# Non-linear Analysis and Stability of the Automotive Suspensions with neo-Hookean Elements 

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#### Abstract

In this work we present two systems with non-linear neo-Hookean components. In such systems there always exists an element with a non-linear characteristic equation. This element can be considered to be a rubber or another equivalent structure. We shall prove that the utilization of a neo-Hookean element will not destroy the properties of the structure, but it riches these properties and it could be a good solution in many cases. The first model describes a quarter of an automobile and the second one is dedicated to a halfautomobile model. We obtain the equilibrium positions, study their stability in the most general case and for the first model we also discuss the stability of the motion. In the paper there are also numerical applications.


Key-Words: - Neo-Hookean, stability, modeling, non-linear

## 1 Introduction

In this work we shall present two systems with nonlinear neo-Hookean components. In such systems there always exists an element with a non-linear characteristic equation. This element can be considered to be a rubber or another equivalent structure.

We shall prove that the utilization of a neoHookean element will not destroy the properties of the structure, it riches these properties and it could be a good solution in many cases.

## 2 Neo-Hookean Suspension for a Quarter of an Automobile

It is known the specialists' preoccupation in the field of automotive to realize cars with high degree of comfort, equipped with suspensions which don't lead to resonance for a large enough set of excitation from the road. A solution in this direction is to equip the automobile with suspensions with neo-Hookean elements.

### 2.1 Formulation of the problem

We shall consider a quarter of an automobile schematized as two masses that oscillate in the vertical direction (Fig. 1).

The mass $m_{1}$ characterizes the wheel and the all other elements jointed to it, and the mass $m_{2}$ marks the quarter of the automobile.


Fig. 1. The model quarter of automotive.
The mass $m_{1}$ is linked to the road by the linear spring of stiffness $k$, and the mass $m_{2}$ is linked to the mass $m_{1}$ by the neo-Hookean element denoted by ENH for which the elastic force reads

$$
\begin{equation*}
F_{e}=k_{1} z-\frac{k_{2}}{z^{2}}, \tag{1}
\end{equation*}
$$

where $k_{1}$ and $k_{2}$ are two strict positive constants, and $z$ is the elongation of the neo-Hookean element.

### 2.2 The equations of motion

Isolating the two masses $m_{1}$ and $m_{2}$ (fig. 2) and writing for each one the Newton's law, it results

$$
\begin{align*}
& m_{1} \ddot{1}_{1}=-k z_{1}+k_{1}\left(z_{2}-z_{1}\right)-\frac{k_{2}}{\left(z_{2}-z_{1}\right)^{2}}+m_{1} g ; \\
& m_{2} \ddot{z}_{2}=-k_{1}\left(z_{2}-z_{1}\right)+\frac{k_{2}}{\left(z_{2}-z_{1}\right)^{2}}+m_{2} g . \tag{2}
\end{align*}
$$

Denoting now


Fig. 2. Isolation of the two masses.

$$
\begin{equation*}
\xi_{1}=z_{1} ; \xi_{2}=z_{2} ; \xi_{3}=\dot{z}_{1} ; \xi_{4}=\dot{z}_{2} \tag{3}
\end{equation*}
$$

the system (2) of the equations of motion can be written as a system of four first order non-linear differential equations, that is

$$
\begin{align*}
& \frac{\mathrm{d} \xi_{1}}{\mathrm{~d} t}=\xi_{3}  \tag{4a}\\
& \frac{\mathrm{~d} \xi_{2}}{\mathrm{~d} t}=\xi_{4} \tag{4b}
\end{align*}
$$

$$
\frac{\mathrm{d} \xi_{3}}{\mathrm{~d} t}=\frac{1}{m_{1}} \times
$$

$$
\begin{equation*}
\times\left[-k \xi_{1}+k_{1}\left(\xi_{2}-\xi_{1}\right)-\frac{k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{2}}+m_{1} g\right] \tag{4c}
\end{equation*}
$$

$$
\frac{\mathrm{d} \xi_{4}}{\mathrm{~d} t}=\frac{1}{m_{2}} \times
$$

$$
\begin{equation*}
\times\left[-k_{1}\left(\xi_{2}-\xi_{1}\right)-\frac{k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{2}}+m_{2} g\right] . \tag{4d}
\end{equation*}
$$

### 2.3 The determination of the equilibrium positions

The equilibrium positions are found at the nullclines' intersection of the system (4). Equating with 0 the right-hand expressions of the equations (4), results the system
$\xi_{3}=0 ; \xi_{4}=0$;
$-k \xi_{1}+k_{1}\left(\xi_{2}-\xi_{1}\right)-\frac{k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{2}}+m_{1} g=0 ;$
$-k_{1}\left(\xi_{2}-\xi_{1}\right)+\frac{k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{2}}+m_{2} g=0$.
Summing now the last two relations (5), one obtains
$k \xi_{1}=\left(m_{1}+m_{2}\right) g$,
wherefrom
$\xi_{1}=\frac{m_{1}+m_{2}}{k} g$.
The last relation (5) offers
$k_{1}\left(\xi_{2}-\xi_{1}\right)^{3}-m_{2} g\left(\xi_{2}-\xi_{1}\right)^{2}-k_{2}=0$.
Let us denote by $f\left(\xi_{2}\right)$ the function
$f\left(\xi_{2}\right)=k_{1}\left(\xi_{2}-\xi_{1}\right)^{3}-m_{2} g\left(\xi_{2}-\xi_{1}\right)^{2}-k_{2}$.
The derivative of the function $f$ is
$f^{\prime}\left(\xi_{2}\right)=3 k_{1}\left(\xi_{2}-\xi_{1}\right)^{2}-2 m_{2} g\left(\xi_{2}-\xi_{1}\right)$,
which, equated to 0 , leads us to
$\left(\xi_{2}-\xi_{1}\right)\left[3 k_{1}\left(\xi_{2}-\xi_{1}\right)-2 m_{2} g\right]=0$.
The roots of the derivative are
$\xi_{2}^{(1)}=\xi_{1} ; \xi_{2}^{(2)}=\frac{2 m_{2} g}{3 k_{1}}+\xi_{1}$.
For $\xi_{2}^{(1)}$ we have
$f\left(\xi_{2}^{(1)}\right)=-k_{2}<0$,
according to the hypothesis $k_{2}>0$, and for $\xi_{2}^{(2)}$ we can write

$$
\begin{align*}
& f\left(\xi_{2}^{(2)}\right)=k_{1}\left(\frac{2 m_{2} g}{3 k_{1}}\right)^{3}-m_{2} g\left(\frac{2 m_{2} g}{3 k_{1}}\right)^{2}-k_{2}=  \tag{14}\\
= & \frac{8 m_{2}^{3} g^{3}}{27 k_{1}^{2}}-\frac{4 m_{2}^{3} g^{3}}{9 k_{1}^{2}}-k_{2}=\frac{-4 m_{2}^{3} g^{3}}{27 k_{1}}-k_{2}<0 .
\end{align*}
$$

In addition, we also have
$\xi_{2}^{(1)}>0 ; \xi_{2}^{(2)}>0$.
Graphically, the situation is presented in the figure3.


Fig. 3. The graphic of the function $f\left(\xi_{2}\right)$.
We deduce that the equation $f\left(\xi_{2}\right)=0$ has only one root $\xi_{2}>\xi_{2}^{(2)}$.

### 2.4 The stability of the equilibrium

Let us consider the system (4) of the moving equations and let us rewrite it as
$\frac{\mathrm{d} \xi_{i}}{\mathrm{~d} t}=f_{i}\left(\xi_{1}, \xi_{2}, \xi_{3}, \xi_{4}\right) ; i=\overline{1,4}$,
where

$$
\begin{align*}
& f_{1}\left(\xi_{1}, \xi_{2}, \xi_{3}, \xi_{4}\right)=\xi_{3} ; f_{2}\left(\xi_{1}, \xi_{2}, \xi_{3}, \xi_{4}\right)=\xi_{4} ; \\
& =\frac{1}{m_{1}}\left[-k \xi_{1}+k_{1}\left(\xi_{2}-\xi_{2}, \xi_{3}\right)-\frac{k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{2}}+m_{1} g\right] ; \\
& \quad f_{4}\left(\xi_{1}, \xi_{2}, \xi_{3}, \xi_{4}\right)=\frac{1}{m_{2}} \\
& \times\left[-k\left(\xi_{2}-\xi_{1}\right)+\frac{k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{2}}+m_{2} g\right] .
\end{align*}
$$

Let us denote by
$j_{k l}=\frac{\partial f_{k}}{\partial \xi_{l}} ; k=\overline{1,4} ; l=\overline{1,4}$.
We have
$j_{11}=0 ; j_{12}=0 ; j_{13}=1 ; j_{14}=0$,
$j_{21}=0 ; j_{22}=0 ; j_{23}=0 ; j_{24}=1$,
$j_{31}=\frac{1}{m_{1}}-\left[-k-k_{1}-\frac{2 k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{3}}\right]$;
$j_{32}=\frac{1}{m_{1}}\left[\frac{2 k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{3}}+k_{1}\right] ; j_{33}=0 ; j_{34}=0$,
$j_{41}=\frac{1}{m_{2}}\left[k_{1}+\frac{2 k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{3}}\right] ;$
$j_{42}=\frac{1}{m_{2}}\left[-\frac{2 k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{3}}-k_{1}\right] ; j_{43}=0$;
$j_{44}=0$.
The characteristic equation
$\operatorname{det}([\mathbf{J}]-\varphi[\mathbf{I}])=0$,
in which $[\mathbf{J}]$ is the Jacobi matrix,

$$
[\mathbf{J}]=\left[j_{k l}\right]_{l=\overline{1,4}, 4}^{l},
$$

and $[\mathbf{I}]$ is the fourth order unity matrix, offers us

$$
\begin{align*}
& \left|\begin{array}{cccc}
-\varphi & 0 & 1 & 0 \\
0 & -\varphi & 0 & 1 \\
j_{31} & j_{32} & -\varphi & 0 \\
j_{41} & j_{42} & 0 & -\varphi
\end{array}\right|=0,  \tag{25}\\
& -\varphi\left|\begin{array}{ccc}
-\varphi & 0 & 1 \\
j_{32} & -\varphi & 0 \\
j_{42} & 0 & -\varphi
\end{array}\right|+\left|\begin{array}{ccc}
0 & -\varphi & 1 \\
j_{31} & j_{32} & 0 \\
j_{41} & j_{42} & -\varphi
\end{array}\right|=0,  \tag{26}\\
& \varphi^{4}-\left(j_{42}+j_{31}\right) \varphi^{2}+j_{31} j_{42}-j_{41} j_{32}=0 . \tag{27}
\end{align*}
$$

The discriminate of this bi-square equation is

$$
\begin{align*}
\Delta= & \left(j_{42}+j_{31}\right)^{2}-4\left(j_{31} j_{42}-j_{41} j_{32}\right)=  \tag{28}\\
& =\left(j_{42}-j_{31}\right)^{2}+4 j_{41} j_{32}>0,
\end{align*}
$$

because both $j_{41}$ and $j_{32}$ are strict positive expressions.

