A Strain-based Non-linear Elastic Model for Geomaterials

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Abstract: - The non-linear elastic behaviour of geomaterials has traditionally been modeled using stress-dependent elastic models. The approach used by many researchers is not rigorous where the "elastic" models can create energy and can predict permanent deformation under certain circumstances. Other researchers have used rigorous hyperelastic models where energy is conserved by the elastic portion of the model, but these models have stress-dependent stiffness which can slow computation and lead to iterative solutions in finite-element or other strain-based computation techniques. To overcome these problems a rigorous elastic model is investigated which predicts similar behaviour to hyperelastic models, but has the advantage of being elastic strain based which allows more direct solution in strain-based finite-element or other computational codes.

Key-Words: - Elasticity, Geomaterials, Non-linear, Finite Element, Elastic strain

1 Introduction

The resilient response of geomaterials has received much attention over the years. Initial models were linear elastic, but these presented a poor representation of granular material response. In the late 1960s and early 1970s bulk stress dependent, non-linear elastic models were proposed [1]. These early models (and many later models) were based on observations of laboratory test results and can create energy and, possibly more importantly, permanent strains on certain stress paths. Because of the large number of load applications in repeated load applications (e.g. earthquake engineering), permanent strains predicted by and elastic model are considered unacceptable.

To overcome these limitations, a number of hyperelastic models were developed [2,3]. These models do not predict permanent strains and always conserve energy, but do not always provide realistic predictions of response as they can predict a negative bulk modulus or increasing shear modulus with increasing shear stress, trends that are not noted during laboratory testing. One reason for the mismatch in theory and observed response is the hyperelastic models ignore hysteretic energy dissipation during loading.

Most current non-linear elastic models are stress-based models (stiffness is defined as a function of stress state), which simplifies parameter determination during stress controlled laboratory tests, but can significantly increase computational time and create convergence problems when analysing field loading conditions. Models based on simple laboratory tests are often unable to represent behaviour under field loading conditions.

This paper discusses the requirements for a non-linear elastic model, and investigates a new strain-based model for geomaterials.

2 Problem Formulation

An example of a non-rigorous elastic model is shown for a model where the stiffness is stress dependent [1]. This and other similar formulations are probably the most popular for modelling geomaterials and the model formulation is characterised by a non-linear secant Young's modulus ($E=E_R p'^n$) and a constant Poisson's ratio (ν). To determine the effect of complex stress paths on model predictions, it is easiest to rewrite the model in terms of shear modulus (G) and bulk modulus (K), thereby enabling the shear and volumetric behaviour to be independently considered:

$$K = \frac{E}{3(1-2\nu)} = \frac{E_R p'^n}{3(1-2\nu)} \quad : \quad \Delta\varepsilon_p^e = \frac{\Delta p'}{K} \qquad (1)$$

$$G = \frac{E}{2(1+\nu)} = \frac{E_R p'^n}{2(1+\nu)} \quad : \quad \Delta\varepsilon_s^e = \frac{\Delta s}{2G} \qquad (2)$$

where E_R and n are material constants, p' is the mean effective stress, s is shear stress (difference in vertical and horizontal stress under triaxial conditions), ε_p^e is the elastic volumetric strain, and ε_s^e is the elastic shear strain (difference between axial and radial strain for triaxial conditions). If the model in Equations (1) and (2) is taken on the stress path in Fig. 1, the model predicts permanent elastic shear strains as detailed in Table 1, indicating it is not a rigorous elastic model.

Table 1: "Elastic" shear strains predicted by Equation (2).

Loading stage from Fig. 1	*"Elastic" shear strains*
1	$\dfrac{(s_b - s_a)(1+\nu)}{E_R p_a'^{\,n}}$
2	0
3	$\dfrac{(s_a - s_b)(1+\nu)}{E_R p_b'^{\,n}}$
4	0
Total	$\dfrac{(p_a'^{\,n} - p_b'^{\,n})(s_a - s_b)}{E_R p_a'^{\,n} p_b'^{\,n} / (1+\nu)}$

As shown in Table 1, the model predicts permanent elastic shear strains along the stress path considered, indicating it is not a true elastic model. Although not demonstrated in this paper, the model can also create or dissipate energy on the stress path, thereby violating the laws of thermodynamics.

There are, however, numerous rigorous non-linear elastic models which can overcome this limitation but which have other problems with implementation because they are stress-based.

Implementation of non-linear elastic models subjected to field loading conditions is most likely to be performed using the finite element method Finite element theory is complex, and the discussion presented here is not intended to be comprehensive, but rather highlights the aspects of the theory that are important for modelling non-linear elastic material models. A more complete discussion on the finite element method can be found in numerous texts on the subject, e.g. [4]. It should be noted that throughout this paper small-strain behaviour is assumed which is appropriate for this field.

