Creep Analysis of Thin Rotating Disc under Plane Stress with no Edge load

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Abstract: - Creep stresses and strain rates for a transversely isotropic and isotropic materials have been obtained for a thin rotating disc using Seth's transition theory. Results obtained have been discussed numerically and depicted graphically. It is seen that a disc made of transversely isotropic material rotating with higher angular speed increases the possibility of fracture at the bore as compared to a disc made of isotropic material and possibility of fracture further decreases with the increase in measure N. The deformation is significant for transversely isotropic disc for the measure N =7.

Key words: Creep, Transition, Stresses, Strain Rates, Transversely Isotropic and Rotating Disc.

Nomenclature of Symbols:

a and b : Internal and external radii of disc
D : a constant
R : Radial distance
x, y, z : Cartesian co-ordinates
u, v, w : Displacement components
r, θ, z : Polar co-ordinates
e_{ij} and T_{ij} : Strain and stress tensor
e_{ii} : First strain invariant
β : Function of *r* only
P : Function of β only
C_{ij} : Material constants
λ *and* μ : Lame's constants
$R = (r / b), \quad R_0 = (a / b)$
$\sigma_r = T_{rr} / C_{66}$ - Radial stress components
$\sigma_\theta = T_{\theta\theta} / C_{66}$ -Circumferential stress components

1. Introduction

This paper is concerned with the analysis of a transversely isotropic thin rotating disk. There are many applications of such type of rotating disks, such as in turbines, rotors and with the advent of computers, disk drives. Naturally, with all these applications and interest, there has been much research in this field. The analytical procedures presently available are restricted to problems with simplest configurations. The use of rotating disk in machinery and structural applications has generated considerable interest in many problems in domain of solid mechanics. Solutions for thin isotropic disks can be found in most of standard creep text books [1-4]. Han [5] has investigated elastic and plastic stresses for isotropic materials with variable thickness. Wang [6] has investigated deformation of elastic half rings. Enescu[7] give some numerical methods for determining stresses in rolling bearings while Mahri[8] calculated the stresses by using finite element method for wind turbine rotors. Wahl [9] has investigated creep deformation in rotating disks assuming small deformation, incompressibility condition, Tresca's yield criterion, its associated flow rule and a power strain law. Transition theory [10-11] does not require any assumptions like a yield criterion, incompressibility condition and thus poses and solves a more general problem from which cases pertaining to above assumption can be worked out. Transition theory uses the concept of generalized principal strain measure which simplifies the constitutive equations by prescribing a priori the order of the measure of deformation and helps to achieve better agreement between the theoretical and experimental results. It has been shown that for the uni-axial case a generalized measure, which includes all the known measures of Cauchy, Green, Almansi, Hencky and others, is given by [12]. The generalized principal strain measure [12] is defined as

$$e_{ii} = \int_0^{e_{ii}^A} \left[1-2e_{ii}^A\right]^{\frac{n}{2}-1} de_{ii}^A = \frac{1}{n}\left[1-\left(1-2e_{ii}^A\right)^{\frac{n}{2}}\right] \qquad (1)$$

where n is the measure and e_{ij}^A is the principal Almansi finite strain components [13]. This theory is applied to a large number of elastic-plastic and creep problems [14-20]. In this paper an attempt has been made to study the behaviour of transversely isotropic thin rotating disc using transition theory [10-11].

2. Governing Equations

We consider a thin disc of constant density made of transversely isotropic material with internal and external radii 'a' and 'b' respectively. The disc is rotating with angular velocity 'ω' about an axis perpendicular to its plane and passing through the centre of the disc. The disc is thin and is effectively in a state of plane stress.

The displacement components in cylindrical polar co-ordinates are given [13] by,

$$u = r(1-\beta), \; v = 0, \; w = dz, \qquad (2)$$

where β is a function of $r = \sqrt{x^2+y^2}$ only and d is a constant.

The finite components of strain [13] are

$$\begin{aligned} e_{rr}^A &= \frac{1}{2}[1-(r\beta'+\beta)^2], \\ e_{\theta\theta}^A &= \frac{1}{2}[1-\beta^2], \\ e_{zz}^A &= \frac{1}{2}[1-(1-d)^2], \\ e_{r\theta}^A &= e_{\theta z}^A = e_{zr}^A = 0 \end{aligned} \qquad (3)$$

where $\beta' = d\beta/dr$

Substituting (3) in (1), the generalized components of strain are

$$e_{rr} = \frac{1}{n}[1-(r\beta'+\beta)^n], \; e_{\theta\theta} = \frac{1}{n}[1-\beta^n]$$

$$e_{zz} = \frac{1}{n}[1-(1-d)^n], e_{r\theta} = e_{\theta z} = e_{zr} = 0 \qquad (4)$$

The stress-strain relations for transversely isotropic material are

$$T_{rr} = C_{11}e_{rr} + (C_{11}-2C_{66})e_{\theta\theta} + C_{13}e_{zz}$$

$$T_{\theta\theta} = (C_{11}-2C_{66})e_{rr} + C_{11}e_{\theta\theta} + C_{13}e_{zz}$$

$$T_{zz} = C_{13}e_{rr} + C_{13}e_{\theta\theta} + C_{33}e_{zz} = 0$$

$$e_{zz} = -\frac{C_{13}}{C_{33}}e_{rr} - \frac{C_{13}}{C_{33}}e_{\theta\theta}$$

$$T_{zr} = T_{\theta z} = T_{r\theta} = 0 \qquad (5)$$

where C_{ij} are material constants.