On the other hand,
$j_{42}<0 ; j_{31}<0$
and
$\sqrt{\Delta}<-\left(j_{42}+j_{31}\right)$.
Indeed, the relation (30), keeping into account that $\sqrt{\Delta}>0$ and $-\left(j_{42}+j_{31}\right)>0$, reads in the equivalent form
$\Delta<\left(j_{42}+j_{31}\right)^{2}$.
It results successively
$\left(j_{42}+j_{31}\right)^{2}-4\left(j_{31} j_{42}-j_{41} j_{32}\right)<\left(j_{42}+j_{31}\right)^{2}$,
$j_{31} j_{42}-j_{41} j_{32}>0$,
$\frac{1}{m_{1} m_{2}}\left[-k-k_{1}-\frac{2 k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{3}}\right] \times$
$\times\left[-\frac{2 k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{3}}+k_{1}\right]>0$,
$\frac{1}{m_{1} m_{2}} k\left[\frac{2 k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{3}}+k_{1}\right]>0$,
the last relation being obviously true from the above discussion.

The roots of the bi-squared equation read
$\varphi_{1,2}^{2}=\frac{j_{42}+j_{31} \pm \sqrt{\Delta}}{2}$
and one observes that
$\varphi_{1}^{2}<0 ; \varphi_{2}^{2}<0$,
that is, the roots of the characteristic equation are pure imaginary,
$\varphi_{1}=i \sqrt{-\frac{j_{42}+j_{31}+\sqrt{\Delta}}{2}} ;$
$\varphi_{2}=-i \sqrt{-\frac{j_{42}+j_{31}+\sqrt{\Delta}}{2}} ;$
$\varphi_{3}=i \sqrt{-\frac{j_{42}+j_{31}-\sqrt{\Delta}}{2}} ;$
$\varphi_{4}=-i \sqrt{-\frac{j_{42}+j_{31}-\sqrt{\Delta}}{2}}$.
Therefore, the equilibrium is simply stable.
Let us consider the equilibrium position $\left(\xi_{1}, \xi_{2}, 0,0\right)$ and a deviation $\left(u_{1}, u_{2}, u_{3}, u_{4}\right)$ sufficiently small in its norm.

Keeping into account that $\left(\xi_{1}, \xi_{2}, 0,0\right)$ is a solution of the differential equations system (4), it results the system in deviations

$$
\begin{align*}
& \frac{\mathrm{d} u_{1}}{\mathrm{~d} t}=u_{3}  \tag{39a}\\
& \frac{\mathrm{~d} u_{2}}{\mathrm{~d} t}=u_{4},  \tag{39b}\\
& \frac{\mathrm{~d} u_{3}}{\mathrm{~d} t}=\frac{1}{m_{1}}\left[-k\left(\xi_{1}+u_{1}\right)+k_{1}\right. \\
& \quad \times\left(\xi_{2}-\xi_{1}+u_{2}-u_{1}\right) \\
& \left.\quad-\frac{k_{2}}{\left(\xi_{2}-\xi_{1}+u_{2}-u_{1}\right)^{2}}+m_{1} g\right]-\frac{1}{m_{1}}  \tag{39c}\\
& \times\left[-k \xi_{1}+k_{1}\left(\xi_{2}-\xi_{1}\right)+\frac{k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{2}}+m_{1} g\right] \\
& \quad \frac{\mathrm{d} u_{4}}{\mathrm{~d} t}=\frac{1}{m_{2}}\left[-k_{1}\left(\xi_{2}-\xi_{1}+u_{2}-u_{1}\right)\right. \\
& \left.\quad+\frac{k_{2}}{\left(\xi_{2}-\xi_{1}+u_{2}-u_{1}\right)^{2}}+m_{2} g\right]  \tag{39d}\\
& -\frac{1}{m_{2}}\left[-k_{1}\left(\xi_{2}-\xi_{1}\right)+\frac{k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{2}}+m_{2} g\right]
\end{align*}
$$

We can make the approximation

$$
\begin{align*}
& \frac{1}{\left(\xi_{2}-\xi_{1}+u_{2}-u_{1}\right)^{2}} \\
\approx & \frac{1}{\left(\xi_{2}-\xi_{1}\right)^{2}}-\frac{2\left(u_{2}-u_{1}\right)}{\left(\xi_{2}-\xi_{1}\right)^{3}} \tag{39}
\end{align*}
$$

such that the system
$\left(k^{*}=k_{1}+\frac{2 k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{3}}\right)$
$\frac{\mathrm{d} u_{1}}{\mathrm{~d} t}=u_{3} ; \frac{\mathrm{d} u_{2}}{\mathrm{~d} t}=u_{4} ;$
$\frac{\mathrm{d} u_{3}}{\mathrm{~d} t}=\frac{1}{m_{1}}\left[-k u_{1}+k^{*}\left(u_{2}-u_{1}\right)\right] ;$
$\frac{\mathrm{d} u_{4}}{\mathrm{~d} t}=\frac{1}{m_{2}}\left[-k^{*}\left(u_{2}-u_{1}\right)\right]$.
The Jacobi matrix of the system (41) has the expression

$$
[\mathbf{J}]=\left|\begin{array}{cccc}
0 & 0 & 1 & 0  \tag{42}\\
0 & 0 & 0 & 1 \\
\frac{-k-k^{*}}{m_{1}} & \frac{k^{*}}{m_{1}} & 0 & 0 \\
\frac{k^{*}}{m_{2}} & -\frac{k^{*}}{m_{2}} & 0 & 0
\end{array}\right|,
$$

the characteristic equation reading

$$
\begin{align*}
& \left|\begin{array}{cccc}
-\varphi & 0 & 1 & 0 \\
0 & -\varphi & 0 & 1 \\
-\frac{k+k^{*}}{m_{1}} & \frac{k^{*}}{m_{1}} & -\varphi & 0 \\
\frac{k^{*}}{m_{2}} & -\frac{k^{*}}{m_{2}} & 0 & -\varphi
\end{array}\right|=0,  \tag{43}\\
& -\varphi\left|\begin{array}{ccc}
-\varphi & 0 & 1 \\
\frac{k^{*}}{m_{1}} & -\varphi & 0 \\
-\frac{k^{*}}{m_{2}} & 0 & -\varphi
\end{array}\right| \\
& +\left|\begin{array}{ccc}
0 & -\varphi & 1 \\
-\frac{k+k^{*}}{m_{1}} & \frac{k^{*}}{m_{1}} & 0 \\
\frac{k^{*}}{m_{2}} & -\frac{k^{*}}{m_{2}}-\varphi
\end{array}\right|=0,  \tag{44}\\
& \varphi^{4}-\left(\frac{k^{*}}{m_{2}}+\frac{k+k^{*}}{m_{1}}\right) \varphi^{2}+\frac{k^{*}}{m_{2}} \frac{k+k^{*}}{m_{1}}=0 . \tag{45}
\end{align*}
$$

The roots of the characteristic equation (45) are

$$
\begin{equation*}
\varphi_{1}^{2}=-\frac{k^{*}}{m_{2}} ; \varphi_{2}^{2}=-\frac{k+k^{*}}{m_{1}} \tag{46}
\end{equation*}
$$

wherefrom
$\varphi_{1}=i \sqrt{\frac{k^{*}}{m_{2}}} ; \varphi_{2}=-i \sqrt{\frac{k^{*}}{m_{2}}} ; \varphi_{3}=i \sqrt{\frac{k+k^{*}}{m_{1}}} ;$
$\varphi_{4}=-i \sqrt{\frac{k+k^{*}}{m_{2}}}$.
THEOREM 1. If $k_{2}>0$, then the system has only one equilibrium position which is stable and not asymptotically stable.

Proof: Since the roots (47) of the characteristic equation (45) are pure imaginary, it follows that the system (41) has the solution
$u_{1}=C_{1} \cos \left(p_{1} t\right)+C_{2} \cos \left(p_{2} t\right)$;
$u_{2}=D_{1} \cos \left(p_{1} t\right)+D_{2} \cos \left(p_{2} t\right)$,
where $C_{1}, C_{2}, D_{1}, D_{2}$ are constants of integration which result from the initial conditions, and $p_{1}, p_{2}$ have the expressions
$p_{1}=\sqrt{\frac{k^{*}}{m_{2}}} ; p_{2}=\sqrt{\frac{k+k^{*}}{m_{1}}}$.
The solutions (48) are bounded and it follows that the equilibrium is stable but not asymptotically stable.

### 2.5 The small oscillations around the position of stable equilibrium

Let us return to the system (4) and let us make the approximation

$$
\begin{align*}
& \frac{1}{\left(\xi_{2}-\xi_{1}+u_{2}-u_{1}\right)^{2}} \\
\approx & \frac{1}{\left(\xi_{2}-\xi_{1}\right)^{2}}-\frac{2\left(u_{2}-u_{1}\right)}{\left(\xi_{2}-\xi_{1}\right)^{3}} \tag{50}
\end{align*}
$$

obtained developing the function $g(x)=\frac{1}{x^{2}}$ about a point $x_{0}$ and retaining only the first term of the development.