The finite element method is "equivalent to the minimisation of the total potential energy of the system in terms of a prescribed displacement field" [4]. The manner in which this is usually achieved is through the application of displacements (strains), along with an attempt to minimise the total potential energy of the system. For this to occur, the energy from external forces and displacements must be equal to the internal strain energy, calculated from stresses and strains.

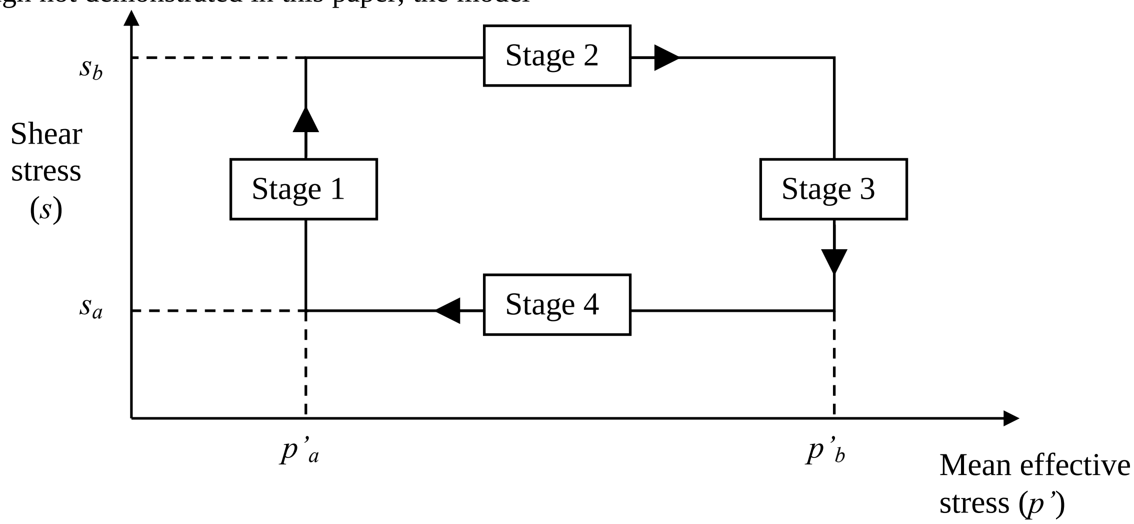

Fig. 1: Stress path of non-zero area.

The relation between the stresses and strains in a material is calculated using a constitutive relation, and the form of this model governs the strain energy of the finite element problem. The basic steps followed by a finite element program are slightly different for linear and non-linear elastic materials, but they can be simplified as shown in Fig. 2. The procedure is iterated through timesteps to ensure compatibility and convergence. As shown, the procedure is strain-based where a strain is inputted to the constitutive relation to calculate a stress.

For linear elastic material models there is a direct approach for computation as there is a linear constitutive relation between strain and stress (Fig. 3), but for non-linear elastic material models the computation is more complex. This is particularly true when the non-linear elastic constitutive model is stress-based, i.e. when the relationship between elastic strains and stresses is a function of stress state. This is particularly important for non-rigorous non-linear elastic models (e.g. Equations (1) and (2)) where there is a non-unique solution to the problem and therefore solution instability (Fig. 4).

If the constitutive relation is a function of stress state, an iterative solution is required where the calculation of stresses is dependent on the stress level. If the constitutive relation is strain-based or constant (as in linear elastic materials), the computation will be significantly quicker as the iterative solution is not required to calculate stresses from strains.

