On the Mechanics of the Particle on a Curve

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Abstract: - In this paper we present a new method to study the mechanics of a particle on a curve without friction. Two aspects are discussed. First is the static developed in linear independent co-ordinates, not necessary orthogonal. The second aspect treats the dynamics in intrinsic co-ordinates, obtaining not only the speed but also the components of the acceleration and the time variation of the generalized co-ordinate on the curve. The theory is applied in practical situations.

Key-Words: - Linear independent co-ordinates, static, dynamics

1 Introduction

In the classical mechanics the linkage of the particle on a curve is seen in two different ways. In static, the curve is defined as an intersection of two surfaces of equations $f(x, y, z) = 0$, respective $g(x, y, z) = 0$. The equilibrium position results together with the normal reaction (defined by its components on two directions in the normal plan) by solving a system of five equations with five unknowns. In dynamics the motion is studied in intrinsic co-ordinates. On the tangent direction results the speed, and on the normal and bi-normal directions result the components of the normal reaction.

If there exists friction between the curve and the particle, the problem is more complicated. In the static, the usual condition is given with the aid of a dot product and a cross product. In the dynamics the differential equations contain non-continuous function (name it, the signature function) so that the moving equations must be treated on distinctive intervals.

In this paper we shall give a unitary approach to the study of the particle's mechanics. We shall prove that in the static the equilibrium position can be separated of the problem of the normal reaction. In addition, this normal reaction results by its components on two directions, neither necessary orthogonal one to another and nor necessary situated in the normal plan, that is, we shall work on three independent directions, not necessary orthogonal one to another. In dynamics, we shall obtain not only the variation of the speed, but also the variation of the acceleration of the particle on the given curve and, of course, the normal reaction.

The problem of the mechanics of a particle on a curve is and old one and many papers deal with it. One type of problems is those which treat celestial mechanics [1, 2]. Another type consists of problems, which lead to the dynamics on a curve, no matter if the authors consider a particle, a rigid body, or a system of rigid bodies [3, 4]. A unitary approach for the case without friction is given in [7].

2 The Static of the Particle on a Curve without Friction

Let us consider the curve (Γ) of parametric equations $x = x(\theta)$, $y = y(\theta)$, $z = z(\theta)$. In this way the position vector $\mathbf{r}$ is

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \mathbf{r}(\theta). \tag{1}$$

Let M be the particle and $\mathbf{F}$ the resultant of the forces which act upon it, resultant being assumed to be function of the position vector $\mathbf{r}$, $\mathbf{F} = \mathbf{F}(\mathbf{r})$ and therefore $\mathbf{F} = \mathbf{F}(\theta)$, where θ is a real parameter (fig. 1).

We shall denote by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ the vectors given by

$$\mathbf{a} = \frac{\mathrm{d}x}{\mathrm{d}\theta}\mathbf{i} + \frac{\mathrm{d}y}{\mathrm{d}\theta}\mathbf{j} + \frac{\mathrm{d}z}{\mathrm{d}\theta}\mathbf{k}, \tag{2}$$

$$\mathbf{b} = \frac{\mathrm{d}^2x}{\mathrm{d}\theta^2}\mathbf{i} + \frac{\mathrm{d}^2y}{\mathrm{d}\theta^2}\mathbf{j} + \frac{\mathrm{d}^2z}{\mathrm{d}\theta^2}\mathbf{k}, \tag{3}$$

$$\mathbf{c} = \frac{\mathrm{d}^3x}{\mathrm{d}\theta^3}\mathbf{i} + \frac{\mathrm{d}^3y}{\mathrm{d}\theta^3}\mathbf{j} + \frac{\mathrm{d}^3z}{\mathrm{d}\theta^3}\mathbf{k} \tag{4}$$

and we shall assume that these three vectors are linear independents, that is $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \neq 0$. Let us observe that the vector $\mathbf{a}$ is on the tangent direction to the curve (Γ), but the vectors $\mathbf{b}$ and $\mathbf{c}$ are arbitrary.

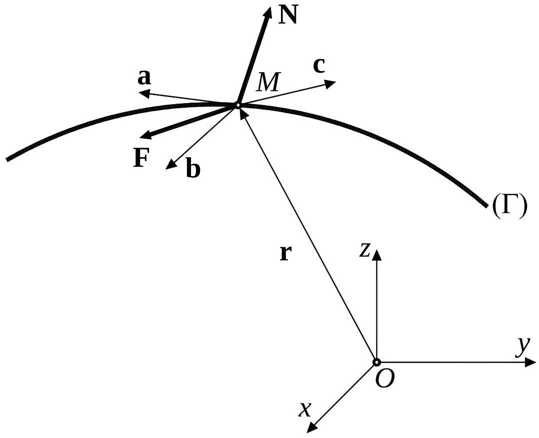

Fig. 1. The static without friction.

If M is an equilibrium position, then the normal reaction is orthogonal to the vector $\mathbf{a}$, so that $\mathbf{N}$ will have components only on the directions of the vector $\mathbf{b}$ and $\mathbf{c}$,

$$\mathbf{N} = N_b\mathbf{b} + N_c\mathbf{c}. \tag{5}$$

Because M is an equilibrium position, results the equation

$$\mathbf{F} + \mathbf{N} = \mathbf{0}, \tag{6}$$

where

$$\mathbf{F} = F_a\mathbf{a} + F_b\mathbf{b} + F_c\mathbf{c}. \tag{7}$$

The relation (6) will be multiplied successively by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ obtaining

$$\mathbf{Fa} + N_b\mathbf{ba} + N_c\mathbf{ca} = 0,\ \mathbf{Fb} + N_b\mathbf{b}^2 + N_c\mathbf{cb} = 0,$$

$$\mathbf{Fc} + N_b\mathbf{bc} + N_c\mathbf{c}^2 = 0, \tag{8}$$

The last two relations (8) offer us

$$N_b = \frac{\begin{vmatrix} -\mathbf{Fb} & \mathbf{bc} \\ -\mathbf{Fc} & \mathbf{c}^2 \end{vmatrix}}{\begin{vmatrix} \mathbf{b}^2 & \mathbf{bc} \\ \mathbf{bc} & \mathbf{c}^2 \end{vmatrix}},\ N_c = \frac{\begin{vmatrix} \mathbf{b}^2 & -\mathbf{Fb} \\ \mathbf{bc} & -\mathbf{Fc} \end{vmatrix}}{\begin{vmatrix} \mathbf{b}^2 & \mathbf{bc} \\ \mathbf{bc} & \mathbf{c}^2 \end{vmatrix}}, \tag{9}$$

that is, N_b and N_c are linear functions of $\mathbf{Fb}$ and $\mathbf{Fc}$,

$$N_b = \lambda_b\mathbf{Fb} + \lambda_c\mathbf{Fc},\ N_c = \mu_b\mathbf{Fb} + \mu_c\mathbf{Fc}. \tag{10}$$