The system (39) takes now the form

$$
\begin{align*}
\frac{\mathrm{d} u_{1}}{\mathrm{~d} t} & =u_{3} ; \frac{\mathrm{d} u_{2}}{\mathrm{~d} t}=u_{4} ; \\
\frac{\mathrm{d} u_{3}}{\mathrm{~d} t} & =\frac{1}{m_{1}}\left\{\left[-k-k_{1}-\frac{2 k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{3}}\right] u_{1}\right. \\
& \left.+\left[k_{1}+\frac{2 k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{3}}\right] u_{2}\right\} ; \\
\frac{\mathrm{d} u_{4}}{\mathrm{~d} t} & =\frac{1}{m_{2}}\left\{\left[\left[k_{1}+\frac{2 k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{3}}\right] u_{1}\right.\right.  \tag{51}\\
+ & {\left.\left[-k_{1}-\frac{2 k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{3}}\right] u_{2}\right\} . }
\end{align*}
$$

Let us denote
$a_{1}=-k-k_{1}-\frac{2 k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{3}} ; b_{1}=k_{1}+\frac{2 k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{3}} ;$
$a_{2}=k_{1}+\frac{2 k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{3}} ; b_{2}=-k_{1}-\frac{2 k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{3}}$,
the system (51) reading
$m_{1} \ddot{u}_{1}=a_{1} u_{1}+b_{1} u_{2} ; m_{2} \ddot{u}_{2}=a_{2} u_{1}+b_{2} u_{2}$.
For the system (53) we shall look for solutions in the form
$u_{1}=\alpha \cos (\omega t) ; u_{2}=\beta \cos (\omega t) ; \alpha^{2}+\beta^{2} \neq 0$.
It follows
$\alpha \omega^{2} m_{1}=a_{1} \alpha+b_{1} \beta ;-\beta \omega^{2} m_{2}=a_{2} \alpha+b_{2} \beta$.
The system (55) has non-zero solution if its determinant is equal to 0 , that is
$\left|\begin{array}{cc}a_{1}+m_{1} \omega^{2} & b_{1} \\ a_{2} & b_{2}+m_{2} g \omega^{2}\end{array}\right|=0$,
wherefrom
$\left(a_{1}+m_{1} \omega^{2}\right)\left(b_{2}+m_{2} \omega\right)^{2}-a_{2} b_{1}=0$.
Let us denote by $a$ the expression
$a=k_{1}+\frac{2 k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{3}}$,
such that
$a_{1}=-a-k ; b_{1}=a ; a_{2}=a ; b_{2}=-a$.
The equation (57) reads now
$\left(-a-k+m_{1} \omega^{2}\right)\left(-a+m_{2} \omega^{2}\right)-a^{2}=0$
or, equivalently,

$$
\begin{equation*}
m_{1} m_{2} \omega^{4}+\omega^{2}\left(-a m_{1}-a m_{2}-k m_{2}\right)+a k=0 . \tag{60}
\end{equation*}
$$

It results

$$
\begin{gather*}
\omega_{1,2}^{2}=\frac{a\left(m_{1}+m_{2}\right)+k m_{2}}{2 m_{1} m_{2}}  \tag{61}\\
\pm \frac{\sqrt{\left[a\left(m_{1}+m_{2}\right)-k m_{2}\right]^{2}+4 m_{1} m_{2} a^{2}}}{2 m_{1} m_{2}} \tag{62}
\end{gather*} .
$$

### 2.6 Comparison with the linear case

The linear case is defined by
$k_{2}=0$.
In this situation, the parameter $a$ takes its minimum value, which is

$$
\begin{equation*}
a=k_{1} . \tag{64}
\end{equation*}
$$

Let us consider the expression (62), which gives the eigenpulsations and let $\omega_{1}^{2}$ be the value corresponding to the sign - and $\omega_{2}^{2}$ the value corresponding to the sign + .

If $a$ increases, then $\omega_{2}^{2}$ will increase, such that we can always write
$\omega_{2}>\left(\omega_{2}\right)_{l}$,
where the index $l$ signifies linear.
On the other hand, the increasing of $a$ implies both the increasing of the expression $a\left(m_{1}+m_{2}\right)$ and the expression under the radical, the increasing of the expression $a\left(m_{1}+m_{2}\right)$ being greater than that of the expression under the radical. It therefore results
$\omega_{1}>\left(\omega_{1}\right)_{l}$.
Finally, we found that the use of the neoHookean element leads to the displacement of the fundamental pulsation in an increasing sense.

In addition, the increasings of the two fundamental pulsations $\omega_{1}$ and $\omega_{2}$ are not equal, in the sense that $\omega_{2}$ increases more than $\omega_{1}$. This thing is mathematically written by the relation
$\omega_{2}-\omega_{1}>\left(\omega_{2}\right)_{l}-\left(\omega_{1}\right)_{l}$.
In this way, the safety domain where the resonance doesn't appear increases and it goes to superior value comparing to those in the linear case.

### 2.7 Numerical application

Let us consider a numerical case defined by $k=4 \cdot 10^{5}[\mathrm{~N} / \mathrm{m}] ; k_{1}=5 \cdot 10^{4}[\mathrm{~N} / \mathrm{m}]$;
$k_{2}=5\left[\mathrm{Nm}^{2}\right] ; m_{1}=25[\mathrm{~kg}] ; m_{2}=350[\mathrm{~kg}]$;
$g=9.8065\left[\mathrm{~m} / \mathrm{s}^{2}\right]$.
The static elongations (the equilibrium position) are given by


c)

d)

Fig. 4. The graphics of variation for different characteristic variables; a) $\xi_{1}=\xi_{1}(t)$ for $0 \leq t \leq 1[\mathrm{~s}] ; \mathrm{b}) \xi_{2}=\xi_{2}(t)$ for $0 \leq t \leq 1[\mathrm{~s}]$; c) $\xi_{3}=\xi_{3}(t)$ for $0 \leq t \leq 1[\mathrm{~s}]$; d) $\xi_{4}=\xi_{4}(t)$ for $0 \leq t \leq 1[\mathrm{~s}]$.

The initial values are
$\xi_{1}^{0}=0.009[\mathrm{~m}] ; \xi_{2}^{0}=0.09[\mathrm{~m}] ; \xi_{3}^{0}=0[\mathrm{~m} / \mathrm{s}] ;$
$\xi_{4}^{0}=0[\mathrm{~m} / \mathrm{s}]$,
and the parameter $a$ from the expression (58) is
$a=67419.68[\mathrm{~N} / \mathrm{m}]$.
In figure 4 were drawn the graphics of variation for different characteristic variables obtained by numerical simulation.

One can obviously see the quasi-periodic character of the presented variations.

The formula (62) offers us the eigenpulsations in the non-linear case

$$
\begin{equation*}
\omega_{1}=12.83\left[\mathrm{~s}^{-1}\right] ; \quad \omega_{2}=136.84\left[\mathrm{~s}^{-1}\right] \tag{72}
\end{equation*}
$$

In the linear case one obtains the values
$\left(\omega_{1}\right)_{l}=11.26\left[\mathrm{~s}^{-1}\right] ; \quad \omega_{2}=134.22\left[\mathrm{~s}^{-1}\right]$.
The breadth of the non-dangerous eigenpulsations is in the non-linear case
$\Delta \omega=\omega_{2}-\omega_{1}=124.01\left[\mathrm{~s}^{-1}\right]$,
and in the linear case
$(\Delta \omega)_{l}=\left(\omega_{2}\right)_{l}-\left(\omega_{1}\right)_{l}=122.96\left[\mathrm{~s}^{-1}\right]$.
The results are in good agreement with the above theoretical considerations.

### 2.8 The stability of the motion

Let us consider that $\xi_{i}, i=\overline{1,4}$ is a solution of the system (4) of the moving equations and let $\left(u_{1}, u_{2}, u_{3}, u_{4}\right)$ be a deviation sufficiently small in its norm.

The system in deviations has the expression
$\frac{\mathrm{d} u_{1}}{\mathrm{~d} t}=u_{3} ; \frac{\mathrm{d} u_{2}}{\mathrm{~d} t}=u_{4} ;$

$$
\frac{\mathrm{d} u_{3}}{\mathrm{~d} t}=\frac{1}{m_{1}}\left[-k u_{1}+k_{1}\left(u_{2}-u_{1}\right)\right.
$$

$$
\begin{gather*}
\left.-\frac{k_{2}}{\left(\xi_{2}-\xi_{1}+u_{2}-u_{1}\right)^{2}}+\frac{k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{2}}\right] \\
\frac{\mathrm{d} u_{4}}{\mathrm{~d} t}=\frac{1}{m_{2}}\left[-k_{1}\left(u_{2}-u_{1}\right)\right. \\
\left.-\frac{k_{2}}{\left(\xi_{2}-\xi_{1}+u_{2}-u_{1}\right)}+\frac{k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{2}}\right] . \tag{76}
\end{gather*}
$$

Let us denote by $g\left(u_{1}, u_{2}, u_{3}, u_{4}\right)$ the functions in the right-hand terms of the expressions (76) and by $j_{k l}$ the partial derivatives
$j_{k l}=\frac{\partial g_{k}}{\partial u_{l}} ; k=\overline{1,4} ; l=\overline{1,4}$.
We have
$j_{11}=0 ; j_{12}=0 ; j_{13}=1 ; j_{14}=0$,
$j_{21}=0 ; j_{22}=0 ; j_{23}=0 ; j_{24}=1$,
$j_{31}=\frac{1}{m_{1}}\left[-k-k_{1}-\frac{2 k_{2}}{\left(\xi_{2}-\xi_{1}+u_{2}-u_{1}\right)^{3}}\right]$;
$j_{32}=\frac{1}{m_{1}}\left[k_{1}+\frac{2 k_{2}}{\left(\xi_{2}-\xi_{1}+u_{2}-u_{1}\right)^{3}}\right] ; j_{33}=0$;
$j_{34}=0$,
$j_{41}=\frac{1}{m_{2}}\left[k_{1}+\frac{2 k_{2}}{\left(\xi_{2}-\xi_{1}+u_{2}-u_{1}\right)^{3}}\right] ;$
$j_{42}=\frac{1}{m_{2}}\left[-k_{1}-\frac{2 k_{2}}{\left(\xi_{2}-\xi_{1}+u_{2}-u_{1}\right)^{3}}\right] ; j_{43}=0$;
$j_{44}=0$.
We denote by $a$ the expression
$a=k_{1}+\frac{2 k_{2}}{\left(\xi_{2}-\xi_{1}+u_{2}-u_{1}\right)^{3}}$,
such that we can write
$j_{31}=\frac{-k-a}{m_{1}} ; j_{32}=\frac{a}{m_{1}} ; j_{41}=\frac{a}{m_{2}}$;
$j_{42}=\frac{-a}{m_{2}}$.
The characteristic equation
$\operatorname{det}([\mathbf{J}]-\varphi[\mathbf{I}])=0$,
where $[\mathbf{J}]$ is the Jacobi matrix,

$$
\begin{equation*}
[\mathbf{J}]=\left[j_{k l}\right]_{l=1,4}^{k=1,4}, \tag{85}
\end{equation*}
$$

and $[\mathbf{I}]$ is the fourth order unity matrix, takes the form

$$
\left|\begin{array}{cccc}
-\varphi & 0 & 1 & 0  \tag{86}\\
0 & -\varphi & 0 & 1 \\
-\frac{a+k}{m_{1}} & \frac{a}{m_{1}} & -\varphi & 0 \\
\frac{a}{m_{2}} & -\frac{a}{m_{2}} & 0 & -\varphi
\end{array}\right|=0
$$