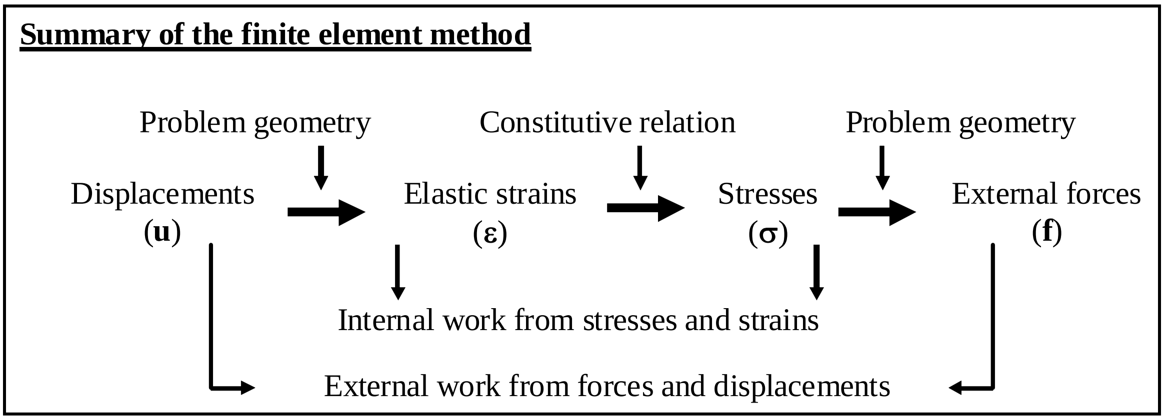

Fig. 2: Summary of the finite element method

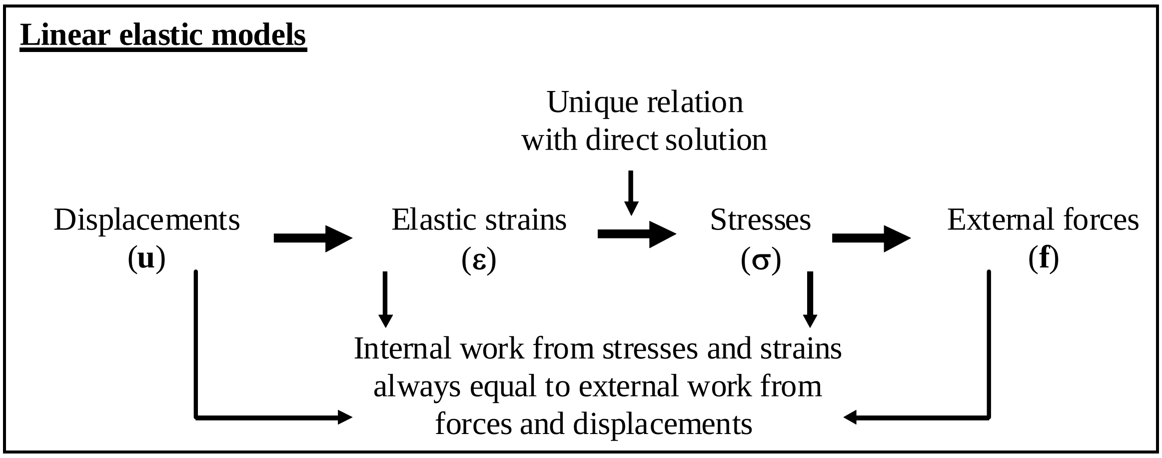

Fig. 3: Implementation of the finite element method for linear elastic models

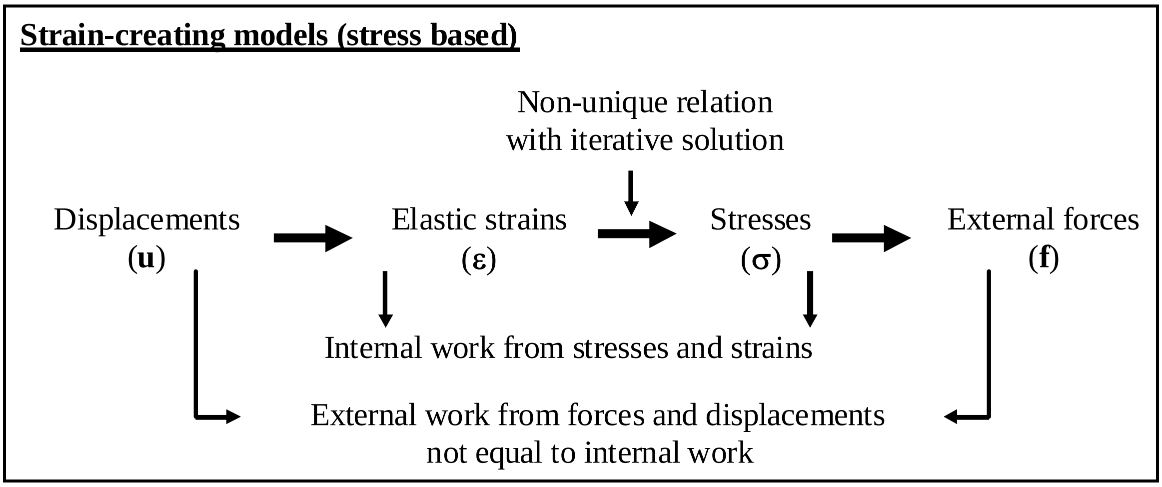

Fig. 4: Implementation of the finite element method for non-rigorous non-linear elastic models

3 Strain-based non-linear elastic models for geomaterials

In order to eliminate the iterative solution required to determine stress from strain in a stress-based model, and to improve convergence in a finite element package, a strain-based non-linear elastic model can be formulated for geomaterials. The model requirements are:

- Strain-based for rapid computation and improved convergence in a finite element analysis.
- Energy and elastic strain conserving under all reasonable loading conditions.
- Few model parameters with each parameter making physical sense, preferably related to parameters used for existing stress-based model
- Able to accurately model the small-strain elastic behaviour of geomaterials.

The first two requirements were addressed by creating a strain-based hyperelastic model where a strain energy density function ($U(\varepsilon)$) is defined and the stresses can be determined by differentiating this function, as discussed in Section 3.1. The third and fourth requirements are addressed in the model formulation which is discussed in Section 3.2. A comparison between the implementation of a stress-based and strain-based hyperelastic formulation is presented in Figs 5 and 6.

