_2 and we add it to the last equation (124) resulting

$$-\left(L_1+L_2\right)d_1\left(\xi_3+L_1\xi_4-\xi_1\right)^3+\frac{\left(L_1+L_2\right)e_1}{\left(\xi_3+L_1\xi_4-\xi_1\right)^2}+L_2Mg=0 \quad (131)$$

or, equivalently,

$$\left(\xi_3+L_1\xi_4-\xi_1\right)^3-\frac{L_2}{\left(L_1+L_2\right)d_1}\times Mg\left(\xi_3+L_1\xi_4-\xi_1\right)^2-\frac{e_1}{d_1}=0. \quad (132)$$

Let us consider for the beginning the equation (130) and let us denote

$$z=\xi_3-L_2\xi_4-\xi_2;\ \alpha=\frac{L_1}{\left(L_1+L_2\right)d_1}Mg;\ \beta=\frac{e_2}{d_2};\ \alpha>0;\ \beta>0, \quad (133)$$

resulting the relation

$$z^3-\alpha z^2-\beta=0. \quad (134)$$

In the sequence of the coefficients for the equation (134) there exists only one variation of sign and applying the Descartes theorem, it results that the equation (134) has only one positive real root. Making the change of variable $z\mapsto -z$, one obtains the equation

$$z^3+\alpha z^2+\beta=0 \quad (135)$$

for which there exists no variation of sign in the sequence of the coefficients. Applying again the Descartes theorem, it results that the equation (135) has no positive real root and therefore the equation (134) has no negative real root. In the end, we obtained that the equation (134) has only one real root, thus the equation (130) has one real root, too. Let us denote this root by z_1.

Proceeding in an analogous way, one deduces that the equation (132) has one real root and we denote this root by z_2.

It results the system

$$\xi_3-L_2\xi_4-\xi_2=z_1;\ \xi_3+L_1\xi_4-\xi_1=z_2, \quad (136)$$

for which the solution is

$$\xi_3=\frac{\begin{vmatrix}z_1+\xi_2 & -L_2\\ z_2+\xi_1 & L_1\end{vmatrix}}{\begin{vmatrix}1 & -L_2\\ 1 & L_1\end{vmatrix}}=\frac{L_1\left(z_1+\xi_2\right)+L_2\left(z_2+\xi_1\right)}{L_1+L_2}, \quad (137a)$$

respectively

$$\xi_4 = \frac{\begin{vmatrix} 1 z_1 + \xi_2 \\ 1 z_2 + \xi_1 \end{vmatrix}}{\begin{vmatrix} 1 - L_2 \\ 1 \ L_1 \end{vmatrix}} \quad (137b)$$

$$= \frac{(z_2 + \xi_1) - (z_1 + \xi_2)}{L_1 + L_2}.$$

We obtained that there exists only one equilibrium position defined by the relations (128) and (137).

3.4 Stability of the equilibrium

Let us denote by f_i the expressions in the right-hand side of the relations (122) and let be

$$j_{kl} = \frac{\partial f_k}{\partial \xi_l};\ k = \overline{1,8};\ l = \overline{1,8}. \quad (138)$$