$-\varphi\left|\begin{array}{ccc}-\varphi & 0 & 1 \\ \frac{a}{m_{1}} & -\varphi & 0 \\ -\frac{a}{m_{2}} & 0 & -\varphi\end{array}\right|+\left|\begin{array}{ccc}0 & -\varphi & 1 \\ -\frac{a+k}{m_{1}} & \frac{a}{m_{1}} & 0 \\ \frac{a}{m_{2}} & -\frac{a}{m_{2}} & -\varphi\end{array}\right|=0$,
$\varphi^{4}+\left(\frac{a}{m_{2}}+\frac{a+k}{m_{1}}\right) \varphi^{2}+\frac{a k}{m_{1} m_{2}}=0$.
The discriminate of this equation is

$$
\begin{equation*}
\left.\pm \sqrt{\left(\frac{a}{m_{2}}+\frac{a+k}{m_{1}}\right)^{2}-\frac{4 a k}{m_{1} m_{2}}}\right] \tag{90}
\end{equation*}
$$

and one can easy see that they are pure imaginary,

$$
\begin{gather*}
\varphi_{1}=\mathrm{i} \times \\
\sqrt{\frac{a}{m_{2}}+\frac{a+k}{m_{1}}+\sqrt{\left(\frac{a}{m_{2}}+\frac{a+k}{m_{1}}\right)^{2}-\frac{4 a k}{m_{1} m_{2}}}}  \tag{91a}\\
\sqrt{\frac{\varphi_{2}=-\mathrm{i} \times}{\frac{a}{m_{2}}+\frac{a+k}{m_{1}}+\sqrt{\left(\frac{a}{m_{2}}+\frac{a+k}{m_{1}}\right)^{2}-\frac{4 a k}{m_{1} m_{2}}}}} \\
\sqrt{\frac{\varphi_{3}=\mathrm{i} \times}{\frac{m_{2}}{m_{2}}+\frac{a+k}{m_{1}}-\sqrt{\left(\frac{a}{m_{2}}+\frac{a+k}{m_{1}}\right)^{2}-\frac{4 a k}{m_{1} m_{2}}}}}  \tag{91b}\\
\sqrt{\frac{\varphi_{4}=-\mathrm{i} \times}{\frac{a}{m_{2}}+\frac{a+k}{m_{1}}-\sqrt{\left(\frac{a}{m_{2}}+\frac{a+k}{m_{1}}\right)^{2}-\frac{a n k}{m_{1} m_{2}}}}}
\end{gather*}
$$

Result the solutions of the system (81) in the form

$$
\begin{align*}
& u_{1}=C_{1} \cos \left(p_{1} t\right)+C_{2} \cos \left(p_{2} t\right) \\
& u_{2}=D_{1} \cos \left(p_{1} t\right)+D_{2} \cos \left(p_{2} t\right) \tag{92}
\end{align*}
$$

in which $C_{1}, C_{2}, D_{1}, D_{2}$ are constants of integration which are obtained from the initial conditions, and $p_{1}, p_{2}$ read as

$$
\begin{gather*}
p_{1}= \\
\sqrt{\frac{\left(\frac{a}{m_{2}}+\frac{a+k}{m_{1}}+\sqrt{\left(\frac{a}{m_{2}}+\frac{a+k}{m_{1}}\right)^{2}-\frac{4 a k}{m_{1} m_{2}}}\right.}{2}} \\
\sqrt{\frac{p_{2}=}{\frac{a}{m_{2}}+\frac{a+k}{m_{1}}-\sqrt{\left(\frac{a}{m_{2}}+\frac{a+k}{m_{1}}\right)^{2}-\frac{4 a k}{m_{1} m_{2}}}}} . \tag{93}
\end{gather*}
$$

THEOREM 2. In the case $k_{2}>0$ the motion is always stable and not asymptotically stable.

Proof: It is obvious from the above discussions.

### 2.9 Numerical application

As numerical application let us consider the case defined by the relations (68), the initial conditions being

$$
\begin{align*}
& \xi_{1}^{0}=0.012[\mathrm{~m}] ; \xi_{2}^{0}=0.022[\mathrm{~m}] ; \xi_{3}^{0}=0[\mathrm{~m} / \mathrm{s}] ; \\
& \xi_{4}^{0}=0[\mathrm{~m} / \mathrm{s}], \tag{94}
\end{align*}
$$

and the deviations
$u_{1}^{0}=-0.001[\mathrm{~m}] ; u_{2}^{0}=-0.001[\mathrm{~m}] ; u_{3}^{0}=0.01[\mathrm{~m} / \mathrm{s}] ;$
$u_{4}^{0}=0.00001[\mathrm{~m} / \mathrm{s}]$.
In the figure 5 we presented the representative diagrams for the unperturbed motion; in the figure 6 for the perturbed motion, and in figure 7 the deviations of the perturbed motion with respect to the unperturbed motion.

### 2.10 Discussion

Until now we considered the case $k_{2}>0$. The Case $k_{2}=0$ is not an interesting one because we arrive to the linear case.

Let us now consider $k_{2}<0$.
One obtains the same value for $\xi_{1}$ given by the formula (7) and the same roots of the derivative $f^{\prime}\left(\xi_{2}\right)$ given by the expressions (12).

For $\xi_{2}^{(1)}$ we find
$f\left(\xi_{2}^{(1)}\right)=-k_{2}>0$,
and for $\xi_{2}^{(2)}$ we have
$f\left(\xi_{2}^{(2)}\right)=-\frac{4 m_{2}^{3} g^{3}}{27 k_{1}^{2}}-k_{2}$.
If $f\left(\xi_{2}^{(2)}\right)>0$, that is
$k_{2}<-\frac{4 m_{2}^{3} g^{3}}{27 k_{1}^{2}}$,
then we are in the situation drawn in the figure 8 , the equation $f\left(\xi_{2}\right)=0$ having only one root $\xi_{2}<\xi_{2}^{(1)}$.


Fig. 5. The representative diagrams for the unperturbed motion; a) $\xi_{1}=\xi_{1}(t)$ for $0 \leq t \leq 1[\mathrm{~s}]$; b) $\xi_{2}=\xi_{2}(t)$ for $0 \leq t \leq 1[\mathrm{~s}]$; c) $\xi_{3}=\xi_{3}(t)$ for $0 \leq t \leq 1[\mathrm{~s}] ; \mathrm{d}) \xi_{4}=\xi_{4}(t)$ for $0 \leq t \leq 1[\mathrm{~s}]$.

If $f\left(\xi_{2}^{(2)}\right)<0$, that is
$k_{2}>-\frac{4 m_{2}^{3} g^{3}}{27 k_{1}^{2}}$,
then we are in the situation presented in the figure 9 , the equation $f\left(\xi_{2}\right)=0$ having now three real roots situated in the intervals $\left(-\infty, \xi_{2}^{(1)}\right),\left(\xi_{2}^{(1)}, \xi_{2}^{(2)}\right)$,


Fig. 6. The representative diagrams for the perturbed motion; a) $\xi_{1}=\xi_{1}(t)$ for $0 \leq t \leq 1[\mathrm{~s}]$; b) $\xi_{2}=\xi_{2}(t)$ for $0 \leq t \leq 1[\mathrm{~s}] ; \mathrm{c}) \xi_{3}=\xi_{3}(t)$ for $\left.0 \leq t \leq 1[\mathrm{~s}] ; \mathrm{d}\right)$

$$
\xi_{4}=\xi_{4}(t) \text { for } 0 \leq t \leq 1[\mathrm{~s}] \text {. }
$$

No matter in what situation we are, the components of the Jacobi matrix have the same expressions given by the formulas (19), (21) and (22), and the characteristic equation reads in the same form (27).

Let us consider for the beginning the situation described in the figure 8 . In this case $\xi_{2}-\xi_{1}<0$, $k_{2}<0$ and it results


Fig. 7. The deviations of the perturbed motion with respect to the unperturbed motion; a) $\Delta \xi_{1}=\Delta \xi_{1}(t)$ for $0 \leq t \leq 1[\mathrm{~s}]$; b) $\Delta \xi_{2}=\Delta \xi_{2}(t)$ for $0 \leq t \leq 1[\mathrm{~s}]$; c) $\Delta \xi_{3}=\Delta \xi_{3}(t)$ for $0 \leq t \leq 1[\mathrm{~s}]$; d) $\Delta \xi_{4}=\Delta \xi_{4}(t)$

$$
\text { for } 0 \leq t \leq 1[\mathrm{~s}] \text {. }
$$

$j_{31}<0 ; j_{32}>0 ; j_{41}<0 ; j_{42}<0$.
The discriminate of the characteristic equation is given by the formula (28) and it is also positive, the expression (30) remaining true.

In addition, the roots of the characteristic equation are pure imaginary, the equilibrium being simply stable.

The system in deviations has again the form (41), the roots of the characteristic equation being pure imaginary and given by the formulas (47).


Fig. 8. The case $f\left(\xi_{2}^{(2)}\right)>0$.


Fig. 9. The case $f\left(\xi_{2}^{(2)}\right)<0$.
THEOREM 3. In the case $k_{2}<0$, $k_{2}<-\frac{4 m_{2}^{3} g^{3}}{27 k_{1}^{2}}$, the equilibrium is always stable and not asymptotically stable.

Proof: It is identical to that of the theorem 1.
We shall go now to the case described in the figure 9 , that is $k_{2}<0, k_{2}>-\frac{4 m_{2}^{3} g^{3}}{27 k_{1}^{2}}$.

Obviously, if $\xi_{2} \in\left(-\infty, \xi_{2}^{(1)}\right)$, the discussion doesn't change being the same as in the previous case.

THEOREM 4. In the case $k_{2}<0$, $k_{2}>-\frac{4 m_{2}^{3} g^{3}}{27 k_{1}^{2}}$, the equilibrium position $\xi_{2} \in\left(-\infty, \xi_{2}^{(1)}\right)$ is always stable and not asymptotically stable.

Proof: It is identical to that of the theorem 1.
Let us now consider the situations for which $\xi_{2} \in\left(\xi_{2}^{(1)}, \xi_{2}^{(2)}\right)$ or $\xi_{2} \in\left(\xi_{2}^{(2)},+\infty\right)$.