Replacing now in the first expression (8) we find

$$\mathbf{Fa} + (\lambda_b\mathbf{Fb} + \lambda_c\mathbf{Fc})\mathbf{ba} + (\mu_b\mathbf{Fb} + \mu_c\mathbf{Fc})\mathbf{ca} = 0, \tag{11}$$

relation from which results the equilibrium position.

3 Application

Application 1. Let us consider the circular helix given by the parametric equations $x = R\cos\theta$, $y = R\sin\theta$, $z = R\theta$. On this helix there is the particle M of mass $\frac{\alpha R}{g}$ acted by its weight and by the force $\mathbf{R}$ given by

$$\mathbf{R} = \alpha\overline{MA}, \tag{12}$$

the point A having the co-ordinates $(0,0,R)$, α being a positive real parameter, and g the gravitational acceleration (fig. 2).

We immediately find the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in the form

$$\mathbf{a} = -R\sin\theta\mathbf{i} + R\cos\theta\mathbf{j} + R\mathbf{k},$$

$$\mathbf{b} = -R\cos\theta\mathbf{i} - R\sin\theta\mathbf{j},$$

$$\mathbf{c} = R\sin\theta\mathbf{i} - R\cos\theta\mathbf{j}, \tag{13}$$

The resultant of the forces which act on the particle reads

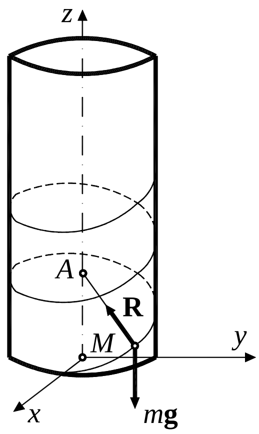

Fig. 2. Application 1.

$$\mathbf{F} = \alpha[-R\cos\theta\mathbf{i} - R\sin\theta\mathbf{j} + (R - R\theta)\mathbf{k}] - \\ -\frac{\alpha R}{g}g\mathbf{k} = \alpha R(-\cos\theta\mathbf{i} - \sin\theta\mathbf{j} - \theta\mathbf{k}). \tag{14}$$

We immediately find

$$\mathbf{b}^2 = R^2,\ \mathbf{bc} = 0,\ \mathbf{c}^2 = R^2,$$

$$\mathbf{ab} = 0,\ \mathbf{ac} = -R^2, \tag{15}$$

$$\mathbf{Fb} = \alpha R^2,\ \mathbf{Fc} = 0,\ \mathbf{Fa} = -\alpha R^2\theta. \tag{16}$$

The expressions (9) offer us

$$N_b = \frac{\begin{vmatrix} -\alpha R^2 & 0 \\ 0 & R^2 \end{vmatrix}}{\begin{vmatrix} R^2 & 0 \\ 0 & R^2 \end{vmatrix}} = -\alpha,$$

$$N_c = \frac{\begin{vmatrix} R^2 & -\alpha R^2 \\ 0 & R^2 \end{vmatrix}}{\begin{vmatrix} R^2 & 0 \\ 0 & R^2 \end{vmatrix}} = 0. \tag{17}$$

The first relation (8) gives us now

$$-\alpha R^2\theta = 0, \tag{18}$$

wherefrom $\theta = 0$, so that the equilibrium position is given by $x = R$, $y = 0$, $z = 0$. For this equilibrium position we have $\mathbf{N} = -\alpha\mathbf{b} = \alpha R\mathbf{i}$.

4 The Static of the Particle on a Curve with Friction

Let us now consider that there exists friction of coefficient of friction μ between the curve and the particle. We decompose the force $\mathbf{F}$ onto the direction of the tangent $\mathbf{F}_t$ and into the normal plan $\mathbf{F}_n$. The component $\mathbf{F}_t$ is equilibrated by the friction force $\mathbf{F}_f$, and the component $\mathbf{F}_n$ by the normal reaction $\mathbf{N}$ (fig. 3)

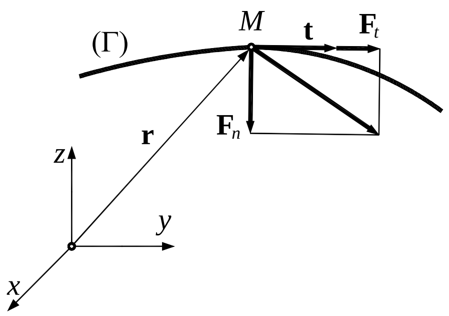

Fig. 3. The static with friction.

We have

$$\mathbf{F}_t = \frac{\mathbf{F}\cdot\mathbf{t}}{|\mathbf{t}|^2}\mathbf{t}, \tag{19}$$

$$\mathbf{F}_n = \mathbf{F} - \mathbf{F}_t = \mathbf{F} - \frac{\mathbf{F}\cdot\mathbf{t}}{|\mathbf{t}|^2}\mathbf{t}, \tag{20}$$

where $\mathbf{t}$ is a vector tangent to the curve at the point M.

The inequality of the friction

$$\left|\mathbf{F}_f\right| \le \mu|\mathbf{N}| \tag{21}$$

leads us to

$$\left|\frac{\mathbf{F}\cdot\mathbf{t}}{|\mathbf{t}|^2}\mathbf{t}\right| \le \left|\mathbf{F} - \frac{\mathbf{F}\cdot\mathbf{t}}{|\mathbf{t}|^2}\mathbf{t}\right|, \tag{22}$$

relation from which we obtain the equilibrium positions.