Using equation (4) in equation (5) we have

$$T_{rr} = \frac{A}{n}[2-\beta^n(1+(1+P)^n)] - \frac{2C_{66}}{n}(1-\beta^n)$$

$$T_{\theta\theta} = \frac{A}{n}[2-\beta^n(1+(1+P)^n)] - \frac{2C_{66}}{n}\left[1-\beta^n(1+P)^n\right]$$

$$T_{r\theta} = T_{\theta z} = T_{zr} = T_{zz} = 0 \qquad (6)$$

where $A = \left(C_{11} - C_{13}^2/C_{33}\right)$.

The equations of equilibrium are all satisfied except

$$\frac{d}{dr}(rT_{rr}) - T_{\theta\theta} + \rho\omega^2 r^2 = 0 \qquad (7)$$

where ρ is the density of the material.

Using equation (6) in equation (7), we get a non-linear differential equation in β as,

$$\beta^{n+1}P(1+P)^{n-1} = \left[\frac{\rho\omega^2 r^2}{A} + \beta^n\left\{\frac{2C_{66}}{nA}(1+nP-(1+P)^n) - P(1+(1+P)^n)\right\}\right]\frac{d\beta}{dP} \qquad (8)$$

where $r\beta' = \beta P$.

The transition points of β in equation (8) are $P \to -1$ *and* $P \to \pm\infty$.

The boundary conditions are

$$T_{rr} = 0 \text{ at } r = a \text{ and } r = b \qquad (9)$$

3. Solution through Principal Stress Difference

It has been shown that the asymptotic solution through the principal stress difference [13-14, 16, 18-19] at the transition point $P \to -1$ leads to creep state. The transition function R is defined as

$$R = T_{rr} - T_{\theta\theta} = \beta^n \frac{2C_{66}}{n}[1-(P+1)^n] \qquad (10)$$

Taking logarithmic differentiation of above equation, we have

$$\frac{d}{dr}\left(\log R\right)=\frac{2C_{66}}{Rr}\begin{bmatrix}\beta^n P-P\beta^n(P+1)^n\\ -P\beta^{n+1}(P+1)^{n-1}\dfrac{dP}{d\beta}\end{bmatrix} \quad (11)$$

Substituting the value of $dP/d\beta$ from equation (8) in equation (11) and taking asymptotic value $P\rightarrow -1$, we get

$$\frac{d}{dr}\left(\log R\right)=\frac{1}{r}\left[-2n-\frac{n\rho r^2\omega^2}{A\beta^n}+C_2(n-1)\right] (12)$$

where $C_2=2C_{66}/A$

Asymptotic value of β as $P\rightarrow -1$ is $D/r, D$ being a constant.

Integrating with respect to r, we have

$$R=A_1 r^g \exp(f) \quad (13)$$

where A_1 is a constant of integration and

$$f=-\frac{n\rho\omega^2 r^{n+2}}{AD^n(n+2)};\ g=-2n+C_2(n-1)$$

Substituting equation (13) in equation (7), we get

$$T_{rr}=A_2-A_1\int r^{g-1}\exp(f)\ dr-\frac{\rho r^2\omega^2}{2} \quad (14)$$

where A_2 is constant of integration.

Using boundary conditions (9) in equation (14), we get

$$A_1=\frac{\dfrac{\rho w^2}{2}\left(a^2-b^2\right)}{\int\limits_{r=a}^{b} r^{g-1}\exp(f)\ dr}$$

$$A_2=\frac{\rho\omega^2}{2}(a^2-b^2)\frac{\left\{\int\limits_{r=b} r^{g-1}\exp(f)\ dr\right\}}{\left\{\int\limits_{r=a}^{b} r^{g-1}\exp(f)\ dr\right\}}+\frac{\rho b^2\omega^2}{2}$$

Substituting values of A_1 and A_2 in equation (14), we get

$$T_{rr}=\frac{\dfrac{\rho\omega^2}{2}(a^2-b^2)}{\left\{\int\limits_a^b r^{g-1}\exp(f)\ dr\right\}}\left\{\int\limits_r^b r^{g-1}\exp(f)\ dr\right\} \quad (15)$$

$$+\frac{\rho\omega^2}{2}\left(b^2-r^2\right)$$

From equation (15) and equation (10), we get

$$T_{\theta\theta}=T_{rr}-\frac{\dfrac{\rho\omega^2}{2}\left(a^2-b^2\right)}{\int\limits_a^b r^{g-1}\exp(f)\ \ dr} r^g\ \exp(f) \quad (16)$$

Now we introduce the following non-dimensional quantities

$$R=\frac{r}{b}\ ,\ R_0=\frac{a}{b}\ ,\ \Omega^2=\frac{\rho\omega^2 b^2}{C_{66}}\ ,\ \sigma_r=\frac{T_{rr}}{C_{66}}\ ,$$

$$\sigma_\theta=\frac{T_{\theta\theta}}{C_{66}}$$

Using these in equations (14) and (15), we get

$$\sigma_r=\frac{\Omega^2\left(R_0^2-1\right)}{2\int\limits_{R_0}^{1} R^{g-1}\exp f_1 dR}\int\limits_R^1 R^{g-1}\exp f_1\ dR+\frac{\Omega^2\left(1-R^2\right)}{2} \quad (17)$$

$$\sigma_\theta=\sigma_r\ -\ \frac{\Omega^2\left(R_0^2-1\right)\left(R^g\ \exp f_1\right)}{2\int\limits_{R_0}^{1} R^{g-1}\exp f_1 dR} \quad (18)$$

where $f_1=-\dfrac{n\Omega^2\left(\dfrac{b}{D}\right)^n R^{n+2}}{(n+2)\dfrac{C_{11}}{C_{66}}\left(1-\dfrac{C_{13}^2}{C_{11}C_{33}}\right)}$