In this case, the inequality $\sqrt{\Delta}=\left|j_{42}+j_{31}\right|$
is equivalent, according to formula (35), with
$\frac{2 k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{3}}+k_{1}>0$,
or, equivalently,

$$
\begin{equation*}
k_{2}<-\frac{k_{1}\left(\xi_{2}-\xi_{1}\right)^{3}}{2} . \tag{103}
\end{equation*}
$$

Let be the equilibrium position $\xi_{2} \in\left(\xi_{2}^{(2)}, \infty\right)$. From the expression (12) one deduces

$$
\begin{equation*}
\xi_{2}-\xi_{1}>\xi_{2}^{(2)}-\xi_{1}=\frac{2 m_{2} g}{3 k_{1}} \tag{104}
\end{equation*}
$$

Keeping into account the formula (104) and the fact that $k_{2}<0$, the relation (102) leads us to

$$
\begin{align*}
& \frac{2 k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{3}}+k_{1}>\frac{2 k_{2}}{\left(\frac{2 m_{2} g}{3 k_{1}}\right)^{3}}+k_{1}  \tag{105}\\
& =\frac{27 k_{1}^{3} k_{2}}{4 m_{2}^{3} g^{3}}+k_{1}=k_{1}\left(1+\frac{27 k_{1}^{2} k_{2}}{4 m_{2}^{3} g^{3}}\right) \\
& \text { But, since } k_{2}>-\frac{4 m_{2}^{3} g^{3}}{27 k_{1}^{2}}, \text { the relation } \tag{105}
\end{align*}
$$

offers us

$$
\begin{equation*}
\frac{2 k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{3}}+k_{1} \tag{106}
\end{equation*}
$$

$>k_{1}\left(1-\frac{27 k_{1}^{2}}{4 m_{2}^{3} g^{3}} \frac{4 m_{2}^{3} g^{3}}{27 k_{1}^{2}}\right)=0$.
Therefore we proved that for $\xi_{2}>\xi_{2}^{(2)}$, the relation (102) is always true.

On the other hand, the condition
$j_{42}+j_{31}<0$
leads us to
$\left(\frac{1}{m_{1}}+\frac{1}{m_{2}}\right)\left[-\frac{2 k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{3}}-k_{1}\right]-\frac{k}{m_{1}}<0$.
One observes that if the formula (102) is true, then the formula (108) is also true. Therefore, we have to verify only the condition (102).

THEOREM 5. In the case $k_{2}<0$, $k_{2}>-\frac{4 m_{2}^{3} g^{3}}{27 k_{1}^{2}}$, the equilibrium position $\xi_{2}>\xi_{2}^{(2)}$ is always stable and not asymptotically stable.

Proof: It is obvious from the above discussion.
Let us consider now the root $\xi_{2} \in\left(\xi_{2}^{(1)}, \xi_{2}^{(2)}\right)$.
THEOREM 6. In the case $k_{2}<0$, $k_{2}>-\frac{4 m_{2}^{3} g^{3}}{27 k_{1}^{2}}$, the equilibrium position $\xi_{2} \in\left(\xi_{2}^{(1)}, \xi_{2}^{(2)}\right)$ is always unstable.

Proof: From the second formula (5) we deduce
$\frac{k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{2}}=k_{1}\left(\xi_{2}-\xi_{1}\right)-m_{2} g$,
wherefrom

$$
\begin{equation*}
\frac{2 k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{3}}+k_{1}=3 k_{1}-\frac{2 m_{2} g}{\xi_{2}-\xi_{1}} . \tag{110}
\end{equation*}
$$

But $\xi_{2} \in\left(\xi_{2}^{(1)}, \xi_{2}^{(2)}\right)$ and from the expression (13) we have
$\xi_{2}-\xi_{1}<\frac{2 m_{2} g}{3 k_{1}}$.
The relation (110) offers now
$\frac{2 k_{2}}{\left(\xi_{2}-\xi_{1}\right)^{3}}+k_{1}<3 k_{1}-\frac{2 m_{2} g}{\frac{2 m_{2} g}{3 k_{1}}}=0$.
It results that the equilibrium is unstable and the theorem is proved.

## 3 Neo-Hookean Suspension for a Half of an Automobile

### 3.1 Mathematical model

We shall now present the study of the motion for four degrees of freedom system that models a half of an automobile. The model is presented in figure 10. This model consists of the masses $m_{1}$ and $m_{2}$, which mark the wheels of the automobile, masses linked to the ground by linear elastic springs of stiffness $k_{1}$ and $k_{2}$, respectively. By wheels is attached the chassis marked by the bar $A B$ of mass $M$. The linking of the chassis is made by the nonlinear neo-Hookean elastic elements by elastic stiffness $d_{1}, e_{1}$, respectively $d_{2}, e_{2}$. The elastic force that appears in such element is given by
$F=d_{i} z_{i}-\frac{e_{i}}{z_{i}^{2}}$,
where $i=\overline{1,2}, \quad z_{i}$ marks the elongation of the respective element, and $d_{i}>0, e_{1}>0, i=\overline{1,2}$.

The four degrees of freedom of the system were selected as follows: $q_{1}, q_{2}$ the elongations of the linear springs, $q_{3}$ the displacement in the vertical direction of the gravity centre $G$ of the chassis and $q_{4}$ the rotation of the chassis with respect to the horizontal.

We assume that there are known the dimensions $L_{1}$ and $L_{2}$ that define the position of the gravity centre $G$ of the chassis with respect to the two
wheels and $J$ the moment of the inertia with respect to a horizontal axis that passes through its gravity centre.


Fig. 10. The mathematical model.

### 3.2 The equations of motion

The kinetic energy of the system has the expression

$$
\begin{equation*}
T=\frac{1}{2} m_{1} \dot{q}_{1}^{2}+\frac{1}{2} m_{2} \dot{q}_{2}^{2}+\frac{1}{2} M \dot{q}_{3}^{2}+\frac{1}{2} J \dot{q}_{4}^{2} . \tag{114}
\end{equation*}
$$

The forces, which appear in the system, derive from a potential, hence the potential energy reads

$$
\begin{gather*}
V=\frac{1}{2} k_{1} q_{1}^{2}-m_{1} g q_{1}+\frac{1}{2} k_{2} q_{2}^{2} \\
-m_{2} g q_{2}+\frac{1}{2} d_{1}\left(L_{1} q_{4}-q_{1}+q_{3}\right)^{2} \\
+\frac{e_{1}}{L_{1} q_{4}-q_{1}+q_{3}}+\frac{d_{2}}{2}\left(q_{3}-L_{2} q_{4}-q_{2}\right)^{2}  \tag{115}\\
+\frac{e_{2}}{q_{3}-L_{2} q_{4}-q_{2}}-M g q_{3},
\end{gather*}
$$

$g$ being the gravitational acceleration.
We successively calculate
$\frac{\partial T}{\partial \dot{q}_{1}}=m_{1} \dot{q}_{1} ; \frac{\partial T}{\partial \dot{q}_{2}}=m_{2} \dot{q}_{2} ; \frac{\partial T}{\partial \dot{q}_{3}}=M \dot{q}_{3} ;$
$\frac{\partial T}{\partial \dot{q}_{4}}=J \dot{q}_{4}$,
$\frac{\mathrm{d}}{\mathrm{d} t}\left(\frac{\partial T}{\partial \dot{q}_{1}}\right)=m_{1} \ddot{q}_{1} ; \frac{\mathrm{d}}{\mathrm{d} t}\left(\frac{\partial T}{\partial \dot{q}_{2}}\right)=m_{2} \ddot{q}_{2} ;$
$\frac{\mathrm{d}}{\mathrm{d} t}\left(\frac{\partial T}{\partial \dot{q}_{3}}\right)=M \ddot{q}_{3} ; \frac{\mathrm{d}}{\mathrm{d} t}\left(\frac{\partial T}{\partial \dot{q}_{4}}\right)=J \ddot{q}_{4}$,

$$
\begin{align*}
& \frac{\partial T}{\partial q_{1}}=0 ; \frac{\partial T}{\partial q_{2}}=0 ; \frac{\partial T}{\partial q_{3}}=0 ; \frac{\partial T}{\partial q_{4}}=0,  \tag{118}\\
& \frac{\partial V}{\partial q_{1}}=k_{1} q_{1}-m_{1} g-d_{1}\left(L_{1} q_{4}-q_{1}+q_{3}\right)  \tag{119a}\\
& +\frac{e_{1}}{\left(L_{1} q_{4}-q_{1}+q_{3}\right)^{2}}, \\
& \frac{\partial V}{\partial q_{2}}=k_{2} q_{2}-m_{2} g-d_{2}\left(q_{3}-L_{2} q_{4}-q_{2}\right)  \tag{119b}\\
& +\frac{e_{2}}{\left(q_{3}-L_{2} q_{4}-q_{2}\right)^{2}}, \\
& \frac{\partial V}{\partial q_{3}}=d_{1}\left(L_{1} q_{4}-q_{1}+q_{3}\right) \\
& -\frac{e_{1}}{\left(L_{1} q_{4}-q_{1}+q_{3}\right)^{2}}+d_{2}\left(q_{3}-L_{2} q_{4}-q_{2}\right)  \tag{119c}\\
& -\frac{e_{2}}{\left(q_{3}-L_{2} q_{4}-q_{2}\right)^{2}}-M g, \\
& \frac{\partial V}{\partial q_{4}}=L_{1} d_{1}\left(L_{1} q_{4}-q_{1}+q_{3}\right) \\
& -\frac{L_{1} e_{1}}{\left(L_{1} q_{4}-q_{1}+q_{3}\right)^{2}}  \tag{119d}\\
& -L_{2} d_{2}\left(q_{3}-L_{2} q_{4}-q_{2}\right) \\
& +\frac{L_{2} e_{2}}{\left(q_{3}-L_{2} q_{4}-q_{2}\right)^{2}},
\end{align*}
$$

such that the Lagrange equations read

$$
\begin{gather*}
m_{1} \ddot{q}_{1}+k_{1} q_{1}-m_{1} g-d_{1}\left(L_{1} q_{4}-q_{1}+q_{3}\right) \\
\quad+\frac{e_{1}}{\left(L_{1} q_{4}-q_{1}+q_{3}\right)^{2}}=0,  \tag{120a}\\
m_{2} \ddot{q}_{2}+k_{2} q_{2}-m_{2} g-d_{2}\left(q_{3}-L_{2} q_{4}-q_{2}\right) \\
\quad+\frac{e_{2}}{\left(q_{3}-L_{2} q_{4}-q_{2}\right)^{2}}=0,  \tag{120b}\\
M \ddot{q}_{3}+d_{1}\left(L_{1} q_{4}-q_{1}+q_{3}\right) \\
-\frac{e_{1}}{\left(L_{1} q_{4}-q_{1}+q_{3}\right)^{2}}+d_{2}\left(q_{3}-L_{2} q_{4}-q_{2}\right)  \tag{120c}\\
\quad-\frac{e_{2}}{\left(q_{3}-L_{2} q_{4}-q_{2}\right)^{2}}-M g=0, \\
J \ddot{q}_{4}+L_{1} d_{1}\left(L_{1} q_{4}-q_{1}+q_{3}\right) \\
\quad-\frac{L_{1} e_{1}}{\left(L_{1} q_{4}-q_{1}+q_{3}\right)^{2}}  \tag{120d}\\
-L_{2} d_{2}\left(q_{3}-L_{2} q_{4}-q_{2}\right) \\
+\frac{L_{2} e_{2}}{\left(q_{3}-L_{2} q_{4}-q_{2}\right)^{2}}=0 .
\end{gather*}
$$