5 Application

Application 2. Let us consider again the example given in the application 1 and let be μ the coefficient of friction between the helix and the particle.

We find that

$$\mathbf{F} = \alpha\mathbf{R}(-\mathbf{cos}\theta\mathbf{i} - \mathbf{sin}\theta\mathbf{j} - \theta\mathbf{k}), \tag{23}$$

$$\mathbf{t} = -R\sin\theta\mathbf{i} + R\cos\theta\mathbf{j} + R\mathbf{k}, \tag{24}$$

$$|\mathbf{t}|^2 = 2R^2, \tag{25}$$

$$\mathbf{F}\cdot\mathbf{t} = -\alpha\mathbf{R}^2\theta, \tag{26}$$

$$\frac{\mathbf{F}\cdot\mathbf{t}}{|\mathbf{t}|^2}\mathbf{t} = -\frac{\alpha}{2}R\theta(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j} + \mathbf{k}), \tag{27}$$

$$\begin{aligned}\mathbf{F} - \frac{\mathbf{F}\cdot\mathbf{t}}{|\mathbf{t}|^2}\mathbf{t} = \alpha R\Bigg[&\left(-\cos\theta - \frac{\theta\sin\theta}{2}\right)\mathbf{i} + \\ &+\left(-\sin\theta + \frac{\theta\cos\theta}{2}\right)\mathbf{j} - \frac{\theta}{2}\mathbf{k}\Bigg]\end{aligned}, \tag{28}$$

$$\left|\frac{\mathbf{F}\cdot\mathbf{t}}{|\mathbf{t}|^2}\mathbf{t}\right|^2 = \frac{\alpha R|\theta|}{\sqrt{2}},\ \left|\mathbf{F} - \frac{\mathbf{F}\cdot\mathbf{t}}{|\mathbf{t}|^2}\mathbf{t}\right| = \frac{\alpha R}{\sqrt{2}}\sqrt{2+\theta^2}, \tag{29}$$

so that the inequity (22) becomes

$$|\theta| \le \sqrt{2+\theta^2} \tag{30}$$

or, equivalently,

$$0 \le 2, \tag{31}$$

that is, the particle stays at rest onto the whole helix.

5 The Dynamics of the Particle on a Curve without Friction

Let us consider the curve (Γ) given by the parametric equations $x = x(\theta)$, $y = y(\theta)$, $z = z(\theta)$ and let us denote by $\mathbf{F}$ the resultant of the forces which act upon the particle M of mass m. Let M_0 be the initial position of the particle. In the point M we consider the unit vectors of the tangent $\mathbf{t}$, normal $\mathbf{n}$, and bi-normal $\mathbf{b}$. We also denote by $\mathbf{v}$ the speed of the particle and by $\mathbf{a}$ its acceleration (fig. 4). Obviously,

$$\mathbf{v} = v\mathbf{t},\ \mathbf{a} = a_t\mathbf{t} + a_n\mathbf{n}. \tag{32}$$

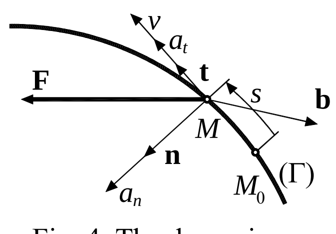

Fig. 4. The dynamics.

From the relation

$$a_t = \frac{\mathrm{d}v}{\mathrm{d}t} \quad (33)$$

we find successively

$$a_t = \frac{\mathrm{d}v}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}s} = v\frac{\mathrm{d}v}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}s}, \quad (34)$$

so that the moving equation on the tangent direction reads

$$mv\frac{\mathrm{d}v}{\mathrm{d}\theta} = F_t\frac{\mathrm{d}s}{\mathrm{d}\theta}, \quad (35)$$

where

$$\mathbf{F} = F_t\mathbf{t} + F_n\mathbf{n} + F_b\mathbf{b}. \quad (36)$$

On the other hand,

$$\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta = L(\theta)\mathrm{d}\theta, \quad (37)$$

the notation being obvious, so that the equation (35) becomes

$$\frac{1}{2}m\frac{\mathrm{d}(v^2)}{\mathrm{d}\theta} = F_t(\theta)L(\theta), \quad (38)$$

and by integration one obtains

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \int_{\theta_0}^{\theta} F_t(\theta)L(\theta)\mathrm{d}\theta. \quad (39)$$

The last expression leads us to

$$v = \pm\sqrt{v_0^2 + \frac{2}{m}\int_{\theta_0}^{\theta} F_t(\theta)L(\theta)\mathrm{d}\theta} = \pm M(\theta), \quad (40)$$

the notation being again obvious.

The component a_n of the acceleration is

$$a_n = \frac{v^2}{\rho}, \quad (41)$$

where ρ is the curvature radius,

$$\rho = \frac{\left[\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2\right]^{\frac{3}{2}}}{\sqrt{A^2 + B^2 + C^2}}, \quad (42)$$

with

$$A = \frac{\mathrm{d}x}{\mathrm{d}\theta}\frac{\mathrm{d}^2y}{\mathrm{d}\theta^2} - \frac{\mathrm{d}y}{\mathrm{d}\theta}\frac{\mathrm{d}^2x}{\mathrm{d}\theta^2},\ B = \frac{\mathrm{d}y}{\mathrm{d}\theta}\frac{\mathrm{d}^2z}{\mathrm{d}\theta^2} - \frac{\mathrm{d}z}{\mathrm{d}\theta}\frac{\mathrm{d}^2y}{\mathrm{d}\theta^2},$$
$$C = \frac{\mathrm{d}z}{\mathrm{d}\theta}\frac{\mathrm{d}^2x}{\mathrm{d}\theta^2} - \frac{\mathrm{d}x}{\mathrm{d}\theta}\frac{\mathrm{d}^2z}{\mathrm{d}\theta^2}, \quad (43)$$

Therefore, we have

$$a_t = \frac{F_t(\theta)}{m},\ a = \sqrt{a_t^2 + a_n^2},$$
$$a^2 = \frac{F_t^2(\theta)}{m^2} + \frac{v^4}{\rho^2}. \quad (44)$$

But

$$a_t = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t}, \quad (45)$$

so that we obtain

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = a_t\frac{\mathrm{d}\theta}{\mathrm{d}v} = \frac{\frac{F_t(\theta)}{m}}{\pm\frac{\mathrm{d}M(\theta)}{\mathrm{d}\theta}}, \quad (46)$$

where we kept into account the expression (40).