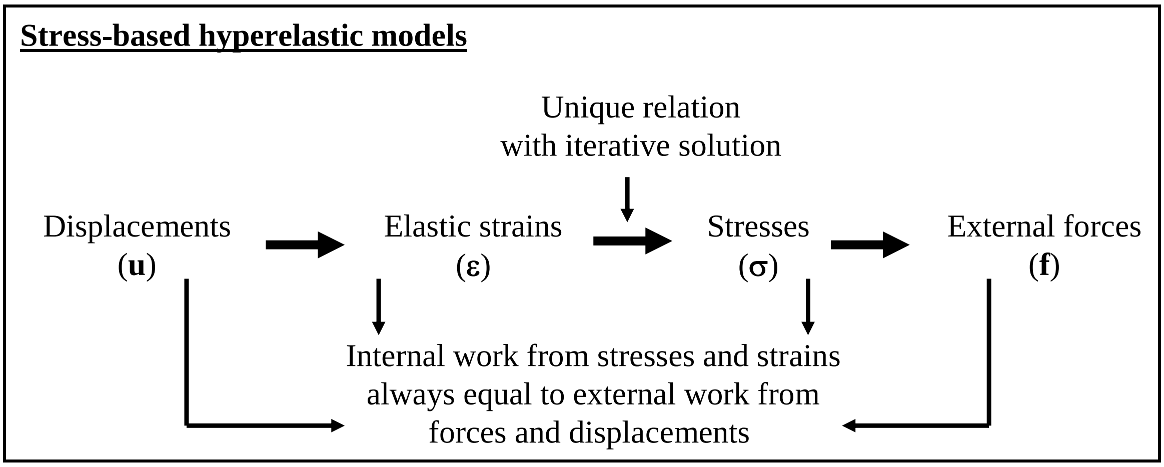

Fig. 5: Implementation of the finite element method for stress-based hyperelastic models

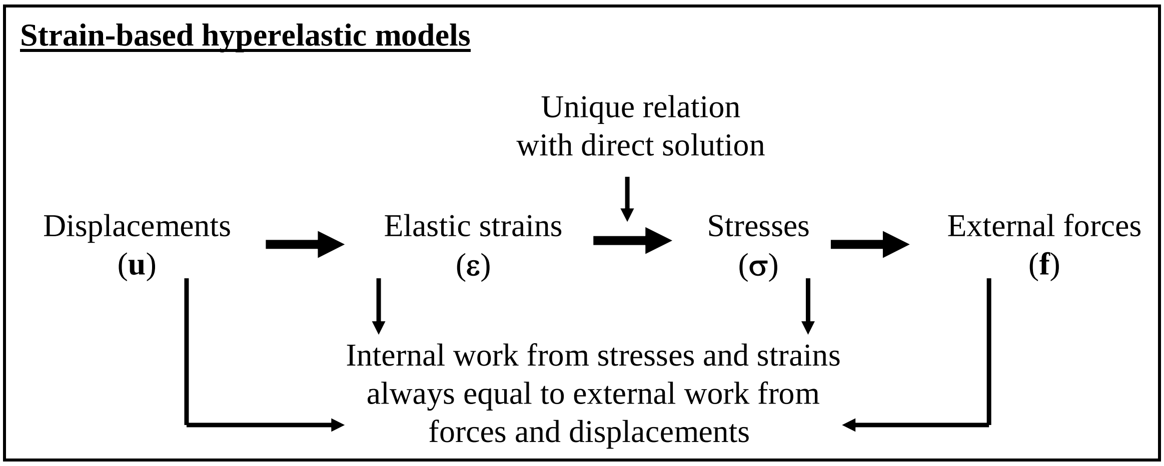

Fig. 6: Implementation of the finite element method for strain-based hyperelastic models

3.1 Energy and strain conservation

While it is fairly simple to formulate a constitutive model that does not predict permanent strains, it is also imperative that any model either conserves or dissipates the applied energy, but never creates it. Any energy creation would be a violation of the second law of thermodynamics, and will result in an incorrect solution in a finite element program which is using the minimum potential energy to solve the problem. During loading, geomaterials materials dissipate energy from hysteretic / plastic response during loading and unloading cycles. Experimental evidence has shown this behaviour is largely rate-independent in granular materials, with the stiffness largely independent of loading rate [5].

There are numerous energy-dissipation models for soils (e.g. [6]), with most of these originally developed for earthquake engineering purposes. They are generally valid only for constant mean effective stress conditions and are therefore of little use for three dimensional modelling applications where shear stresses and mean effective stresses undergo simultaneous changes. A rigorous implementation of hysteretic energy dissipation is extremely complex as any model must ensure resilient strains are conserved and that energy is always dissipated (and never created). This will greatly increase solution time in a finite element analysis and it is therefore proposed that constitutive models for unbound geomaterials first ensure that no energy is created before including energy dissipation. Should the reader require more information on rigorous implementation of rate-independent energy dissipation, this is available in [7].