We have

$$j_{11} = 0;\ j_{12} = 0;\ j_{13} = 0;\ j_{14} = 0;\ j_{15} = 1;$$
$$j_{16} = 0;\ j_{17} = 0;\ j_{18} = 0, \quad (139)$$
$$j_{21} = 0;\ j_{22} = 0;\ j_{23} = 0;\ j_{24} = 0;\ j_{25} = 0;$$
$$j_{26} = 1;\ j_{27} = 0;\ j_{28} = 0, \quad (140)$$
$$j_{31} = 0;\ j_{32} = 0;\ j_{33} = 0;\ j_{34} = 0;\ j_{35} = 0;$$
$$j_{36} = 0;\ j_{37} = 1;\ j_{38} = 0, \quad (141)$$
$$j_{41} = 0;\ j_{42} = 0;\ j_{43} = 0;\ j_{44} = 0;\ j_{45} = 0;$$
$$j_{46} = 0;\ j_{47} = 0;\ j_{48} = 1, \quad (142)$$
$$j_{51} = -\frac{k_1}{m_1} - \frac{d_1}{m_1} - \frac{2e_1}{m_1(L_1\xi_4 - \xi_1 + \xi_3)^3};\ j_{52} = 0;$$
$$j_{53} = \frac{d_1}{m_1} + \frac{2e_1}{m_1(L_1\xi_4 - \xi_1 + \xi_3)^3};$$
$$j_{54} = \frac{d_1 L_1}{m_1} + \frac{2e_1 L_1}{(L_1\xi_4 - \xi_1 + \xi_3)^3};\ j_{55} = 0;\ j_{56} = 0;$$
$$j_{57} = 0;\ j_{58} = 0, \quad (143)$$
$$j_{61} = 0;\ j_{62} = -\frac{k_2}{m_2} - \frac{d_2}{m_2} - \frac{2e_2}{m_2(\xi_3 - L_2\xi_4 - \xi_2)^3};$$
$$j_{63} = \frac{d_2}{m_2} + \frac{2e_2}{m_2(\xi_3 - L_2\xi_4 - \xi_2)^3};$$
$$j_{64} = -\frac{d_2 L_2}{m_2} - \frac{2e_2 L_2}{m_2(\xi_3 - L_2\xi_4 - \xi_2)^3};\ j_{65} = 0;$$
$$j_{66} = 0;\ j_{67} = 0;\ j_{68} = 0, \quad (144)$$
$$j_{71} = \frac{d_1}{M} + \frac{2e_1}{M(L_1\xi_4 - \xi_1 + \xi_3)^3};$$
$$j_{72} = \frac{d_2}{M} + \frac{2e_2}{M(\xi_3 - L_2\xi_4 - \xi_2)^3};$$
$$j_{73} = -\frac{d_1}{M} - \frac{2e_1}{M(L_1\xi_4 - \xi_1 + \xi_3)^3} - \frac{d_2}{M} - \frac{2e_2}{M(\xi_3 - L_2\xi_4 - \xi_2)^3};$$
$$j_{74} = -\frac{d_1 L_1}{M} - \frac{2L_1 e_1}{M(L_1\xi_4 - \xi_1 + \xi_3)^3} + \frac{d_2 L_2}{M} + \frac{2e_2 L_2}{M(\xi_3 - L_2\xi_4 - \xi_2)^3};$$
$$j_{75} = 0;\ j_{76} = 0;\ j_{77} = 0;\ j_{78} = 0, \quad (145)$$
$$j_{81} = \frac{L_1 d_1}{J} + \frac{2L_1 e_1}{J(L_1\xi_4 - \xi_1 + \xi_3)^3};$$
$$j_{82} = -\frac{L_2 d_2}{J} - \frac{2L_2 e_2}{J(\xi_3 - L_2\xi_4 - \xi_2)^3};$$
$$j_{83} = -\frac{L_1 d_1}{J} - \frac{2L_1 e_1}{J(L_1\xi_4 - \xi_1 + \xi_3)^3} + \frac{L_2 d_2}{J} + \frac{2L_2 e_2}{J(\xi_3 - L_2\xi_4 - \xi_2)^3};$$
$$j_{84} = -\frac{L_1^2 d_1}{J} - \frac{2L_1^2 e_1}{J(L_1\xi_4 - \xi_1 + \xi_3)^3} - \frac{L_2^2 d_2}{J} - \frac{2L_2^2 e_2}{J(\xi_3 - L_2\xi_4 - \xi_2)^3};\ j_{85} = 0;$$
$$j_{86} = 0;\ j_{87} = 0;\ j_{88} = 0. \quad (146)$$

The characteristic equation

$$\det(\mathbf{J} - \lambda\mathbf{I}) = 0, \quad (147)$$

where $\mathbf{J}$ is the Jacobi matrix

$$\mathbf{J} = [j_{kl}]_{k,l=\overline{1,8}}, \quad (148)$$

and $\mathbf{I}$ is the eight-order unity matrix, reads

$$\begin{vmatrix} -\lambda & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\lambda & 0 & 0 & 0 & 1 \\ j_{51} & 0 & j_{53} & j_{54} & -\lambda & 0 & 0 & 0 \\ 0 & j_{62} & j_{63} & j_{64} & 0 & -\lambda & 0 & 0 \\ j_{71} & j_{72} & j_{73} & j_{74} & 0 & 0 & -\lambda & 0 \\ j_{81} & j_{82} & j_{83} & j_{84} & 0 & 0 & 0 & -\lambda \end{vmatrix} = 0. \quad (149)$$

Multiplying the columns five, six, seven and eight by λ and summing the obtained results to the columns one, two, three and four, respectively, one deduces the equation