Let us denote

$$
\begin{align*}
& \xi_{1}=q_{1} ; \xi_{2}=q_{2} ; \xi_{3}=q_{3} ; \xi_{4}=q_{4} ; \xi_{5}=\dot{q}_{1} ; \\
& \xi_{6}=\dot{q}_{2} ; \xi_{7}=\dot{q}_{3} ; \xi_{8}=\dot{q}_{4} \tag{121}
\end{align*}
$$

obtaining a system of eight first order non-linear differential equations

$$
\begin{align*}
& \frac{\mathrm{d} \xi_{1}}{\mathrm{~d} t}=\xi_{5} ; \frac{\mathrm{d} \xi_{2}}{\mathrm{~d} t}=\xi_{6} ; \frac{\mathrm{d} \xi_{3}}{\mathrm{~d} t}=\xi_{7} ; \frac{\mathrm{d} \xi_{4}}{\mathrm{~d} t}=\xi_{8} ; \\
& \begin{aligned}
& \frac{\mathrm{d} \xi_{5}}{\mathrm{~d} t}= \frac{1}{m_{1}}\left[-k_{1} \xi_{1}+m_{1} g+d_{1}\left(L_{1} \xi_{4}-\xi_{1}+\xi_{3}\right)\right. \\
&\left.\quad-\frac{e_{1}}{\left(L_{1} \xi_{4}-\xi_{1}+\xi_{3}\right)^{2}}\right] ; \\
& \frac{\mathrm{d} \xi_{6}}{\mathrm{~d} t}= \frac{1}{m_{2}}\left[-k_{2} \xi_{2}+m_{2} g+d_{2}\left(\xi_{3}-L_{2} \xi_{4}-\xi_{2}\right)\right. \\
&\left.\quad-\frac{e_{2}}{\left(\xi_{3}-L_{2} \xi_{4}-\xi_{2}\right)^{2}}\right] ; \\
& \begin{aligned}
& \frac{\mathrm{d} \xi_{7}}{\mathrm{~d} t}= \frac{1}{M}\left[-d_{1}\left(L_{1} \xi_{4}-\xi_{1}+\xi_{3}\right)+\frac{e_{1}}{\left(L_{1} \xi_{4}-\xi_{1}+\xi_{3}\right)^{2}}\right. \\
&\left.-d_{2}\left(\xi_{3}-L_{2} \xi_{4}-\xi_{2}\right)+\frac{e_{2}}{\left(\xi_{3}-L_{2} \xi_{4}-\xi_{2}\right)^{2}}+M g\right] \\
& \frac{\mathrm{d} \xi_{8}}{\mathrm{~d} t}= \frac{1}{J}\left[-L_{1} d_{1}\left(L_{1} \xi_{4}-\xi_{1}+\xi_{3}\right)\right. \\
&+\frac{L_{1} e_{1}}{\left(L_{1} \xi_{4}-\xi_{1}+\xi_{3}\right)^{2}} \\
& \quad+ L_{2} d_{2}\left(\xi_{3}-L_{2} \xi_{4}-\xi_{2}\right) \\
& \quad-\frac{L_{2} e_{2}}{\left.\left(\xi_{3}-L_{2} \xi_{4}-\xi_{2}\right)^{2}\right] .}
\end{aligned}
\end{aligned} .
\end{align*}
$$

### 3.3 The equilibrium positions

These are obtained at the intersection of the nullclines, resulting the system

$$
\begin{align*}
& \xi_{5}=0 ; \xi_{6}=0 ; \xi_{7}=0 ; \xi_{8}=0,  \tag{123}\\
& -k_{1} \xi_{1}+m_{1} g+d_{1}\left(L_{1} \xi_{4}-\xi_{1}+\xi_{3}\right) \\
& \quad-\frac{e_{1}}{\left(L_{1} \xi_{4}-\xi_{1}+\xi_{3}\right)^{2}}=0,  \tag{124a}\\
& -k_{2} \xi_{2}+m_{2} g+d_{2}\left(\xi_{3}-L_{2} \xi_{4}-\xi_{2}\right) \\
& \quad-\frac{e_{2}}{\left(\xi_{3}-L_{2} \xi_{4}-\xi_{2}\right)^{2}}=0,  \tag{124b}\\
& -d_{1}\left(L_{1} \xi_{4}-\xi_{1}+\xi_{3}\right)+\frac{e_{1}}{\left(L_{1} \xi_{4}-\xi_{1}+\xi_{3}\right)^{2}} \\
& \quad-d_{2}\left(\xi_{3}-L_{2} \xi_{4}-\xi_{2}\right)  \tag{124c}\\
& \quad+\frac{e_{2}}{\left(\xi_{3}-L_{2} \xi_{4}-\xi_{2}\right)^{2}}+M g=0,
\end{align*}
$$

$$
\begin{gather*}
-L_{1} d_{1}\left(L_{1} \xi_{4}-\xi_{1}+\xi_{3}\right)+\frac{L_{1} e_{1}}{\left(L_{1} \xi_{4}-\xi_{1}+\xi_{3}\right)^{2}} \\
+L_{2} d_{2}\left(\xi_{3}-L_{2} \xi_{4}-\xi_{2}\right)  \tag{124d}\\
-\frac{L_{2} e_{2}}{\left(\xi_{3}-L_{2} \xi_{4}-\xi_{2}\right)^{2}}=0
\end{gather*}
$$

Adding the first three relations (124), one obtains the equation
$-k_{1} \xi_{1}-k_{2} \xi_{2}+\left(m_{1}+m_{2}+M\right) g=0$.
Multiplying the first relation (124) by $L_{1}$, the second relation (124) by $-L_{2}$ and summing the results at the last expression (124), we deduce

$$
\begin{equation*}
-L_{1} k_{1} \xi_{1}+L_{2} k_{2} \xi_{2}+\left(L_{1} m_{1}-L_{2} m_{2}\right) g=0 . \tag{126}
\end{equation*}
$$

The relations (125) and (126) form a linear system of two equations with two unknowns ( $\xi_{1}$ and $\xi_{2}$ )
$k_{1} \xi_{1}+k_{2} \xi_{2}=\left(m_{1}+m_{2}+M\right) g ;$
$L_{1} k_{1} \xi_{1}-L_{2} k_{2} \xi_{2}=\left(L_{1} m_{1}-L_{2} m_{2}\right) g$,
the solution of this system being

$$
\begin{align*}
\xi_{1} & =\frac{\left|\begin{array}{cc}
\left(m_{1}+m_{2}+M\right) g & k_{2} \\
\left(L_{1} m_{1}-L_{2} m_{2}\right) g-L_{2} k_{2}
\end{array}\right|}{\left|\begin{array}{c}
k_{1} \\
k_{2} \\
L_{1} k_{1}-L_{2} k_{2}
\end{array}\right|}  \tag{128a}\\
& =\frac{m_{1}\left(L_{1}+L_{2}\right)+L_{2} M}{\left(L_{1}+L_{2}\right) k_{1}} g, \\
\xi_{2} & =\frac{\left|\begin{array}{c}
k_{1}\left(m_{1}+m_{2}+M\right) g \\
L_{1} k_{1}\left(L_{1} m_{1}-L_{2} m_{2}\right) g
\end{array}\right|}{\left|\begin{array}{c}
k_{1} \\
k_{2} \\
L_{1} k_{1}-L_{2} k_{2}
\end{array}\right|} \\
& =\frac{m_{2}\left(L_{1}+L_{2}\right)+L_{1} M}{\left(L_{1}+L_{2}\right) k_{2}} g .
\end{align*}
$$

We multiply now the third equation (124) by - $L_{1}$ and we add it to the last equation (124) obtaining

$$
\begin{gather*}
\left(L_{1}+L_{2}\right) d_{2}\left(\xi_{3}-L_{2} \xi_{4}-\xi_{2}\right) \\
-\frac{\left(L_{1}+L_{2}\right) e_{2}}{\left(\xi_{3}-L_{2} \xi_{4}-\xi_{2}\right)^{2}}-L_{1} M g=0 \tag{129}
\end{gather*}
$$

or, equivalently,
$\left(\xi_{3}-L_{2} \xi_{4}-\xi_{2}\right)^{3}-\frac{L_{1}}{\left(L_{1}+L_{2}\right) d_{2}}$
$\times M g\left(\xi_{3}-L_{2} \xi_{4}-\xi_{2}\right)^{2}-\frac{e_{2}}{d_{2}}=0$.
We multiply the third equation (124) by $L_{2}$ and we add it to the last equation (124) resulting

$$
\begin{align*}
& -\left(L_{1}+L_{2}\right) d_{1}\left(\xi_{3}+L_{1} \xi_{4}-\xi_{1}\right)^{3} \\
+ & \frac{\left(L_{1}+L_{2}\right) e_{1}}{\left(\xi_{3}+L_{1} \xi_{4}-\xi_{1}\right)^{2}}+L_{2} M g=0 \tag{131}
\end{align*}
$$

or, equivalently,

$$
\begin{align*}
& \left(\xi_{3}+L_{1} \xi_{4}-\xi_{1}\right)^{3}-\frac{L_{2}}{\left(L_{1}+L_{2}\right) d_{1}}  \tag{132}\\
& \times M g\left(\xi_{3}+L_{1} \xi_{4}-\xi_{1}\right)^{2}-\frac{e_{1}}{d_{1}}=0 .
\end{align*}
$$

Let us consider for the beginning the equation (130) and let us denote

$$
\begin{align*}
& z=\xi_{3}-L_{2} \xi_{4}-\xi_{2} ; \alpha=\frac{L_{1}}{\left(L_{1}+L_{2}\right) d_{1}} M g ; \\
& \beta=\frac{e_{2}}{d_{2}} ; \alpha>0 ; \beta>0, \tag{133}
\end{align*}
$$

resulting the relation

$$
\begin{equation*}
z^{3}-\alpha z^{2}-\beta=0 \tag{134}
\end{equation*}
$$

In the sequence of the coefficients for the equation (134) there exists only one variation of sign and applying the Descartes theorem, it results that the equation (134) has only one positive real root. Making the change of variable $z \mapsto-z$, one obtains the equation

$$
\begin{equation*}
z^{3}+\alpha z^{2}+\beta=0 \tag{135}
\end{equation*}
$$

for which there exists no variation of sign in the sequence of the coefficients. Applying again the Descartes theorem, it results that the equation (135) has no positive real root and therefore the equation (134) has no negative real root. In the end, we obtained that the equation (134) has only one real root, thus the equation (130) has one real root, too. Let us denote this root by $z_{1}$.