The easiest rigorous method of ensuring energy is conserved in a non-linear elastic model is through the use of hyperelasticity. A hyperelastic constitutive model is derived from a potential function and will ensure that when implemented in a finite element code, the internal strain energy will equal the energy from external forces and displacements. Hyperelastic models can be either stress-based or strain-based, and they consist of an arbitrary function (V(**σ**) or U(ε)) which, when differentiated with respect to any stress or elastic strain direction (σ'_{ij} or ε^e_{ij} in direction *ij*), gives the corresponding strain or stress related to that variable:

$$\varepsilon^e_{ij} = \frac{\partial V}{\partial \sigma'_{ij}} \quad : \quad \sigma'_{ij} = \frac{\partial U}{\partial \varepsilon^e_{ij}} \qquad (3)$$

Hyperelastic models will always conserve energy and strains, regardless of the stress path followed, and are therefore considered a rigorous method of implementing non-linear elastic models in a finite element code.

The use of hyperelasticity in the formulation of a strain-based elastic model is discussed below.

3.1 Model formulation

Many stress-based elastic models have been validated the following relation between mean effective stress (p') and bulk elastic strain (ε_p^e) under hydrostatic conditions (e.g. [1], [2], [3]):

$$\varepsilon^e_p = \frac{p'}{K_R p'^n} \text{ therefore} \qquad (4)$$

$$p' = \left(K_R \varepsilon^e_p\right)^{\frac{1}{1-n}} \qquad (5)$$

These same models have a corresponding shear modulus (under hydrostatic conditions) with the following form:

$$G = G_R p'^n \qquad (6)$$

By substituting Equation (5) into Equation (6), it is possible to obtain the general form of a model that will have similar constants and behaviour to stress-based hyperelastic models, but this model will be strain-based which will improve solution time and convergence in a finite element package. The complete form of the hyperelastic model is determined from the strain energy density function ($U(\varepsilon)$) that was calculated using the model requirements listed earlier :

$$U = G_R \left(K_R \varepsilon_p^e\right)^{\frac{n}{1-n}} \varepsilon_{s,ij}^e \varepsilon_{s,ij}^e + \left(K_R \varepsilon_p^e\right)^{\frac{1}{1-n}} \varepsilon_p^e \frac{1-n}{2-n} \quad (7)$$

The shear stress and mean effective stress (and therefore secant shear and bulk moduli) can be determined by partially differentiating U with respect to the shear strains and volumetric strains respectively:

$$s_{ij} = 2G_R \left(K_R \varepsilon_p^e\right)^{\frac{n}{1-n}} \varepsilon_{s,ij}^e \quad (8)$$

$$p' = \left(K_R \varepsilon_p^e\right)^{\frac{1}{1-n}} + \frac{nG_R}{1-n} \left(K_R \varepsilon_p^e\right)^{\frac{n}{1-n}} \left(\frac{\varepsilon_{s,ij}^e \varepsilon_{s,ij}^e}{\varepsilon_p^e} \right) \quad (9)$$

The model described above is a three-dimensional isotropic hyperelastic model, and uses the repeated index summation convention used in solid mechanics. The model appears complex, but this is mainly because it was formulated so the parameters are identical to those from other stress-based models under hydrostatic conditions [2,3]. Under these conditions, each parameter has the same physical meaning as in Equations (1) and (2) and as used for other models [1,2]. The parameter n controls the increase in stiffness with an increase in mean effective stress, and the parameters G_R and K_R give the shear and bulk moduli of the soil at a mean effective stress of one stress unit. To ensure the model parameters are independent of units used, it is suggested that the stresses be normalised by atmospheric pressure before implementation in a finite element package, resulting in G_R and K_R being unit independent.

The behaviour predicted by the model, compared with the models with those used by other researchers [1,2] is illustrated in Figs 3 and 4.

As shown in Fig. 7, because the model formulations are equivalent under hydrostatic conditions, there is no difference between the strain based model and the stress based models used by other researchers.

There is, however, a difference when shear is applied (Fig. 8) where the strain-based model predicts a decrease in stiffness with increased shear. This trend in secant behaviour has been observed during testing [5]. The decrease in stiffness with shearing is often attributed to energy dissipation, but in the case of the model presented here, no energy is dissipated. The increase in stiffness with increased shear for the rigorous stress-based model [2] is not observed in practice.