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ j_{51}-\lambda^2 & 0 & j_{53} & j_{54} & -\lambda & 0 & 0 & 0 \\ 0 & j_{62}-\lambda^2 & j_{63} & j_{64} & 0 & -\lambda & 0 & 0 \\ j_{71} & j_{72} & j_{73}-\lambda^2 & j_{74} & 0 & 0 & -\lambda & 0 \\ j_{81} & j_{82} & j_{83} & j_{84}-\lambda^2 & 0 & 0 & 0 & -\lambda \end{vmatrix} = 0. \tag{150}$$

Developing the determinant after the lines one, two, three and four, it results

$$\begin{vmatrix} j_{51}-\lambda^2 & 0 & j_{53} & j_{54} \\ 0 & j_{62}-\lambda^2 & j_{63} & j_{64} \\ j_{71} & j_{72} & j_{73}-\lambda^2 & j_{74} \\ j_{81} & j_{82} & j_{83} & j_{84}-\lambda^2 \end{vmatrix} = 0 \tag{151}$$

or, equivalently,

$$\begin{aligned} &\left(j_{51}-\lambda^2\right)\begin{vmatrix} j_{62}-\lambda^2 & j_{63} & j_{64} \\ j_{72} & j_{73}-\lambda^2 & j_{74} \\ j_{82} & j_{83} & j_{84}-\lambda^2 \end{vmatrix} \\ &+ j_{53}\begin{vmatrix} 0 & j_{62}-\lambda^2 & j_{64} \\ j_{71} & j_{72} & j_{74} \\ j_{81} & j_{82} & j_{84}-\lambda^2 \end{vmatrix} \\ &- j_{54}\begin{vmatrix} 0 & j_{62}-\lambda^2 & j_{63} \\ j_{71} & j_{72} & j_{73}-\lambda^2 \\ j_{81} & j_{82} & j_{83} \end{vmatrix} = 0. \end{aligned} \tag{152}$$

The relation (151) is a bi-square equation of the fourth order in the unknown λ^2. It also offers the condition for the equilibrium position to be stable or unstable because it imposes a relation of connectivity in the space of the parameters k_1, k_2, d_1, d_2, e_1 and e_2.

3.5 Application

Let us consider the practical case for which

$$k_1 = k_2 = 4\cdot 10^5[\mathrm{N/m}];\ d_1 = d_2 = 5\cdot 10^4[\mathrm{N/m}];$$
$$e_1 = e_2 = 5[\mathrm{Nm^2}];\ L_1 = L_2 = 2[\mathrm{m}];\ M = 900[\mathrm{kg}];$$
$$m_1 = m_2 = 25[\mathrm{kg}];\ g = 10[\mathrm{m/s^2}]. \tag{153}$$

The relations (128) offer

$$\xi_1 = q_1^{ech} = 0.0011875[\mathrm{m}];$$
$$\xi_2 = q_2^{ech} = 0.0011875[\mathrm{m}]. \tag{154}$$

The equation (130) becomes

$$z^3 - \frac{2}{4\cdot 5\cdot 10^4}\cdot 900\cdot 10 z^2 - \frac{5}{5\cdot 10^4} = 0 \tag{155}$$

wherefrom

$$z^3 - 0.09z^2 - 0.0001 = 0 \tag{156}$$

with the solution

$$z = z_1^{ech} = 0.1[\mathrm{m}]. \tag{157}$$

In an analogous way we find

$$z_2^{ech} = 0.1[\mathrm{m}]. \tag{158}$$

The expressions (137) offer

$$\xi_3 = q_3^{ech} = 0.1011875[\mathrm{m}], \tag{159}$$
$$\xi_4 = 0.1011875[\mathrm{m}]. \tag{160}$$

The partial derivatives read

$$j_{51} = -18400;\ j_{52} = 0;\ j_{53} = 2400;$$
$$j_{54} = 4800, \tag{161}$$
$$j_{61} = 0;\ j_{62} = -18400;\ j_{63} = 2400;$$
$$j_{64} = -4800, \tag{162}$$
$$j_{71} = 66.667;\ j_{72} = 66.667;\ j_{73} = -133.333;$$
$$j_{74} = 0, \tag{163}$$
$$j_{81} = 100;\ j_{82} = -100;\ j_{83} = 0;$$
$$j_{84} = -400. \tag{164}$$