6 Applications

Application 3. Let us consider again the application given at paragraph 3. In this case we have

$$\mathbf{F} = \alpha R(-\cos\theta\mathbf{i} - \sin\theta\mathbf{j} - \theta\mathbf{k}), \quad (47)$$

$$\mathbf{t} = \frac{-\sin\theta\mathbf{i} + \cos\theta\mathbf{j} + \mathbf{k}}{\sqrt{2}}, \quad (48)$$

$$\mathbf{n}_1 = \frac{\mathrm{d}^2x}{\mathrm{d}\theta^2}\mathbf{i} + \frac{\mathrm{d}^2y}{\mathrm{d}\theta^2}\mathbf{j} + \frac{\mathrm{d}^2z}{\mathrm{d}\theta^2}\mathbf{k} = $$
$$= -R\cos\theta\mathbf{i} - R\sin\theta\mathbf{j}, \quad (49)$$

$$\mathbf{b}_1 = \frac{\mathrm{d}^3x}{\mathrm{d}\theta^3}\mathbf{i} + \frac{\mathrm{d}^3y}{\mathrm{d}\theta^3}\mathbf{j} + \frac{\mathrm{d}^3z}{\mathrm{d}\theta^3}\mathbf{k} = $$
$$= R\sin\theta\mathbf{i} - R\cos\theta\mathbf{j}, \quad (50)$$

$$\mathbf{n}_1\mathbf{t} = 0, \quad (51)$$

$$\mathbf{n} = \frac{\mathbf{n}_1}{\|\mathbf{n}_1\|} = -\cos\theta\mathbf{i} - \sin\theta\mathbf{j}, \quad (52)$$

$$\mathbf{b}_1\mathbf{t} = -\frac{R}{\sqrt{2}},\ \mathbf{b}_1\mathbf{n} = 0, \quad (53)$$

$$\mathbf{b} = \frac{\mathbf{b}_1 - (\mathbf{b}_1\mathbf{t})\mathbf{t} - (\mathbf{b}_1\mathbf{n})\mathbf{n}}{\|\mathbf{b}_1 - (\mathbf{b}_1\mathbf{t})\mathbf{t} - (\mathbf{b}_1\mathbf{n})\mathbf{n}\|} = \frac{\sin\theta\mathbf{i} - \cos\theta\mathbf{j}}{\sqrt{2}}, \quad (54)$$

$$F_t = \mathbf{Ft} = -\frac{\alpha R\theta}{\sqrt{2}}, \quad (55)$$

$$L(\theta) = R\sqrt{2}, \quad (56)$$

$$\int_{\theta_0}^{\theta} F_t(\theta)L(\theta)\mathrm{d}\theta = \frac{\alpha R^2}{2}(\theta_0^2 - \theta^2). \quad (57)$$

Further on, we shall consider that at $t = 0$ we have $\theta_0 = 0$, that is $x = R$, $y = 0$, $z = 0$ and $\mathbf{v}_0 = c_0(\mathbf{j} + \mathbf{k})$, where c_0 is a non-zero real constant. In these conditions, the expression (57) becomes

$$\int_{\theta_0}^{\theta} F_t(\theta)L(\theta)\mathrm{d}\theta = \int_{0}^{\theta} F_t(\theta)L(\theta)\mathrm{d}\theta = -\frac{\alpha R^2}{2}\theta^2. \quad (58)$$

From the relation (40) we find

$$v = \pm\sqrt{2c_0^2 - \frac{2g}{\alpha R}\frac{\alpha R^2}{2}\theta^2} = \pm\sqrt{2c_0^2 - gR\theta^2}\,. \quad (59)$$

The expressions A, B and C from the formula (43) become now

$$A = R^2,\ B = R^2\sin\theta,\ C = -R^2\cos\theta, \quad (60)$$

the radius of curvature writing

$$\rho = \frac{\left(R^2\sin^2\theta + R^2\cos^2\theta + R^2\right)^{\frac{3}{2}}}{\sqrt{R^4 + R^4\sin^2\theta + R^4\cos^2\theta}} = 2R\,. \quad (61)$$

One deduces now the expressions

$$a_n = \frac{v^2}{\rho} = \frac{2c_0^2 - gR\theta^2}{2R}, \quad (62)$$

$$\alpha_t = -\frac{\alpha R\theta}{\sqrt{2}}\frac{g}{\alpha R} = -\frac{\theta g}{\sqrt{2}}, \quad (63)$$

$$a = \frac{\sqrt{\left(2c_0^2 - gR\theta^2\right)^2 + 2R^2\theta^2 g^2}}{2R}, \quad (64)$$

$$\frac{d\theta}{dt} = \pm\frac{\sqrt{2c_0^2 - gR\theta^2}}{\sqrt{2}R}\,. \quad (65)$$

From the relation (59) we find that the speed becomes zero for

$$\theta^* = \sqrt{\frac{2c_0^2}{gR}}\,. \quad (66)$$

Let us denote by $G(\theta)$ the function

$$G(\theta) = \frac{\sqrt{2}R}{\sqrt{2c_0^2 - gR\theta^2}}, \quad (67)$$

for which we have

$$\begin{aligned} G'(\theta) &= -2gR^2\theta\left(2c_0^2 - gR\theta^2\right)^{-\frac{3}{2}}, \\ G''(\theta) &= 3\sqrt{2}R^2\theta^2\left(2c_0^2 - gR\theta^2\right)^{-\frac{5}{2}} - \\ &\quad - 2gR^2\sqrt{2}\left(2c_0^2 - gR\theta^2\right)^{-\frac{3}{2}}, \end{aligned} \quad (68)$$

$$G(0) = 0,\ G'(0) = 0,\ G''(0) = -gR^2\left|c_0\right|^{-3}. \quad (69)$$

Expanding in series the relation (65), for small θ, one obtains

$$\mp\frac{gR^2}{\left|c_0\right|^3}\frac{\theta^2}{2}d\theta = dt\,, \quad (70)$$

wherefrom, by integration between $t = 0$, $\theta = 0$ and $t = t$, $\theta = \theta$, it follows

$$\mp\frac{gR^2}{\left|c_0\right|^3}\frac{\theta^3}{6} = t\,. \quad (71)$$

Making now $\theta = \theta^*$, one deduces

$$\frac{gR^2}{\left|c_0\right|^3}\frac{2c_0^2}{gR}\sqrt{\frac{2c_0^2}{gR}}\cdot\frac{1}{6} = \frac{T}{4}, \quad (72)$$

where T is the period of the motion,

One deduces the period of the small oscillations

$$T = \frac{4\sqrt{2}}{3}\sqrt{\frac{R}{g}}\,. \quad (73)$$

Application 4 (The motion of a Planet around the Sun). Let M be a planet of mass m_P which rotates on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$, around the Sun A_1 of mass m_S (fig. 5).