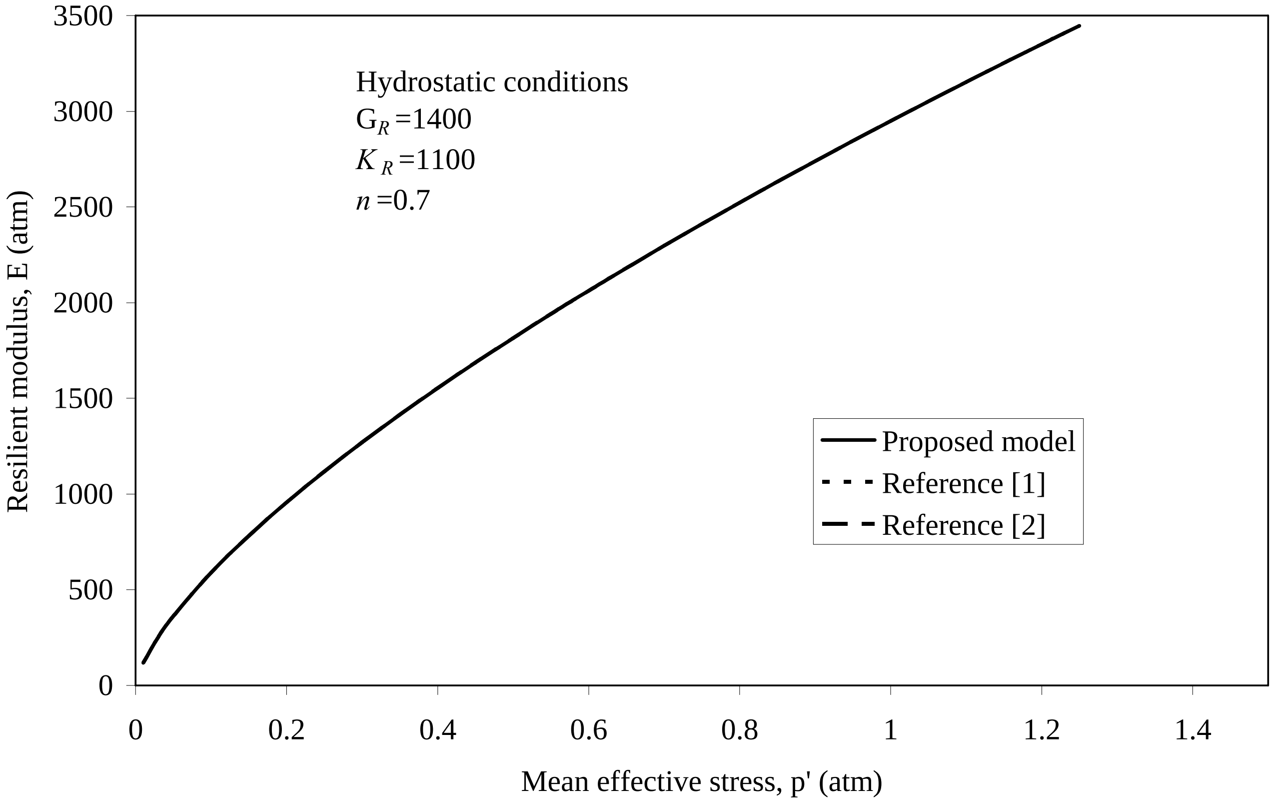

Fig. 7: Model behaviour under hydrostatic conditions (no shear)