For an isotropic material, equation (17) and (18) becomes

$$\sigma_r=\frac{\Omega^2\left(R_0^2-1\right)}{2\int\limits_{R_0}^{1} R^{g_1-1}\exp f_2 dR}\int\limits_R^1 R^{g_1-1}\exp f_2\ dR+\frac{\Omega^2\left(1-R^2\right)}{2} \quad (19)$$

$$\sigma_\theta=\sigma_r\ -\ \frac{\Omega^2\left(R_0^2-1\right)\left(R^{g_1}\ \exp f_2\right)}{2\int\limits_{R_0}^{1} R^{g_1-1}\exp f_2 dR} \quad (20)$$

where $g_1=-2n+\dfrac{\left(n\left(3-2C\right)+1\right)}{\left(2-C\right)}$

$$f_2=-\frac{n\Omega^2 b^n R^{n+2}}{2\left(2-C\right)\left(n+2\right)D^n}$$

For a disc made of incompressible material, i.e. (C $\rightarrow$0), the stresses given by equations (19) and (20), we get

$$\sigma_r = \frac{\Omega^2\left(R_0^2-1\right)}{2\int_{R_0}^{1} R^{g_2-1}\exp f_3 \, dR}\int_R^1 R^{g_2-1}\exp f_3 dR + \frac{\Omega^2\left(1-R^2\right)}{2} \tag{21}$$

$$\sigma_\theta = \sigma_r - \frac{\Omega^2\left(R_0^2-1\right)\left(R^{g_2}\exp f_3\right)}{2\int_{R_0}^{1} R^{g_2-1}\exp f_3 dR} \tag{22}$$

where $g_2 = -\frac{(3n+1)}{2}, \quad f_3 = -\frac{n\Omega^2 b^n R^{n+2}}{4(n+2)D^n}$

These equations are same as obtained by Gupta et.al [16].

4. Strain Rates

The stress-strain relation can be written as,

$$e_{ij} = \left(\frac{A-2C_{66}}{H}\right)\delta_{ij}\Theta - \frac{2\left(A-C_{66}\right)}{H}T_{ij} \tag{23}$$

$$\Theta = T_{11}+T_{22}+T_{33}, \quad H = 4C_{66}\left(C_{66}-A\right),$$

where $A = C_{11} - \frac{C_{13}^2}{C_{33}}$

When the creep sets in, strain should be replaced by strain rates. The stress-strain relation (23) becomes

$$\dot{e}_{ij} = \left(\frac{A-2C_{66}}{H}\right)\delta_{ij}\Theta - \frac{2\left(A-C_{66}\right)}{H}T_{ij} \tag{24}$$

where $\dot{e}_{ij}$ is the strain rate tensor with respect to flow parameter t.

Differentiating second equation of (4) with respect to t, we get

$$\dot{e}_{\theta\theta} = -\beta^{n-1}\dot{\beta} \tag{25}$$

For Swainger measure (n=1), we have from equation (25)

$$\dot{\varepsilon}_{\theta\theta} = -\dot{\beta} \tag{26}$$

From equation (10), the transitional value of β is,

$$\beta = \left[\frac{n}{2C_{66}}\left(T_{rr} - T_{\theta\theta}\right)\right]^{\frac{1}{n}} \tag{27}$$

Using (25), (26) and (27) in equation (24), we get

$$\dot{\varepsilon}_{rr} = \chi\left\{\frac{\alpha\sigma_\theta - \gamma\sigma_r}{\eta}\right\} \tag{28}$$

$$\dot{\varepsilon}_{\theta\theta} = \chi\left\{\frac{\alpha\sigma_\theta - \gamma\sigma_r}{\eta}\right\} \tag{29}$$

$$\dot{\varepsilon}_{zz} = \chi\alpha\left\{\frac{\sigma_r + \gamma\sigma_\theta}{\eta}\right\} \tag{30}$$

where $\chi = \left[\frac{n}{2}(\sigma_r - \sigma_\theta)\right]^{\frac{1}{n}-1}$, $\gamma = 1 - \frac{C_{13}^2}{C_{11}C_{33}}$

$$\alpha = \left(1 - \frac{C_{13}^2}{C_{11}C_{33}} - 2\frac{C_{66}}{C_{11}}\right), \; \eta = 4\left(\frac{C_{66}}{C_{11}} - 1 + \frac{C_{13}^2}{C_{11}C_{33}}\right)$$

For isotropic material, equations (28), (29) and (30) becomes

$$\dot{\varepsilon}_{rr} = \frac{\chi}{2}\left(\frac{2-C}{3-2C}\right)\left[\sigma_r - \left(\frac{1-C}{2-C}\right)\sigma_\theta\right] \tag{31}$$

$$\dot{\varepsilon}_{\theta\theta} = \frac{\chi}{2}\left(\frac{2-C}{3-2C}\right)\left[\sigma_\theta - \left(\frac{1-C}{2-C}\right)\sigma_r\right] \tag{32}$$

$$\dot{\varepsilon}_{zz} = -\frac{\chi}{2}\left(\frac{2-C}{3-2C}\right)\left[\left(\frac{1-C}{2-C}\right)\left(\sigma_r + \sigma_\theta\right)\right] \tag{33}$$