Results the characteristic equation

$$\begin{vmatrix} -18400-\lambda^2 & 0 & 2400 & 4800 \\ 0 & -18400-\lambda^2 & 2400 & -4800 \\ 66.667 & 66.667 & -133.333-\lambda^2 & 0 \\ 100 & -100 & -\lambda^2 & -400-\lambda^2 \end{vmatrix} = 0, \tag{165}$$

wherefrom

$$\lambda^8 + 37333.333\lambda^6 + 356959994.7\lambda^4 + $$
$$+ 1.5872\cdot 10^{11}\lambda^2 + 1.36533\cdot 10^{13} = 0. \tag{166}$$

We denote

$$\lambda^2 = \eta \tag{167}$$

and one obtains the four-order equation

$$\eta^4 + 37333.333\eta^3 + 356959994.7\eta^2 +$$
$$+ 1.5872\cdot 10^{11}\eta + 1.36533\cdot 10^{13} = 0. \tag{168}$$

The solving of this equation is made by the Lobacevski–Graeffe method for which for the equation

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0 \tag{169}$$

the passing from the step p to the step $p+1$ takes place with the formulas

$$a_0^{(p+1)} = \left[a_0^{(p)}\right]^2;\ a_1^{(p+1)} = -\left\{\left[a_1^{(p)}\right]^2 - 2a_0^{(p)}a_2^{(p)}\right\};$$
$$a_2^{(p+1)} = \left[a_2^{(p)}\right]^2 - 2a_1^{(p)}a_3^{(p)} + 2a_0^{(p)}a_4^{(p)};$$
$$a_3^{(p+1)} = -\left\{\left[a_3^{(p)}\right]^2 - 2a_2^{(p)}a_4^{(p)}\right\};\ a_4^{(p+1)} = \left[a_4^{(p)}\right]^2. \tag{170}$$

We shall create the next table.

Table 1. The solving of equation (168) by the Lobacevski–Graeffe method.

Step	a_0	a_1	a_2
0	1	37333.333	356959997.7
1	1	–679857763.5	$1.156 \cdot 10^{17}$
2	1	$-2.310 \cdot 10^{17}$	$1.33 \cdot 10^{34}$
3	1	$-2.668 \cdot 10^{34}$	$1.78 \cdot 10^{68}$

Step	a_3	a_4
0	$1.5872 \cdot 10^{11}$	$1.36533 \cdot 10^{13}$
1	$-1.54 \cdot 10^{22}$	$1.864 \cdot 10^{26}$
2	$-1.95 \cdot 10^{44}$	$3.47 \cdot 10^{52}$
3	$-3.73 \cdot 10^{88}$	$1.21 \cdot 10^{105}$

Let be the function $h : \mathbb{R} \to \mathbb{R}$,

$$h(\eta) = \eta^4 + 37333.333\eta^3 + 356959994.7\eta^2 + 1.5872 \cdot 10^{11}\eta + 1.36533 \cdot 10^{13} \quad (171)$$

for which

$$h'(\eta) = 4\eta^3 + 112000\eta^2 + 713919989.4\eta + 1.5872 \cdot 10^{11}, \quad (172)$$

$$h''(\eta) = 12\eta^2 + 224000\eta + 713919989.4. \quad (173)$$

The equation $h''(\eta) = 0$ has the roots

$$\eta_{1,2} = \frac{-224000}{24} \pm \frac{\sqrt{224000^2 - 4 \cdot 12 \cdot 713919989.4}}{24}, \quad (174)$$

wherefrom

$$\eta_1 = -4078.07;\ \eta_2 = -14588.6. \quad (175)$$

In addition,

$$h'(\eta_1) = -1.161 \cdot 10^{12} < 0;$$

$$h'(\eta_2) = 1.161 \cdot 10^{12} > 0, \quad (176)$$

such that the equation $h'(\eta) = 0$ has three distinct real roots. We also have

$$h(-230) \approx -4.42 \cdot 10^{12} < 0;$$

$$h(-1) \approx 1.36 \cdot 10^{13} > 0;$$

$$h(-18000) \approx -3.8 \cdot 10^{12} < 0;$$

$$h(-5000) \approx 4.1 \cdot 10^{15} > 0 \quad (177)$$

and therefore the equation $h(\eta) = 0$ has four distinct negative real roots.