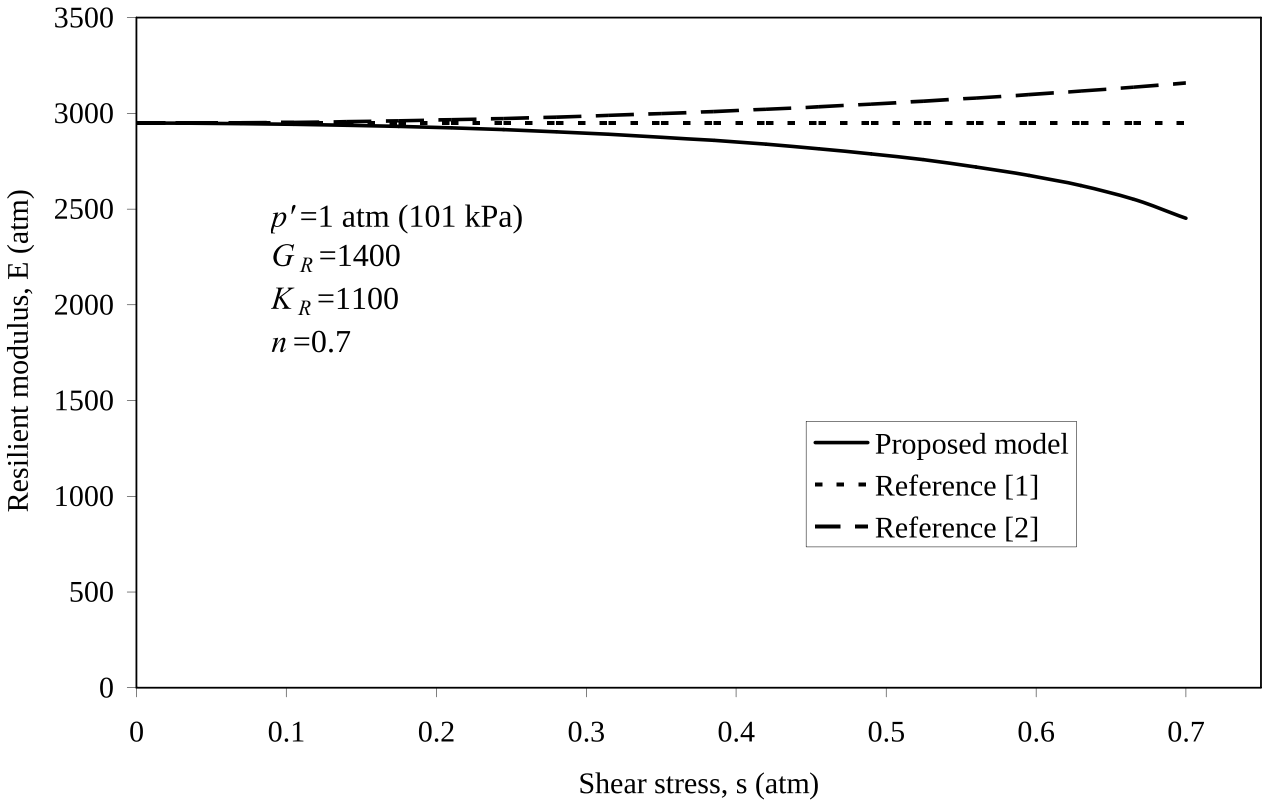

Fig. 8: Model behaviour under constant mean effective stress

3.2 Combining with other strain-based models

The most common models for energy dissipation of geomaterials under repeated loading are described using modulus degradation where the reduction in stiffness is a result of shear strains, such as the model proposed by Hardin and Drnevich [6]. While these common models are very simplistic and should not be considered a rigorous implementation of hysteretic response, the use of a strain-based energy conserving elastic model allows easy combination with a strain-based energy dissipating model and allows rapid computation of stresses from total strains. The current method of using a mixed model with a stress-based energy conserving model and a strain-based energy dissipating model results in a complicated and slow overall solution technique.

The energy dissipation model proposed by Hardin and Drnevich [6] is an empirical model presented in the form of modulus degradation where the ratio between the secant shear modulus (G) and hydrostatic shear modulus ($G_{\max}$) is given by the following equation:

$$\frac{G}{G_{\max}} = \frac{1}{1+\gamma/\gamma_r} \qquad (10)$$

Where γ is the engineering shear strain (twice the pure shear strain, i.e. γ=2ε_s), and γ_r is a reference strain related to the material strength and hydrostatic stiffness.

The original form of the model provided an empirical relationship where $G_{\max}$ could be calculated from the voids ratio and mean effective stress. However, this required the mean effective stress to remain unchanged during analysis and any generated pore pressures or vertical accelerations were therefore ignored in the analysis.

It is possible to use the energy dissipation model with other models for the peak stiffness, and for this it was assumed that a soil had a peak friction angle of 30 degrees. Under simple shear conditions with a mean effective stress of 101 kPa, the parameter γ_r proposed in reference [6] is 4.124e-04 for stiffness parameters in Fig. 8. Converting Equation 10 to give shear stress as a function of shear strain as required in a finite element solution (see Fig. 2), the following is obtained:

$$s_{ij} = 2G\varepsilon_{s,ij} = 2\varepsilon_{s,ij}\frac{G_{\max}}{1+2\varepsilon_{s,ij}/4.124e-04} \qquad (11)$$

If a stress-based model is used to determine $G_{\max}$ (e.g. Equation 2), the following is obtained:

$$s_{ij} = 2\varepsilon_{s,ij}\frac{E_R p'^n}{2(1+\nu)\left(1+2\varepsilon_{s,ij}/4.124e-04\right)} \qquad (12)$$

This is a mixed formulation where the stresses predicted are based on both the stress state and strains, and it is therefore impossible to obtain a direct solution and an iterative solution is required.

If the strain-based model in Equation 8 is used, a solution based solely on strains is obtained:

$$s_{ij} = 2\varepsilon_{s,ij}\frac{G_R\left(K_R\varepsilon_p^e\right)^{\frac{n}{1-n}}}{1+2\varepsilon_{s,ij}/4.124e-04} \qquad (13)$$

Although this formulation appears complicated, its implementation in a strain-based finite element code is simple. If a saturated sand with the initial conditions in Fig. 8 is modeled with Equation 13, the response in Fig. 9 is obtained. In this figure the pure shear strains are converted to engineering shear strains to allow comparison wit the original reference, which produces an identical result if the same value of $G_{\max}$ is used.