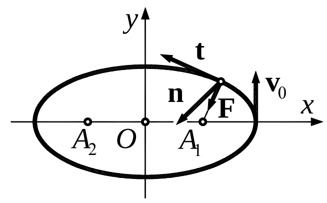

Fig. 5. Application 4.

The parametric equations of the trajectory are

$$x = a\cos\theta,\ y = b\sin\theta, \quad (74)$$

and the attraction force $\mathbf{F}$ has the form (k being the constant of the universal attraction)

$$\mathbf{F} = \frac{km_P m_S}{\left(MA_1\right)^3}\overline{MA_1} \quad (75)$$

where $A_1(c, 0)$ with

$$c = \sqrt{a^2 - b^2} \quad (76)$$

We shall denote by $\lambda = \frac{b}{a}$, so that $c = a\sqrt{1-\lambda^2}$, the expression (75) becoming

$$\mathbf{F} = \frac{km_P m_S}{a^2}\times$$
$$\times\frac{\left(\sqrt{1-\lambda^2} - \cos\theta\right)\mathbf{i} - \lambda\sin\theta\mathbf{j}}{\left[\left(\sqrt{1-\lambda^2} - \cos\theta\right)^2 + \lambda^2\sin^2\theta\right]^{\frac{3}{2}}}\,. \quad (77)$$

We have

$$\frac{dx}{d\theta} = -a\sin\theta,\ \frac{dy}{d\theta} = \lambda a\cos\theta,\ \frac{d^2x}{d\theta^2} = -a\cos\theta,$$
$$\frac{d^2y}{d\theta^2} = -\lambda a\sin\theta, \quad (78)$$

$$\mathbf{t} = \frac{-\sin\theta\mathbf{i} + \lambda\cos\theta\mathbf{j}}{\sqrt{\sin^2\theta + \lambda^2\cos^2\theta}}, \quad (79)$$

$$\mathbf{n} = \frac{-\lambda\cos\theta\mathbf{i} - \sin\theta\mathbf{j}}{\sqrt{\sin^2\theta + \lambda^2\cos^2\theta}}, \quad (80)$$

$$A = \lambda a^2,\ B = 0,\ C = 0, \quad (81)$$

$$\rho = \frac{a}{\lambda}\left(\sin^2\theta + \lambda^2\cos^2\theta\right)^{\frac{3}{2}}\,. \quad (82)$$

We shall assume that at $t = 0$, $\theta = 0$, $v = v_0$, so that we obtain

$$F_t(\theta) = \mathbf{F}\cdot\mathbf{t} = \frac{km_P m_S}{a^2\sqrt{\sin^2\theta + \lambda^2\cos^2\theta}}\times$$
$$\times\frac{\sin\theta\sqrt{1-\lambda^2}\left(1+\cos\theta\sqrt{1-\lambda^2}\right)}{\left[\left(\sqrt{1-\lambda^2}-\cos\theta\right)^2+\lambda^2\sin^2\theta\right]^{\frac{3}{2}}}, \quad (83)$$

$$L(\theta) = a\sqrt{\sin^2\theta + \lambda^2\cos^2\theta}, \quad (84)$$

$$F_t(\theta)L(\theta) = \frac{km_P m_S}{a}\times$$
$$\times\frac{\sin\theta\sqrt{1-\lambda^2}\left(1+\cos\theta\sqrt{1-\lambda^2}\right)}{\left[\left(\sqrt{1-\lambda^2}-\cos\theta\right)^2+\lambda^2\sin^2\theta\right]^{\frac{3}{2}}}. \quad (85)$$

But $F_t(\theta)L(\theta)$ is an odd periodical of period 2π function, so that it results

$$\int_{\theta}^{\theta+2\pi} F_t(\theta)L(\theta)\mathrm{d}\theta = 0, \quad (86)$$

and from the relation (40) we obtain that v is a periodical of period 2π function. This is also true for a, a_t, and a_n.

Finally, we get

$$v = \sqrt{v_0^2 + \frac{2}{m_P}\int_0^{\theta} F_t(\theta)L(\theta)\mathrm{d}\theta}, \quad (87)$$

$$a_n = \frac{v_0^2 + \frac{2}{m_P}\int_0^{\theta} F_t(\theta)L(\theta)\mathrm{d}\theta}{a\left(\sin^2\theta+\lambda^2\cos^2\theta\right)}\lambda, \quad (88)$$

$$a = \sqrt{\frac{F_t^2(\theta)}{m_P^2} + \frac{\left[v_0^2 + \frac{2}{m_P}\int_0^{\theta} F_t(\theta)L(\theta)\mathrm{d}\theta\right]^2}{a^2\left(\sin^2\theta+\lambda^2\cos^2\theta\right)^2}\lambda^2}, \quad (89)$$

$$a_t = \frac{F_t(\theta)}{m_P}. \quad (90)$$

Two particular cases are very important. First is characterized by $a = b$, therefore $\lambda = 1$, that is the ellipse becomes a circle. In this case we get the following relations

$$\mathbf{F} = \frac{kM_P m_S}{a^2}\left(-\cos\theta\mathbf{i} - \sin\theta\mathbf{j}\right), \quad (91)$$

$$\mathbf{t} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j},\ \mathbf{n} = -\cos\theta\mathbf{i} - \sin\theta\mathbf{j}, \quad (92)$$

$$A = a^2,\ B = 0,\ C = 0, \quad (93)$$

$$\rho = a, \quad (94)$$

$$F_t(\theta) = 0, \quad (95)$$

$$L(\theta) = a, \quad (96)$$

$$\int_0^{\theta} F_t(\theta)L(\theta)\mathrm{d}\theta = 0, \quad (97)$$

$$v = v_0 = \mathrm{ct}, \quad (98)$$

$$a_n = \frac{v_0^2}{a},\ a = \frac{v_0^2}{a} = a_n,\ a_t = 0. \quad (99)$$

We obtained the uniform circular motion.