From the table 1 we get

$$\eta_1 = -\sqrt[8]{-\frac{a_1^{(3)}}{a_0^{(3)}}} = -\sqrt[8]{\frac{2.66 \cdot 10^{34}}{1}} \approx -20096, \quad (178a)$$

$$\eta_2 = -\sqrt[8]{-\frac{a_2^{(3)}}{a_1^{(3)}}} = -\sqrt[8]{\frac{1.78 \cdot 10^{68}}{2.66 \cdot 10^{34}}} \approx -16812, \quad (178b)$$

$$\eta_3 = -\sqrt[8]{-\frac{a_3^{(3)}}{a_2^{(3)}}} = -\sqrt[8]{\frac{3.73 \cdot 10^{88}}{1.78 \cdot 10^{68}}} \approx -346.8, \quad (178c)$$

$$\eta_4 = -\sqrt[8]{-\frac{a_4^{(3)}}{a_3^{(3)}}} = -\sqrt[8]{\frac{1.21 \cdot 10^{105}}{3.73 \cdot 10^{88}}} \approx -115.82. \quad (178d)$$

Result the roots of the characteristic equation

$$\lambda_1 \approx 141.76i;\ \lambda_2 \approx -141.76i;\ \lambda_3 \approx 130.05i;\ \lambda_4 \approx -130.05i;\ \lambda_5 \approx 18.62i;\ \lambda_6 \approx -18.62i;\ \lambda_7 \approx 10.76i;\ \lambda_8 \approx -10.76i \quad (179)$$

and all of them are pure imaginary, the equilibrium being simply stable.

4 Conclusions

In this work we presented two different models using rubber type components modeled as non-linear neo–Hookean elements. The first model is a quarter of an automobile model, and the second is the model of a half of automobile. For both models we obtained the differential equations of motion and we studied the equilibrium positions and their stability. For the first model we also studied the stability of the motion. A comparison between the linear and the non-linear case is also performed for both models. We proved that the utilization of the neo–Hookean element leads to the increasing zone where the resonance doesn't appears. For this reason, the neo–Hookean elements can be a valid substitute for the classical linear elements. In our future work we shall develop the models presented, using also linear and non-linear damping elements.

References:

[1] Stănescu, N.–D., Munteanu, L., Chiroiu, V., Pandrea, N., *Dynamical systems. Theory and applications, vol. 1*, The Publishing House of the Romanian Academy, Bucharest (Romania), 2007.

[2] Stănescu, N.–D., Munteanu, L., Chiroiu, V., Pandrea, N., *Dynamical systems. Theory and applications, vol. 2*, The Publishing House of the Romanian Academy, Bucharest (Romania), 2009 (in press).

[3] Pandrea, N., Stănescu, N.–D., *Mechanics*, Didactical and Pedagogical Publishing House, Bucharest (Romania), 2002.

[4] Stănescu, N.–D., *Numerical methods*, Didactical and Pedagogical Publishing House, Bucharest (Romania), 2007.

[5] Stănescu, N.–D., *Some problems of instability of the mechanical systems*, PhD thesis, University of Bucharest, Bucharest, 2008.