These equations are same as obtained by Odquist [2] provided we put $n = 1/N$,

For incompressible materials, i.e. $C \to 0$, equations (31), (32), (33) becomes

$$\dot{\varepsilon}_{rr} = \frac{\chi}{3}\left[\sigma_r - \frac{\sigma_\theta}{2}\right] \tag{34}$$

$$\dot{\varepsilon}_{\theta\theta} = \frac{\chi}{3}\left[\sigma_\theta - \frac{\sigma_r}{2}\right] \tag{35}$$

$$\dot{\varepsilon}_{zz} = \frac{\chi}{6}\left[\sigma_\theta + \sigma_r\right] \tag{36}$$

Numerical Illustration and Discussion

For calculating the stresses and strain rate distribution based on the above analysis, the following values of measure n and angular velocity Ω^2 have been taken as:

$$\Omega^2 = 0.1, 1, 5, 10$$

$$n = 1, 1/3 \textit{ and } 1/7 \; (i.e.\; N = 1, 3 \textit{ and } 7)$$

Elastic constants C_{ij} for transversely isotropic material (Magnesium) and isotropic material (Brass) have been given in table 1.

Table 1: Elastic Constants C_{ij} (in terms of $10^{10}\, N/m^2$)

	C_{44}	C_{11}	C_{12}	C_{13}	C_{33}
TIM (Mg)	1.64	5.97	2.62	2.17	6.17
IM (Brass)	1.0	3.0	1.0	1.0	3.0

Curves have been drawn in figures 1 – 4 between stresses and radii ratio R= r/b for a disc rotating with different angular speed Ω^2 and for measure N = 3, 5 and 7.

For a disc made of transversely isotropic material rotating with angular speed Ω^2 = 0.1, circumferential stress is maximum at the internal surface for measure n = 1/3 (N = 3) as compared to disc made of isotropic material and this value of circumferential stress decreases at the internal surface with increase in measure (i.e., N = 5, 7). The circumferential stress increases at internal surface with increase in angular velocity (Ω^2 = 1, 5, 10). It means that a disc made of transversely isotropic material rotating with higher angular speed increases the possibility of fracture at the bore as compared to a disc made of isotropic material. Possibility of fracture decreases with the increase in measure N. It can be concluded that a rotating disc made of isotropic material for measure N = 7 is on the safer side of the design in comparison to a disc made of transversely isotropic material.

In figures 5–8, curves have been drawn between strain rates and radii ratio (R = r/b) at angular speed Ω^2 and measure N = 3, 5, 7. It has been observed that a transversely isotropic disc experiences a significant deformation for the measure N = 7 as compared to a rotating disc made of isotropic material.

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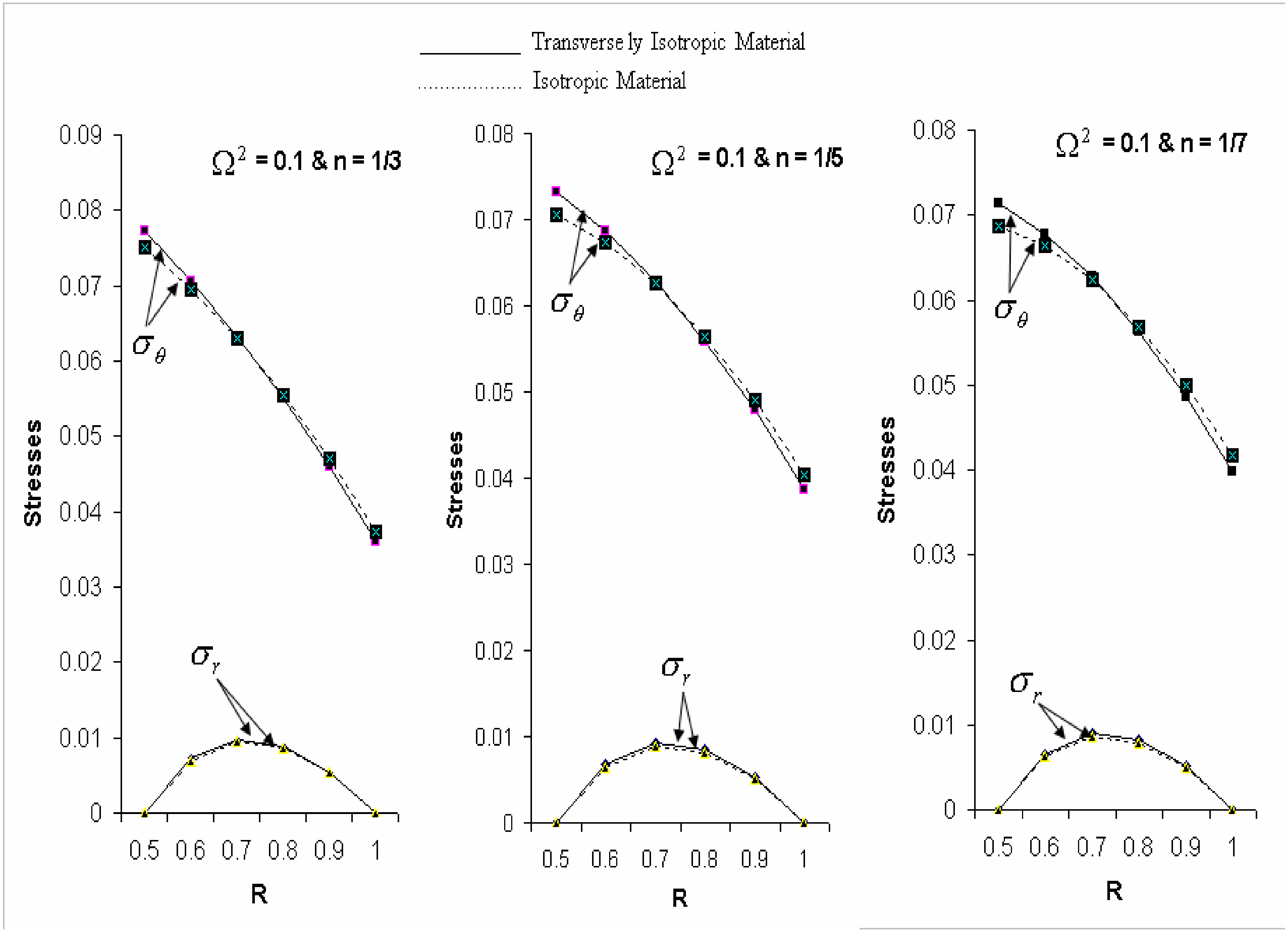