Proceeding in an analogous way, one deduces that the equation (132) has one real root and we denote this root by $z_{2}$.

It results the system

$$
\begin{equation*}
\xi_{3}-L_{2} \xi_{4}-\xi_{2}=z_{1} ; \xi_{3}+L_{1} \xi_{4}-\xi_{1}=z_{2}, \tag{136}
\end{equation*}
$$

for which the solution is

$$
\begin{gather*}
\xi_{3}=\frac{\left|\begin{array}{cc}
z_{1}+\xi_{2}-L_{2} \\
z_{2}+\xi_{1} & L_{1}
\end{array}\right|}{\left|\begin{array}{cc}
1-L_{2} \\
1 & L_{1}
\end{array}\right|}  \tag{137a}\\
=\frac{L_{1}\left(z_{1}+\xi_{2}\right)+L_{2}\left(z_{2}+\xi_{1}\right)}{L_{1}+L_{2}},
\end{gather*}
$$

respectively

$$
\left.\begin{gather*}
\left.\xi_{4}=\frac{\left|\begin{array}{c}
1 z_{1}+\xi_{2} \\
1 z_{2}+\xi_{1}
\end{array}\right|}{\mid 1-L_{2}} \right\rvert\,  \tag{137b}\\
1 L_{1} \mid
\end{gather*} \right\rvert\,
$$

We obtained that there exists only one equilibrium position defined by the relations (128) and (137).

### 3.4 Stability of the equilibrium

Let us denote by $f_{i}$ the expressions in the righthand side of the relations (122) and let be

$$
\begin{equation*}
j_{k l}=\frac{\partial f_{k}}{\partial \xi_{l}} ; k=\overline{1,8} ; l=\overline{1,8} . \tag{138}
\end{equation*}
$$

We have

$$
\begin{align*}
& j_{11}=0 ; j_{12}=0 ; j_{13}=0 ; j_{14}=0 ; j_{15}=1 ; \\
& j_{16}=0 ; j_{17}=0 ; j_{18}=0 ;  \tag{139}\\
& j_{21}=0 ; j_{22}=0 ; j_{23}=0 ; j_{24}=0 ; j_{25}=0 ; \\
& j_{26}=1 ; j_{27}=0 ; j_{28}=0,  \tag{140}\\
& j_{31}=0 ; j_{32}=0 ; j_{33}=0 ; j_{34}=0 ; j_{35}=0 ; \\
& j_{36}=0 ; j_{37}=1 ; j_{38}=0,  \tag{141}\\
& j_{41}=0 ; j_{42}=0 ; j_{43}=0 ; j_{44}=0 ; j_{45}=0 ; \\
& j_{46}=0 ; j_{47}=0 ; j_{48}=1, \\
& j_{51}=-\frac{k_{1}}{m_{1}}-\frac{d_{1}}{m_{1}}-\frac{2 e_{1}}{m_{1}\left(L_{1} \xi_{4}-\xi_{1}+\xi_{3}\right)^{3}} ; j_{52}=0 ; \\
& j_{53}=\frac{d_{1}}{m_{1}}+\frac{2 e_{1}}{m_{1}\left(L_{1} \xi_{4}-\xi_{1}+\xi_{3}\right)^{3}} ;
\end{align*}
$$

$j_{54}=\frac{d_{1} L_{1}}{m_{1}}+\frac{2 e_{1} L_{1}}{\left(L_{1} \xi_{4}-\xi_{1}+\xi_{3}\right)^{3}} ; j_{55}=0 ; j_{56}=0$;
$j_{57}=0 ; j_{58}=0$,
$j_{61}=0 ; j_{62}=-\frac{k_{2}}{m_{2}}-\frac{d_{2}}{m_{2}}-\frac{2 e_{2}}{m_{2}\left(\xi_{3}-L_{2} \xi_{4}-\xi_{2}\right)^{3}} ;$
$j_{63}=\frac{d_{2}}{m_{2}}+\frac{2 e_{2}}{m_{2}\left(\xi_{3}-L_{2} \xi_{4}-\xi_{2}\right)^{3}} ;$
$j_{64}=-\frac{d_{2} L_{2}}{m_{2}}-\frac{2 e_{2} L_{2}}{m_{2}\left(\xi_{3}-L_{2} \xi_{4}-\xi_{2}\right)^{3}} ; j_{65}=0$;
$j_{66}=0 ; j_{67}=0 ; j_{68}=0$,
$j_{71}=\frac{d_{1}}{M}+\frac{2 e_{1}}{M\left(L_{1} \xi_{4}-\xi_{1}+\xi_{3}\right)^{3}} ;$
$j_{72}=\frac{d_{2}}{M}+\frac{2 e_{2}}{M\left(\xi_{3}-L_{2} \xi_{4}-\xi_{2}\right)^{3}} ;$

$$
\begin{align*}
& j_{73}=-\frac{d_{1}}{M}-\frac{2 e_{1}}{M\left(L_{1} \xi_{4}-\xi_{1}+\xi_{3}\right)^{3}} \\
& -\frac{d_{2}}{M}-\frac{2 e_{2}}{M\left(\xi_{3}-L_{2} \xi_{4}-\xi_{2}\right)^{3}} ; \\
& j_{74}=-\frac{d_{1} L_{1}}{M}-\frac{2 L_{1} e_{1}}{M\left(L_{1} \xi_{4}-\xi_{1}+\xi_{3}\right)^{3}} \\
& +\frac{d_{2} L_{2}}{M}+\frac{2 e_{2} L_{2}}{M\left(\xi_{3}-L_{2} \xi_{4}-\xi_{2}\right)^{3}} ; \\
& j_{75}=0 ; j_{76}=0 ; j_{77}=0 ; j_{78}=0,  \tag{145}\\
& j_{81}=\frac{L_{1} d_{1}}{J}+\frac{2 L_{1} e_{1}}{J\left(L_{1} \xi_{4}-\xi_{1}+\xi_{3}\right)^{3}} ; \\
& j_{82}=-\frac{L_{2} d_{2}}{J}-\frac{2 L_{2} e_{2}}{J\left(\xi_{3}-L_{2} \xi_{4}-\xi_{2}\right)^{3}} ; \\
& j_{83}=-\frac{L_{1} d_{1}}{J}-\frac{2 L_{1} e_{1}}{J\left(L_{1} \xi_{4}-\xi_{1}+\xi_{3}\right)^{3}} ; \\
& +\frac{L_{2} d_{2}}{J}+\frac{2 L_{2} e_{2}}{J\left(\xi_{3}-L_{2} \xi_{4}-\xi_{2}\right)^{3}} ; \\
& j_{84}=-\frac{L_{1}^{2} d_{1}}{J}-\frac{2 L_{1}^{2} e_{1}}{J\left(L_{1} \xi_{4}-\xi_{1}+\xi_{3}\right)^{3}} \\
& -\frac{L_{2}^{2} d_{2}}{J}-\frac{2 L_{2}^{2} e_{2}}{J\left(\xi_{3}-L_{2} \xi_{4}-\xi_{2}\right)^{3}} ; \\
& j_{86}=0 ; j_{87}=0 ; j_{88}=0 .  \tag{146}\\
& \text { The characteristic equation } \\
& \text { det }(\mathbf{J}-\lambda \mathbf{I})=0,  \tag{147}\\
& \text { where } \mathbf{J} \text { is the Jacobi matrix } \\
& \mathbf{J}=\left[j_{k l}\right]_{k, l=\overline{1}, 8}, \tag{148}
\end{align*}
$$

and $\mathbf{I}$ is the eight-order unity matrix, reads

$$
\left|\begin{array}{cccccccc}
-\lambda & 0 & 0 & 0 & 1 & 0 & 0 & 0  \tag{149}\\
0 & -\lambda & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -\lambda & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -\lambda & 0 & 0 & 0 & 1 \\
j_{51} & 0 & j_{53} & j_{54} & -\lambda & 0 & 0 & 0 \\
0 & j_{62} & j_{63} & j_{64} & 0 & -\lambda & 0 & 0 \\
j_{71} & j_{72} & j_{73} & j_{74} & 0 & 0 & -\lambda & 0 \\
j_{81} & j_{82} & j_{83} & j_{84} & 0 & 0 & 0 & -\lambda
\end{array}\right|=0 .
$$

Multiplying the columns five, six, seven and eight by $\lambda$ and summing the obtained results to the columns one, two, three and four, respectively, one deduces the equation

$$
\left|\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
j_{51}-\lambda^{2} & 0 & j_{53} & j_{54} & -\lambda & 0 & 0 & 0 \\
0 & j_{62}-\lambda^{2} & j_{63} & j_{64} & 0 & -\lambda & 0 & 0 \\
j_{71} & j_{72} & j_{73}-\lambda^{2} & j_{74} & 0 & 0 & -\lambda & 0 \\
j_{81} & j_{82} & j_{83} & j_{84}-\lambda^{2} & 0 & 0 & 0 & -\lambda
\end{array}\right|
$$

$$
\begin{equation*}
=0 \tag{150}
\end{equation*}
$$

Developing the determinant after the lines one, two, three and four, it results

$$
\left|\begin{array}{cccc}
j_{51}-\lambda^{2} & 0 & j_{53} & j_{54}  \tag{151}\\
0 & j_{62}-\lambda^{2} & j_{63} & j_{64} \\
j_{71} & j_{72} & j_{73}-\lambda^{2} & j_{74} \\
j_{81} & j_{82} & j_{83} & j_{84}-\lambda^{2}
\end{array}\right|=0
$$

or, equivalently,

$$
\begin{align*}
& \left(j_{51}-\lambda^{2}\right)\left|\begin{array}{ccc}
j_{62}-\lambda^{2} & j_{63} & j_{64} \\
j_{72} & j_{73}-\lambda^{2} & j_{74} \\
j_{82} & j_{83} & j_{84}-\lambda^{2}
\end{array}\right| \\
& +j_{53}\left|\begin{array}{ccc}
0 & j_{62}-\lambda^{2} & j_{64} \\
j_{71} & j_{72} & j_{74} \\
j_{81} & j_{82} & j_{84}-\lambda^{2}
\end{array}\right|  \tag{152}\\
& -j_{54}\left|\begin{array}{ccc}
0 & j_{62}-\lambda^{2} & j_{63} \\
j_{71} & j_{72} & j_{73}-\lambda^{2} \\
j_{81} & j_{82} & j_{83}
\end{array}\right|=0 .
\end{align*}
$$

The relation (151) is a bi-square equation of the fourth order in the unknown $\lambda^{2}$. It also offers the condition for the equilibrium position to be stable or unstable because it imposes a relation of connectivity in the space of the parameters $k_{1}, k_{2}$, $d_{1}, d_{2}, e_{1}$ and $e_{2}$.