The second case is the linear one, that is $b \to 0$, $\lambda \to 0$. In this case we get

$$\mathbf{F} = \frac{km_P m_S\mathbf{i}}{a^2\left(1-\cos^2\theta\right)^2}, \quad (100)$$

$$\mathbf{t} = -\mathrm{sign}(\sin\theta)\mathbf{i},\ \mathbf{n} = -\mathrm{sign}(\sin\theta)\mathbf{j}, \quad (101)$$

$$A = 0,\ B = 0,\ C = 0, \quad (102)$$

$$\rho = \infty, \quad (103)$$

$$F_t(\theta) = -\mathrm{sign}(\sin\theta)\frac{km_P m_S}{a^2(1-\cos\theta)^2}, \quad (104)$$

$$L(\theta) = a|\sin\theta|, \quad (105)$$

$$F_t(\theta)L(\theta) = -\frac{km_P m_S\sin\theta}{a(1-\cos\theta)^2}, \quad (106)$$

$$\int_{\theta_0}^{\theta} F_t(\theta)L(\theta)\mathrm{d}\theta = \frac{km_P m_S}{a}\left(\frac{1}{1-\cos\theta} - \frac{1}{1-\cos\theta_0}\right), \quad (107)$$

$$v = \sqrt{v_0^2 + \frac{2km_S}{a}\left(\frac{1}{1-\cos\theta} - \frac{1}{1-\cos\theta_0}\right)}, \quad (108)$$

$$a = \frac{F_t(\theta)}{m_P}. \quad (109)$$

7 The Dynamics of the Particle on a Curve with Friction

The friction force reads

$$\mathbf{F}_f = -\mu\frac{\mathbf{v}}{|\mathbf{v}|}|\mathbf{N}| = -\mu\frac{\mathbf{v}}{|\mathbf{v}|}\sqrt{N_n^2 + N_b^2}, \quad (110)$$

where $\mathbf{v}$ is the velocity of the particle, N_n and N_b being the components of the normal reaction onto the directions of the normal and bi-normal.

From the moving equations

$$m\dot{v} = F_t - \mu(\mathrm{sign}\, v)\sqrt{N_n^2+N_b^2},\ \frac{mv^2}{\rho} = F_n + N_n,$$
$$0 = F_b + N_b, \quad (111)$$

we get

$$m\dot{v} = F_t - \mu(\mathrm{sign}\, v)\sqrt{\left(\frac{mv^2}{\rho} - F_n\right)^2 + F_b^2}. \quad (112)$$

8 Applications

Application 5. Let us consider a heavy ring that moves with friction on a circle of radius R situated in a vertical plan. At the time $t = 0$, $\theta = 0$, $v = v_0$ (fig. 6).

We have

$$F_t = -mg\sin\theta,\ F_n = -mg\cos\theta. \tag{113}$$

The second relation (111) offers us

$$\frac{mv^2}{R} = -mg\cos\theta + N, \tag{114}$$

so that the first relation (111) leads us to

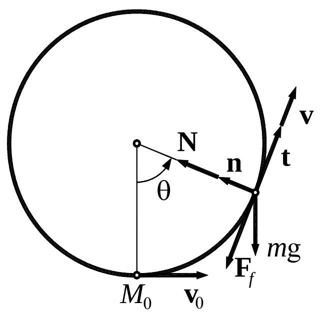

Fig. 6. Application 5.

$$m\dot{v} = -mg\sin\theta - \mu\left(\frac{mv^2}{R} + mg\cos\theta\right). \tag{115}$$

Keeping into account that $v = R\dot{\theta}$, we can write

$$\dot{v} = \frac{\mathrm{d}v}{\mathrm{d}\theta}\dot{\theta} = \frac{1}{2R}\frac{\mathrm{d}\left(v^2\right)}{\mathrm{d}\theta}, \tag{116}$$

wherefrom

$$\frac{\mathrm{d}\left(v^2\right)}{\mathrm{d}\theta} + 2\mu v^2 = -2Rg\left(\sin\theta + \mu\cos\theta\right). \tag{117}$$

The homogenous differential equation reads

$$\frac{\mathrm{d}\left(v^2\right)}{\mathrm{d}\theta} + 2\mu v^2 = 0, \tag{118}$$

and its solution is

$$v_h^2 = Ce^{-2\mu\theta}, \tag{119}$$

where C is a constant of integration which depends on the initial conditions.

For the non-homogenous differential equation we are looking for a solution in the form

$$v_n^2 = A\cos\theta + B\sin\theta, \tag{120}$$

wherefrom

$$\frac{\mathrm{d}\left(v_n^2\right)}{\mathrm{d}\theta} = -A\sin\theta + B\cos\theta. \tag{121}$$

Replacing in the equation (117) and equating the correspondent terms in $\sin$ and $\cos$, we obtain a linear system of two equations with two unknowns,

$$-A + 2\mu B = -2Rg,\ b + 2\mu A = -2\mu Rg. \tag{122}$$

The solution of this system is

$$A = \frac{2Rg}{1 + 4\mu^2}\left(1 - 2\mu^2\right),\ B = -\frac{6\mu Rg}{1 + 4\mu^2}. \tag{123}$$

The solution of the differential equation (117) writes now

$$v^2 = Ce^{-2\mu\theta} + \\ + \frac{2Rg}{1 + 4\mu^2}\left[-3\mu\sin\theta + \left(1 - 2\mu^2\right)\cos\theta\right]. \tag{124}$$

Keeping into account that at $t = 0$, $\theta = 0$, $v^2 = v_0^2$, we obtain the value of the constant C,

$$C = v_0^2 - \frac{2Rg}{1 + 4\mu^2}\left(1 - 2\mu^2\right), \tag{125}$$

and therefore

$$v^2 = \left[v_0^2 - \frac{2Rg}{1 + 4\mu^2}\left(1 - 2\mu^2\right)\right]e^{-2\mu\theta} + \\ + \frac{2Rg}{1 + 4\mu^2}\left[-3\mu\sin\theta + \left(1 - 2\mu^2\right)\cos\theta\right]. \tag{126}$$

The particle stops at the angle θ^* when $v^2 = 0$. The motion continues in the opposite direction if and only if at the angle θ^* the relation

$$\left|mg\sin\theta^*\right| \le \left|mg\cos\theta^*\right| \tag{127}$$

is not fulfilled, that is $\left|\tan\theta^*\right| > \mu$, or, if we denote by φ the friction angle $\left|\tan\theta^*\right| > \tan\varphi$.