Figure 1: Creep Stresses in a Thin Rotating Disc along the Radius R for Angular Speed $\Omega^2 = 0.1$

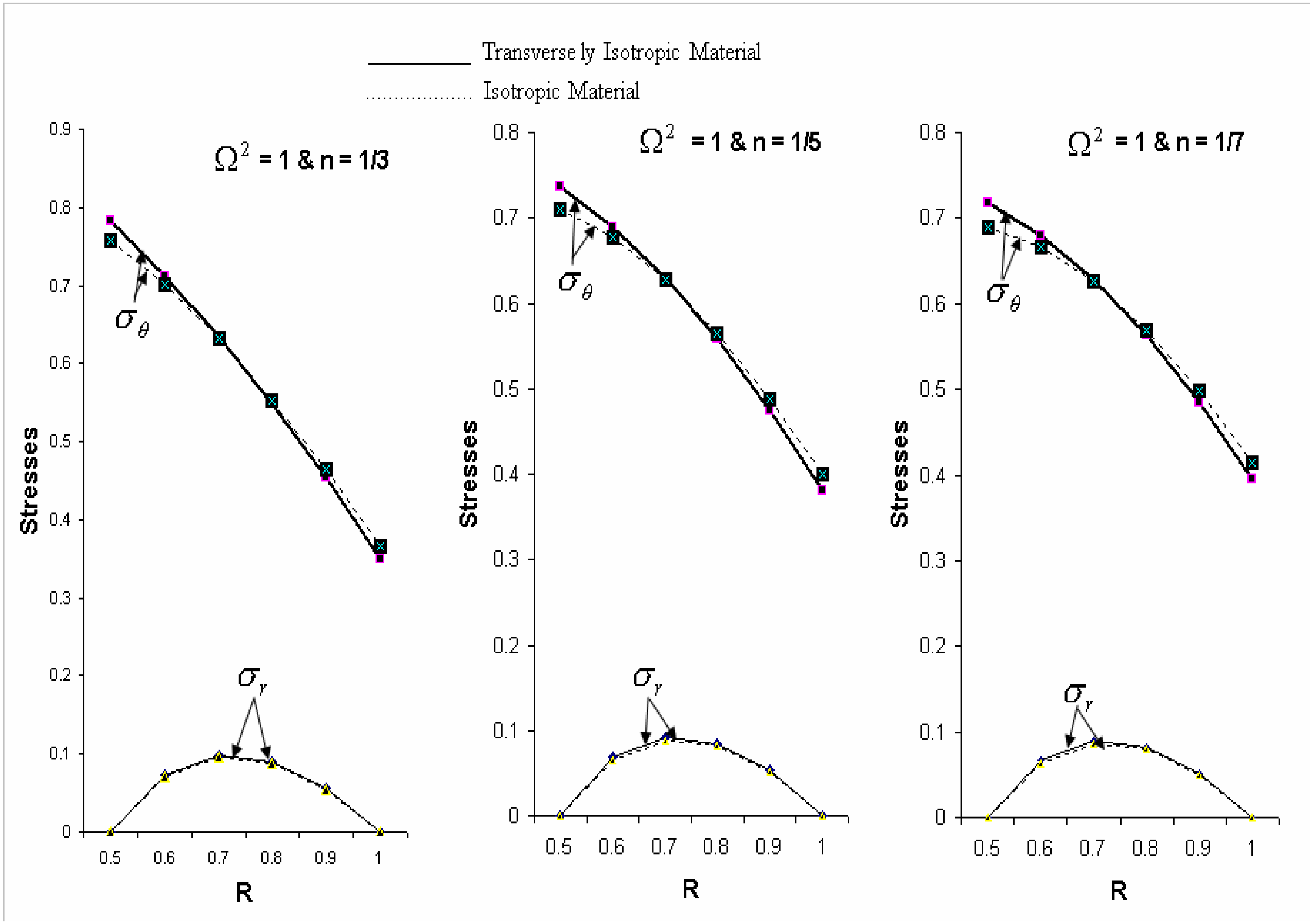


Figure 2: Creep Stresses in a Thin Rotating Disc along the Radius R for Angular Speed $\Omega^2 = 1$