### 3.5 Application

Let us consider the practical case for which
$k_{1}=k_{2}=4 \cdot 10^{5}[\mathrm{~N} / \mathrm{m}] ; d_{1}=d_{2}=5 \cdot 10^{4}[\mathrm{~N} / \mathrm{m}]$;
$e_{1}=e_{2}=5\left[\mathrm{Nm}^{2}\right] ; L_{1}=L_{2}=2[\mathrm{~m}] ; M=900[\mathrm{~kg}]$;
$m_{1}=m_{2}=25[\mathrm{~kg}] ; g=10\left[\mathrm{~m} / \mathrm{s}^{2}\right]$.
The relations (128) offer
$\xi_{1}=q_{1}^{\text {ech }}=0.0011875[\mathrm{~m}]$;
$\xi_{2}=q_{2}^{\text {ech }}=0.0011875[\mathrm{~m}]$.
$z^{3}-\frac{2}{4 \cdot 5 \cdot 10^{4}} \cdot 900 \cdot 10 z^{2}-\frac{5}{5 \cdot 10^{4}}=0$
wherefrom
$z^{3}-0.09 z^{2}-0.0001=0$
with the solution
$z=z_{1}^{\text {ech }}=0.1[\mathrm{~m}]$.
In an analogous way we find
$z_{2}^{e c h}=0.1[\mathrm{~m}]$.
The expressions (137) offer
$\xi_{3}=q_{3}^{e c h}=0.1011875[\mathrm{~m}]$,
$\xi_{4}=0.1011875[\mathrm{~m}]$.
The partial derivatives read
$j_{51}=-18400 ; j_{52}=0 ; j_{53}=2400$;
$j_{54}=4800$,
$j_{61}=0 ; j_{62}=-18400 ; j_{63}=2400$;
$j_{64}=-4800$,
$j_{71}=66.667 ; j_{72}=66.667 ; j_{73}=-133.333$;
$j_{74}=0$,
$j_{81}=100 ; j_{82}=-100 ; j_{83}=0$;
$j_{84}=-400$.
Results the characteristic equation
$\left|\begin{array}{cccc}-18400-\lambda^{2} & 0 & 2400 & 4800 \\ 0 & -18400-\lambda^{2} & 2400 & -4800 \\ 66.667 & 66.667 & -133.333-\lambda^{2} & 0 \\ 100 & -100 & -\lambda^{2} & -400-\lambda^{2}\end{array}\right|$
$=0$,
wherefrom

$$
\begin{align*}
& \lambda^{8}+37333.333 \lambda^{6}+356959994.7 \lambda^{4}+  \tag{166}\\
& +1.5872 \cdot 10^{11} \lambda^{2}+1.36533 \cdot 10^{13}=0
\end{align*}
$$

We denote
$\lambda^{2}=\eta$
and one obtains the four-order equation
$\eta^{4}+37333.333 \eta^{3}+356959994.7 \eta^{2}+$
$+1.5872 \cdot 10^{11} \eta+1.36533 \cdot 10^{13}=0$.
The solving of this equation is made by the Lobacevski-Graeffe method for which for the equation
$a_{0} x^{4}+a_{1} x^{3}+a_{2} x^{2}+a_{3} x+a_{4}=0$
the passing from the step $p$ to the step $p+1$ takes place with the formulas

$$
\begin{align*}
& a_{0}^{(p+1)}=\left[a_{0}^{(p)}\right]^{2} ; a_{1}^{(p+1)}=-\left\{\left[a_{1}^{(p)}\right]^{2}-2 a_{0}^{(p)} a_{2}^{(p)}\right\} ; \\
& a_{2}^{(p+1)}=\left[a_{2}^{(p)}\right]^{2}-2 a_{1}^{(p)} a_{3}^{(p)}+2 a_{0}^{(p)} a_{4}^{(p)} ; \\
& a_{3}^{(p+1)}=-\left[\left[a_{3}^{(p)}\right]^{2}-2 a_{2}^{(p)} a_{4}^{(p)}\right\} ; a_{4}^{(p+1)}=\left[a_{4}^{(p)}\right]^{2} . \tag{170}
\end{align*}
$$

We shall create the next table.

Table 1. The solving of equation (168) by the Lobacevski-Graeffe method.

| Step | $a_{0}$ | $a_{1}$ | $a_{2}$ |
| :---: | :---: | :---: | :---: |
| 0 | 1 | 37333.333 | 356959997.7 |
| 1 | 1 | -679857763.5 | $1.156 \cdot 10^{17}$ |
| 2 | 1 | $-2.310 \cdot 10^{17}$ | $1.33 \cdot 10^{34}$ |
| 3 | 1 | $-2.668 \cdot 10^{34}$ | $1.78 \cdot 10^{68}$ |


| Step | $a_{3}$ | $a_{4}$ |
| :---: | :---: | :---: |
| 0 | $1.5872 \cdot 10^{11}$ | $1.36533 \cdot 10^{13}$ |
| 1 | $-1.54 \cdot 10^{22}$ | $1.864 \cdot 10^{26}$ |
| 2 | $-1.95 \cdot 10^{44}$ | $3.47 \cdot 10^{52}$ |
| 3 | $-3.73 \cdot 10^{88}$ | $1.21 \cdot 10^{105}$ |

Let be the function $h: \mathrm{R} \rightarrow \mathbb{R}$, $h(\eta)=\eta^{4}+37333.333 \eta^{3}+356959994.7 \eta^{2}$

$$
\begin{equation*}
+1.5872 \cdot 10^{11} \eta+1.36533 \cdot 10^{13} \tag{171}
\end{equation*}
$$

for which

$$
\begin{align*}
h^{\prime}(\eta)=4 \eta^{3} & +112000 \eta^{2}+713919989.4 \eta  \tag{172}\\
& +1.5872 \cdot 10^{11} \\
h^{\prime \prime}(\eta)= & 12 \eta^{2} \tag{173}
\end{align*}+224000 \eta+713919989.4 .
$$

The equation $h^{\prime \prime}(\eta)=0$ has the roots

$$
\begin{gather*}
\eta_{1,2}=\frac{-224000}{24} \\
\pm \frac{\sqrt{224000^{2}-4 \cdot 12 \cdot 713919989.4}}{24} \tag{174}
\end{gather*}
$$

wherefrom
$\eta_{1}=-4078.07 ; \eta_{2}=-14588.6$.
In addition,
$h^{\prime}\left(\eta_{1}\right)=-1.161 \cdot 10^{12}<0 ;$
$h^{\prime}\left(\eta_{2}\right)=1.161 \cdot 10^{12}>0$,
such that the equation $h^{\prime}(\eta)=0$ has three distinct real roots. We also have
$h(-230) \approx-4.42 \cdot 10^{12}<0$;
$h(-1) \approx 1.36 \cdot 10^{13}>0$;
$h(-18000) \approx-3.8 \cdot 10^{12}<0$;
$h(-5000) \approx 4.1 \cdot 10^{15}>0$
and therefore the equation $h(\eta)=0$ has four distinct negative real roots.

From the table 1 we get

$$
\begin{align*}
& \eta_{1}=-\sqrt[8]{-\frac{a_{1}^{(3)}}{a_{0}^{(3)}}}=-\sqrt[8]{\frac{2.66 \cdot 10^{34}}{1}} \approx-20096  \tag{178a}\\
& \eta_{2}=-\sqrt[8]{-\frac{a_{2}^{(3)}}{a_{1}^{(3)}}}=-\sqrt[8]{\frac{1.78 \cdot 10^{68}}{2.66 \cdot 10^{34}}} \approx-16812 \tag{178b}
\end{align*}
$$

$\eta_{3}=-\sqrt[8]{-\frac{a_{3}^{(3)}}{a_{2}^{(3)}}}=-\sqrt[8]{\frac{3.73 \cdot 10^{88}}{1.78 \cdot 10^{68}}} \approx-346.8$,
$\eta_{4}=-\sqrt[8]{-\frac{a_{4}^{(3)}}{a_{3}^{(3)}}}=-\sqrt[8]{\frac{1.21 \cdot 10^{105}}{3.73 \cdot 10^{88}}} \approx-115.82$.
Result the roots of the characteristic equation
$\lambda_{1} \approx 141.76 i ; \lambda_{2} \approx-141.76 i ; \lambda_{3} \approx 130.05 i ;$
$\lambda_{4} \approx-130.05 i ; \lambda_{5} \approx 18.62 i ; \lambda_{6} \approx-18.62 i ;$
$\lambda_{7} \approx 10.76 i ; \lambda_{8} \approx-10.76 i$
and all of them are pure imaginary, the equilibrium being simply stable.

## 4 Conclusions

In this work we presented two different models using rubber type components modeled as nonlinear neo-Hookean elements. The first model is a quarter of an automobile model, and the second is the model of a half of automobile. For both models we obtained the differential equations of motion and we studied the equilibrium positions and their stability. For the first model we also studied the stability of the motion. A comparison between the linear and the non-linear case is also performed for both models. We proved that the utilization of the neo-Hookean element leads to the increasing zone where the resonance doesn't appears. For this reason, the neo-Hookean elements can be a valid substitute for the classical linear elements. In our future work we shall develop the models presented, using also linear and non-linear damping elements.

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