Let us consider that

$$v_0^2 = \frac{2Rg}{1 + 4\mu^2}\left(1 - 2\mu^2\right). \tag{128}$$

In this situation the solution (126) of the differential equation (117) takes the form

$$v^2 = \frac{2Rg}{1 + 4\mu^2}\left[-3\mu\sin\theta + \left(1 - 2\mu^2\right)\cos\theta\right]. \tag{129}$$

The particle stops at the angle θ^* given by

$$\tan\theta^* = \frac{1 - 2\mu^2}{3\mu}, \tag{130}$$

and the motion is still possible if and only if

$$\frac{1 - 2\mu^2}{3\mu} > \mu, \tag{131}$$

that is

$$\mu < \frac{1}{\sqrt{5}}. \tag{132}$$

Let us remark that the solution (129) is valid only for the time between 0 and the time when the particle stops. It would be wrong to consider that the particle has an oscillatory motion.

If the motion is without friction, that is $\mu = 0$, the differential equation of the motion (117) becomes

$$\frac{\mathrm{d}(v^2)^2}{\mathrm{d}\theta} = -2Rg\sin\theta, \tag{133}$$

with the solution

$$v^2 = v_0^2 - 2Rg(1 - \cos\theta). \tag{134}$$

From the relation (114) we get the value of the normal reaction,

$$N = \frac{mv^2}{R} + mg\cos\theta, \tag{135}$$

where v^2 is given by the expression (126).

If the link is unilateral, then there exists another condition for the motion, that is $N > 0$.

For the particular case defined by the relation (128), the normal reaction can be negative only if the stop angle θ^* is greater than or equal to $\frac{\pi}{2}$. This condition implies

$$1 - 2\mu^2 \le 0 \tag{136}$$

and therefore

$$\mu \ge \frac{1}{\sqrt{2}}. \tag{137}$$

From the relation (130) we obtain

$$\cos\theta^* = -\frac{1}{\sqrt{1 + \tan^2\theta^*}} = -\frac{3\mu}{\sqrt{1 + 4\mu^4 + 5\mu^2}}, \tag{138}$$

$$\sin\theta^* = \frac{\tan\theta^*}{\sqrt{1 + \tan^2\theta^*}} = \frac{1 - 2\mu^2}{\sqrt{1 + 4\mu^4 + 5\mu^2}}. \tag{139}$$

The velocity is now

$$v^2 = -\frac{12Rg}{1 + 4\mu^2}\frac{1 - 2\mu^2}{\sqrt{1 + 4\mu^4 + 5\mu^2}}, \tag{140}$$

and the normal reaction takes the value

$$N = -\frac{3\mu mg}{1 + 4\mu^2}\frac{5 - 4\mu^2}{\sqrt{1 + 4\mu^4 + 5\mu^2}}. \tag{141}$$

The normal reaction is negative if

$$5 - 4\mu^2 \le 0, \tag{142}$$

that is

$$\mu \ge \frac{\sqrt{5}}{2}. \tag{143}$$

Application 6. Let us study the motion with friction of a heavy material point acted by its own weight and by the force

$$\mathbf{F} = -\xi\mathbf{r} \tag{144}$$

on the cylindrical helix by equations

$$x = R\cos\theta,\ y = R\sin\theta,\ z = R\theta\,\mathrm{tg}\,\alpha. \tag{145}$$

The element of the arc is given by

$$\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta = \frac{R}{\cos\alpha}\mathrm{d}\theta. \tag{146}$$

The unit vector tangent to the helix reads

$$\mathbf{t} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = -\sin\theta\cos\alpha\mathbf{i} + \cos\theta\cos\alpha\mathbf{j} + \sin\alpha\mathbf{k}. \tag{147}$$

From the first Frenet relation,

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \frac{\mathbf{n}}{\rho}, \tag{148}$$

one obtains the unit vector of the normal

$$\mathbf{n} = -\cos\theta\mathbf{i} - \sin\theta\mathbf{j}, \tag{149}$$

and the curvature radius

$$\rho = \frac{R}{\cos^2\alpha}. \tag{150}$$

Expressing the unit vector $\mathbf{b}$ of the bi-normal by the relation

$$\mathbf{b} = \mathbf{t}\times\mathbf{n}, \tag{151}$$

it follows

$$\mathbf{b} = \sin\theta\sin\alpha\mathbf{i} - \cos\theta\sin\alpha\mathbf{j} + \cos\alpha\mathbf{k}. \tag{152}$$

The force $\mathbf{F}$ that acts the particle is

$$\mathbf{F} = -\xi R\cos\theta\mathbf{i} - \xi R\sin\theta\mathbf{j} + (mg - \xi R\theta\,\mathrm{tg}\,\alpha)\mathbf{k} \tag{153}$$

and therefore its components onto the directions of the unit vectors $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ are

$$F_t = \mathbf{F}\cdot\mathbf{t} = \sin\alpha(-\xi R\theta\,\mathrm{tg}\,\alpha + mg), \tag{154}$$

$$F_n = \mathbf{F}\cdot\mathbf{n} = \xi R, \tag{155}$$

$$F_b = \mathbf{F}\cdot\mathbf{b} = \cos\alpha(-\xi R\theta\,\mathrm{tg}\,\alpha + mg). \tag{156}$$