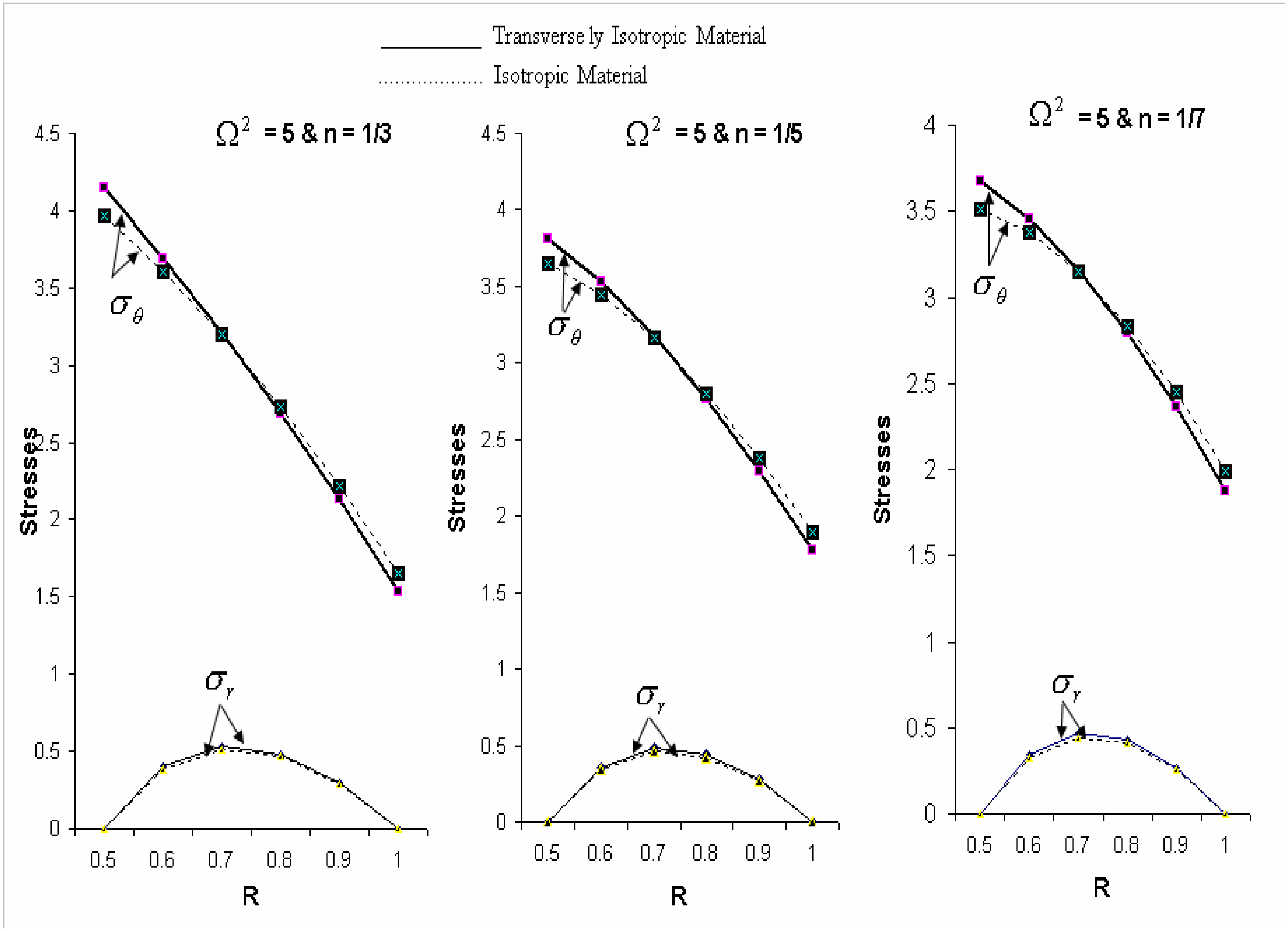
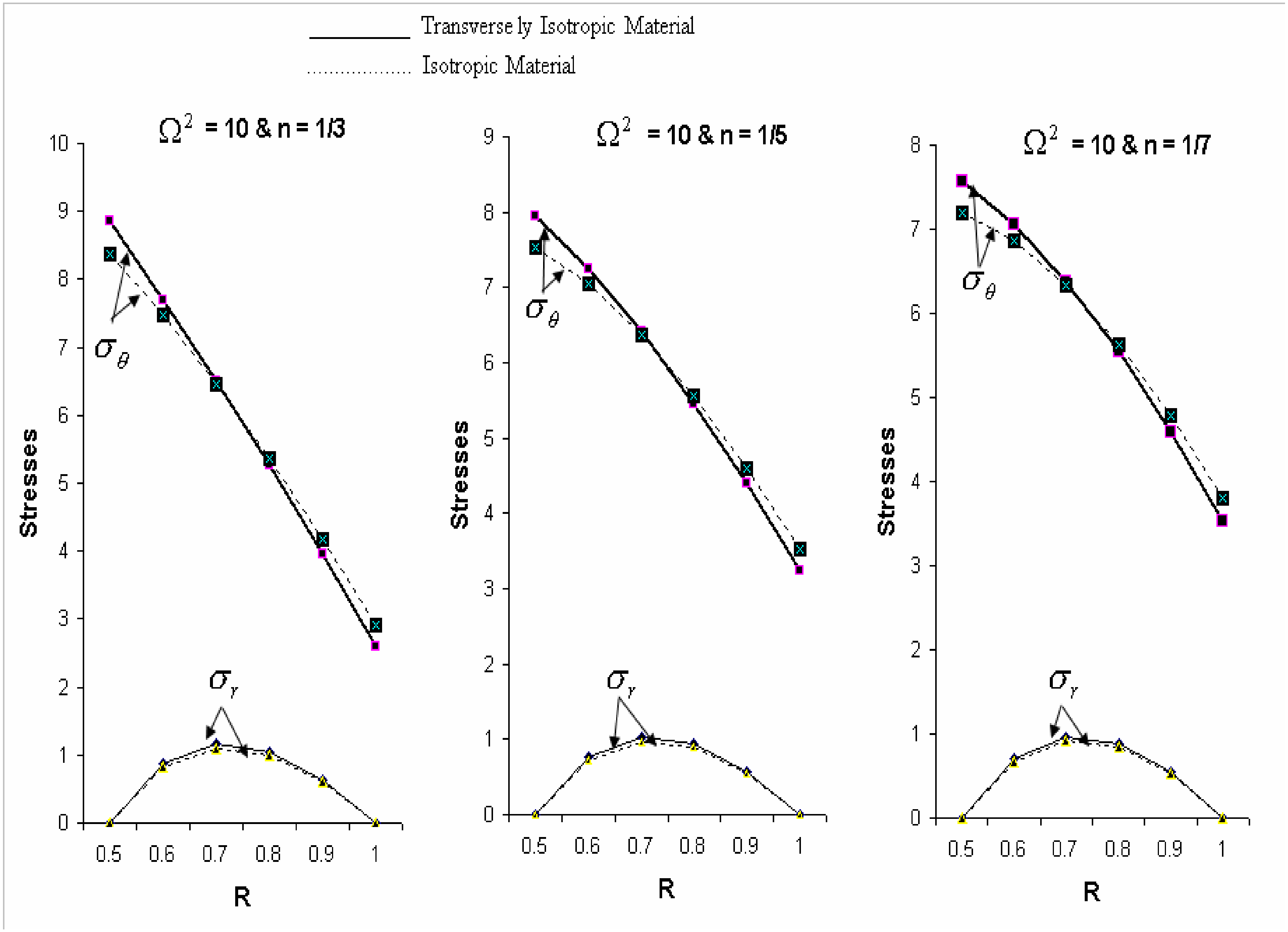