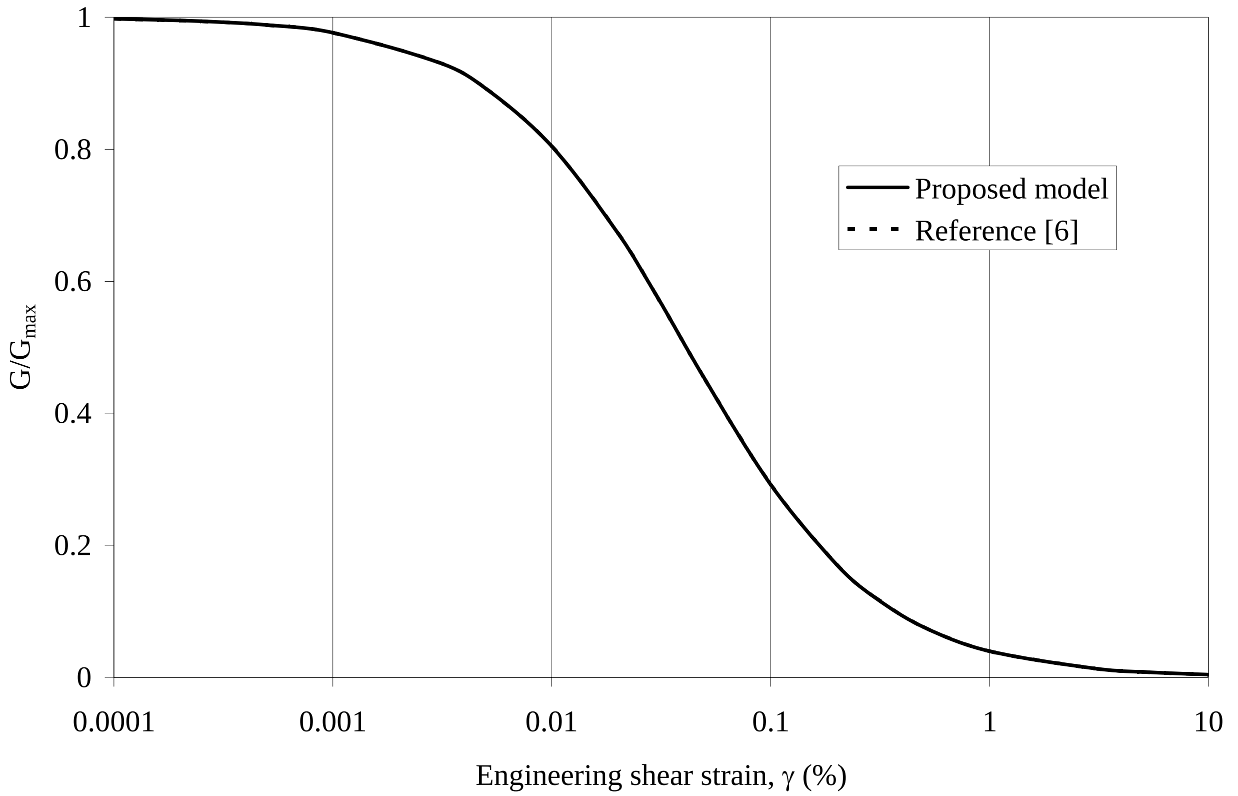

Fig. 9: Predicted model response when combined with Hardin and Drnevich [6] modulus degradation curve.

4 Conclusion

This paper has shown the requirements for implementing non-linear elastic models into a finite element code to model field conditions. Many previous non-linear "elastic" models can predict permanent strains on certain stress paths which is a major concern for repeated load applications. These models can also create energy which violates the laws of thermodynamics and prevents accurate implementation into a finite element code.

Stress-based hyperelastic models will never create energy and will never predict permanent strains and are rigorous constitutive models that can be correctly implemented into a finite element code. As the model is stress-based, its implementation into a strain-based finite element code requires an iterative solution, resulting in slow computation and possible lack of convergence. In addition, many stress-based hyperelastic models predict an increase in resilient modulus with an increase in shear, a trend opposite to observed behaviour.

As a result, a strain-based hyperelastic model is proposed. The model has the same material parameters and predicts the same behaviour under hydrostatic conditions as popular stress based models [1,2], but the behaviour under shearing is different with the new model predicting a decrease in resilient modulus with increased shear, as noted during laboratory testing. The use of a strain-based model also improves convergence and significantly reduces computation time in a finite element analysis.

A strain based elastic model can easily combined with a strain-based modulus degradation model such as that originally proposed by Hardin and Drnevich [6]. This allows the direct computation of stresses from strains without the need for an iterative solution.

There are some problems where this approach will not work, for example when a stress-based hysteretic or yielding model is required, but the use of strain-based hyperelastic models in modelling geomaterials shows promise for certain applications.

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