Writing the velocity as

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{R}{\cos\alpha}\dot{\theta}, \tag{157}$$

the relation (112) becomes

$$\frac{mR\ddot{\theta}}{\cos\alpha} = \sin\alpha(-\xi R\theta\,\mathrm{tg}\,\alpha + mg) - \mu\sqrt{(mR\dot{\theta}^2 - \xi R)^2 + \cos^2\alpha(-\xi R\theta\,\mathrm{tg}\,\alpha + mg)^2}. \tag{158}$$

If the motion is without friction, that is $\mu = 0$, the last differential equation reads

$$\frac{mR\ddot{\theta}}{\cos\alpha} = \sin\alpha(-\xi R\theta\,\mathrm{tg}\,\alpha + mg) \tag{159}$$

or, equivalently,

$$mR\ddot{\theta} = -\xi R\theta \sin^2 \alpha + mg \sin \alpha \cos \alpha . \quad (160)$$

The solution is

$$\theta = C \cos\left(t\sqrt{\frac{\xi}{m}} \sin \alpha + \varphi \right) + \frac{mg}{\xi R} \cos \alpha , \quad (161)$$

where C and φ are two constants of integration which depends on the initial conditions.

In addition, in the case without friction, the force $\mathbf{F}$ is a conservative one because it admits the force potential

$$U_1 = -\frac{\xi}{2}\left(x^2 + y^2 + z^2\right). \quad (162)$$

But the weight is also a conservative function, and from the above arguments it follows that in the case without friction, the mechanical energy has a constant value.

The particle oscillates in the vicinity of the equilibrium position, this position being determined by the minimum of the potential energy. In this point the velocity of the particle has a maximum value in modulus.

If the motion is with friction, then the relation (158) offers us the solution only for the motion between the time 0 and the time when the particle stops. This time is given by the condition

$$\dot{\theta} = 0 \quad (163)$$

and from here we obtain a value θ^* of the parameter θ when the particle stops.

The motion can continue if and only if the position given by θ^* is not a static equilibrium position.

9 Conclusions

In our paper we described a new method for the mechanics of the particle on a curve. The most important thing is that the curve is described in a parametric mode. In the static, the co-ordinates frame is not necessary a tri-orthogonal one, the only thing that is asked is to have the unit vectors of the axes linear independent. In the dynamics the system is selected to be the intrinsic system. We obtained more than the speed of the particle; we also obtained the acceleration of the particle. For the small oscillations around the equilibrium position, the method gives us the period of the motion. In the paper we also presented applications for which the solutions are very difficult in the classical way. The approach is unitary and the theory can be applied to a large class of problems.

From our presentation the reader can easily observe that the most difficult problems appear in the mechanics of the particle with friction. In this situation, there exist, generally speaking, infinite equilibrium positions which define zones of equilibrium on the curve. The possibility of the motion is constrained by the differential equation of the motion, by the physical constraints (the positive value of the normal reaction, for instance) and by the positions where the velocity of the particle becomes zero (if the velocity is zero in a point of static equilibrium, then the particle stays at rest for ever in that point).

References:

[1] M. Zahran, In Orbit Performance of LEO Satellite Electrical Power Subsystem – SW Package for Modelling and Simulation Based on MatLab. 7 GUI, *Proceedings of the 2006 IASME/WSEAS International Conference on Energy & Environmental Systems,* Chalkida, Greece, May, 8-10, 2006, pp. 379-384.

[2] J. A. Lopez, M. Barreda, J. Artes, Integration Algorithms to Construct Semi-analytical Planetary Theories, *Proceedings of the 9th WSEAS International Conference on Applied Mathematics,* Istanbul, Turkey, May, 27-29, 2006, pp. 134-139.

[3] S. Ebetiuc, H. Staab, Applying Differential Geometry to Kinematic Modeling in Mobile Robotics, *Proceedings of the 2005 WSEAS International Conference on Dynamical Systems and Control,* Venice, Italy, November, 2-4, 2005, pp. 106-112.

[4] N. Sidek, N. Sarkar, Dynamic Modeling and Control of Nonholonomic Mobile Robot with Lateral Slip, *Proceedings of the 7th WSEAS International Conference on Signal Processing, Robotics and Automation (ISPRA '08),* University of Cambridge, UK, February, 20-28, 2008.

[5] N.–D. Stănescu, Şt. Tabacu, On the Mechanics of the Particle on a Curve without Friction, *Proceedings of the American Conference on Applied Mathematics (American-Math '09),* Houston, USA, April, 30, May, 2, 2009.

[6] A. H. Nayfeh, D. T. Mook, *Nonlinear oscillations*, John Wiley & Sons, New York, Chichester, Brisbane, Toronto, 1979.

[7] D. K. Arrowsmith, C. M. Place, *An Introduction to Dynamical Systems,* Cambridge University Press, 1994.

[8] H. W. Broer, B. Krauskopf, G. Vegter (eds.), *Global Analysis of Dynamical Systems*, Institute of Physics Publishing, Bristol and Philadelphia, 2001.

[9] J. Guckenheimer, P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcation of Vector Fields,* Springer–Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, Hong Kong, 1983.

[10] L. Perko, *Differential Equations and Dynamical Systems,* Springer–Verlag, New York, Berlin, Heidelberg, 1996.

[11] N.–D. Stănescu, *Numerical methods*, Didactical and Pedagogical Publishing House, Bucharest, 2007.

[12] N. Minorsky, *Nonlinear Oscillations*, Van Nostrand, Princeton, 1962.

[13] C. Hayashi, *Forced Oscillations in Nonlinear Systems*, Nippon, Osaka, 1953.

[14] C. Hayashi, *Nonlinear Oscillations in Physical Systems*, McGraw–Hill, New York, 1964.

[15] N.–D. Stănescu, L. Munteanu, V. Chiroiu, N. Pandrea, *Dynamical systems. Theory and applications*, vol. 1, Romanian Academy Publishing House, Bucharest, 2007.

[16] N.–D. Stănescu, L. Munteanu, V. Chiroiu, N. Pandrea, *Dynamical systems. Theory and applications*, vol. 2, Romanian Academy Publishing House, Bucharest, 2009.

[17] N. Pandrea, N.–D. Stănescu, *Mechanics*, Didactical and Pedagogical Publishing House, Bucharest, 2002.