Figure 3: Creep Stresses in a Thin Rotating Disc along the Radius R for Angular Speed $\Omega^2 = 5$

Figure 4: Creep Stresses in a Thin Rotating Disc along the Radius R for Angular Speed $\Omega^2 = 10$



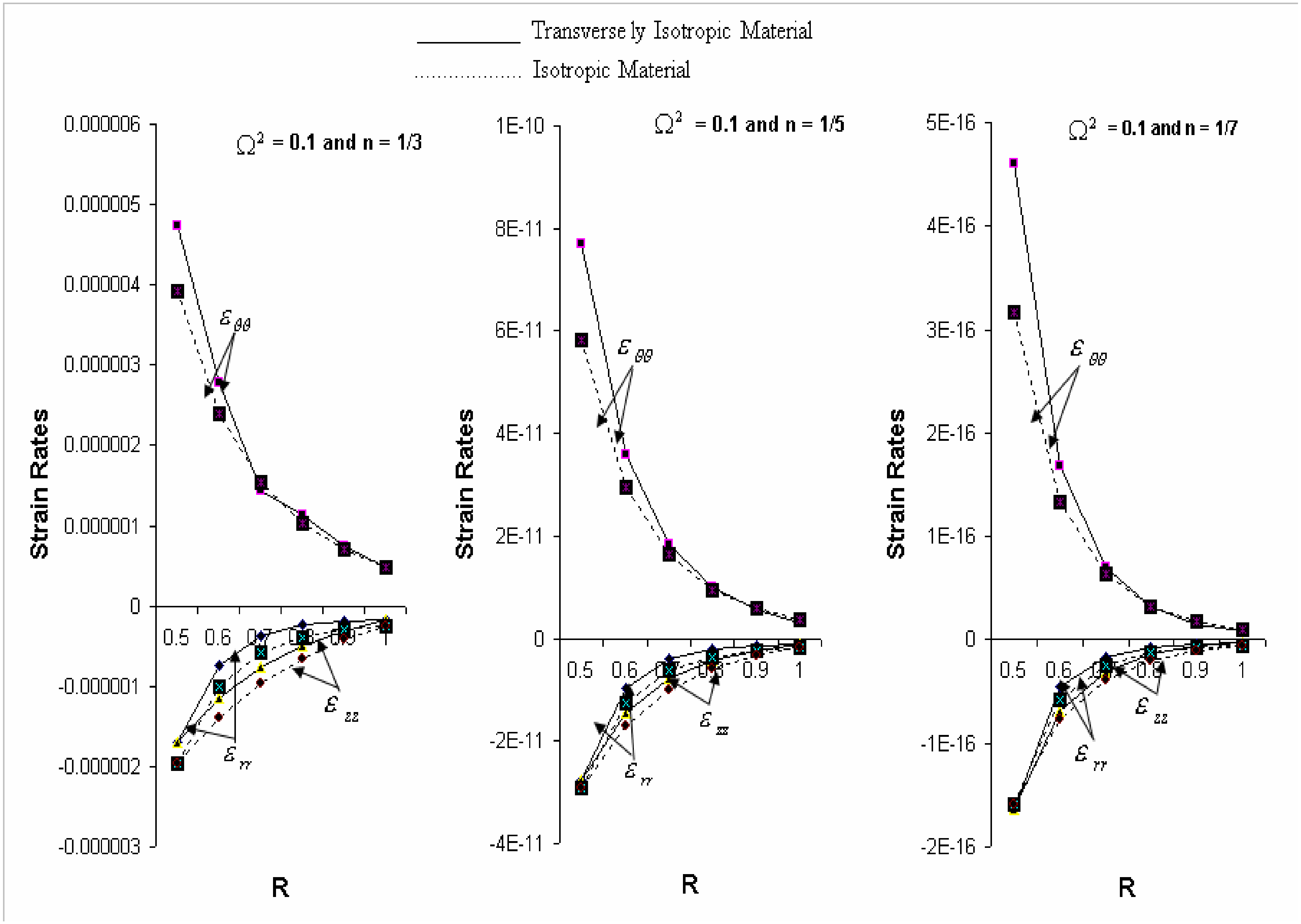