[6] Nayfeh, A., H., Mook, D., T., *Nonlinear oscillations*, John Wiley & Sons, New York, Chichester, Brisbane, Toronto, 1979.
[7] Arrowsmith, D., K., Place, C., M., *An Introduction to Dynamical Systems,* Cambridge University Press, 1994.
[8] Broer, H., W., Krauskopf, B., Vegter, G., (eds.) *Global Analysis of Dynamical Systems*, Institute of Physics Publishing, Bristol and Philadelphia, 2001.
[9] Guckenheimer J., Holmes, P., *Nonlinear Oscillations, Dynamical Systems, and Bifurcation of Vector Fields,* Springer–Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, Hong Kong, 1983.
[10] Perko, L., *Differential Equations and Dynamical Systems,* Springer–Verlag, New York, Berlin, Heidelberg, 1996.
[11] Minorsky, N., *Nonlinear Oscillations*, Van Nostrand, Princeton, 1962.
[12] Hayashi, C., *Forced Oscillations in Nonlinear Systems*, Nippon, Osaka, 1953.
[13] Hayashi, C., *Nonlinear Oscillations in Physical Systems*, McGraw–Hill, New York, 1964.
[14] Stănescu, N.–D., The periodical motions of a automotive with non-linear cubic suspension and harmonic excitation from the road, *Proceedings of the 15th International Congress of Sound and Vibrations*, Daejeon, South Korea, 2008.
[15] Arnold, V., I., (ed.) *Dynamical Systems. Bifurcation Theory and Catastrophe Theory,* Springer–Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, Hong Kong, Barcelona, Budapest, 1994.
[16] Carleson, L., Gamelin, T., W., *Complex Dynamics*, Springer–Verlag, New York, Heidelberg, London, Paris, 1993.
[17] Arnold, V., I., *Catastrophe theory*, Springer–Verlag, Berlin, Heidelberg, New York, Tokyo, 1984.
[18] Carr, J., *Applications of Centre Manifold Theory,* Springer–Verlag, New York, Heidelberg, Berlin, 1981.
[19] Hale, J., K., Koçak, H., *Dynamics and Bifurcations,* Springer–Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, Hong Kong, Barcelona, Budapest, 1991.
[20] Holmes, P., J., *Nonlinear dynamics. Chaos and mechanics*, Applied Mechanics Revue, 43, 1990.
[21] Kuznetsov, Y., A., *Elements of Applied Bifurcation Theory,* Springer–Verlag, New York, Berlin, Heidelberg, London Paris, Tokyo, Hong Kong, Barcelona, Budapest, 1995
[22] Lurie, A., I., *Analytical Mechanics*, Springer–Verlag, Berlin, Heidelberg, New York, Barcelona, Hong Kong, London, Milan, Paris, Tokyo, 2002
[23] Sibirsky, K., S., *Introduction to topological dynamics*, Noordhoff International Publishing, Leyden, The Netherlands, 1975.
[24] Stănescu, N.–D., A study of the small oscillations around the equilibrium position for an automotive with cubic non-linear suspensions, *9th International Congress, Car 2005*, paper Car20051124, 2005, Piteşti.
[25] Stănescu, N.–D., On the periodical motions around the equilibrium position for an automobile with nonlinear suspension, *1st International Conference, ″Computational Mechanics and Virtual Engineering ″COMEC 2005*, Braşov, Romania, 2005.
[26] Stănescu, N.–D., On the study of non-linear suspension, *Annals of the Oradea University, Fascicle of Management and Technological Engineering*, vol. V (XV), 2006.
[27] Stănescu, N.–D., Pandrea, N., Vieru, I., On the dynamics in the case of neo-Hookean automobile's suspensions, *12th IFToMM World Congress*, Besançon (France), June, 18-21, 2007
[28] Stănescu, N.–D., Popa, D., Stan, M., On the stability of the equilibrium for an automotive with neo–Hookean suspension, *2nd International Conference on Experiments/Process/System Modeling/Simulation & Optimization, 2nd IC-EpsMsO*, Athens, 4-7 July, 2007, © IC-EpsMsO.
[29] Stănescu, N.–D., Some aspects concerning the stability of the motion for an automotive with neo–Hookean suspension, *2nd International Conference on Experiments/Process/System Modeling/Simulation & Optimization, 2nd IC-EpsMsO*, Athens, 4-7 July, 2007, © IC-EpsMsO.
[30] Stănescu, N.–D., The study of an automotive with a non-linear cubic stiffness, *9th International Congress, Car 2005*, paper Car20051122, 2005, Piteşti.
[31] Stănescu, N.–D., Vieru, I., On the dynamics and the stability of an automotive with non-linear suspension, *9th International Congress, Car 2005*, paper Car20051123, 2005, Piteşti.
[32] Stănescu, N.–D., Vieru, I., On the stability of the equilibrium positions for an automobile with nonlinear suspension, *1st International Conference, ″Computational Mechanics and Virtual Engineering ″COMEC 2005*, Braşov, Romania, 2005.
[33] Stănescu, N.–D., Pandrea, M., Vieru, I., Pandrea, N., On the rocking vibrations of the automobiles with non-linear elastic stiffness, *Conference on the Dynamics of the Machines*,

Braşov, 2005.
[34] Vieru, I., Şerban, F., Stănescu, N.–D., Marinescu, D., Nicolae, V., Tabacu, Şt., Popa, N., Some possibilities for studying automotive suspensions using finite element method (FEM), *9th International Congress, Car 2005*, paper Car20051142, 2005, Piteşti.
[35] Voinea, R., P., Stroe, I., V., *Introduction in the theory of the dynamical systems,* Romanian Academy Publishing House, Bucharest, 2000.
[36] Wiggins, S., *Introduction to applied nonlinear dynamical systems and chaos,* Springer, New York, 1992.
[37] Stănescu, N.–D., Modelling of an Automobile with neo–Hookean Suspension, *Scientific Bulletin of the University of Piteşti, series Applied Mechanics*, 2008.
[38] Stănescu, N.–D., On the Stability of the Motion for a neo–Hookean System, *Proceedings of the Acústica 2008,* Coimbra, Portugal, 2008.
[39] Stănescu, N.–D., Non-linear neo–Hookean Systems, *Course for the IST, Lisbon, Master of Science in Acoustic*, 2008.