Figure 5: Strain Rates in a Thin Rotating Disc along the Radius R for Angular Speed $\Omega^2 = 0.1$

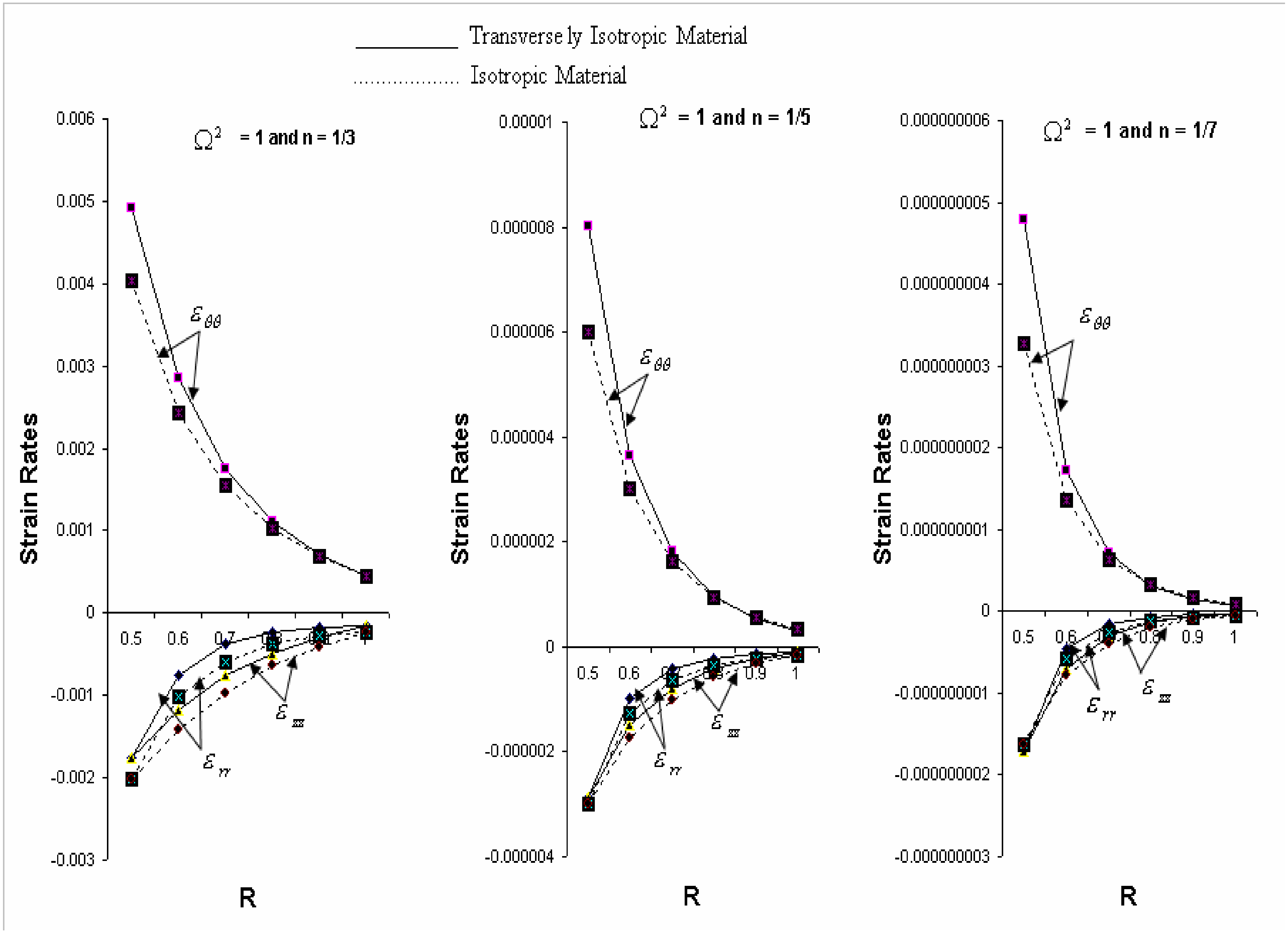


Figure 6: Strain Rates in a Thin Rotating Disc along the Radius R for Angular Speed $\Omega^2 = 1$

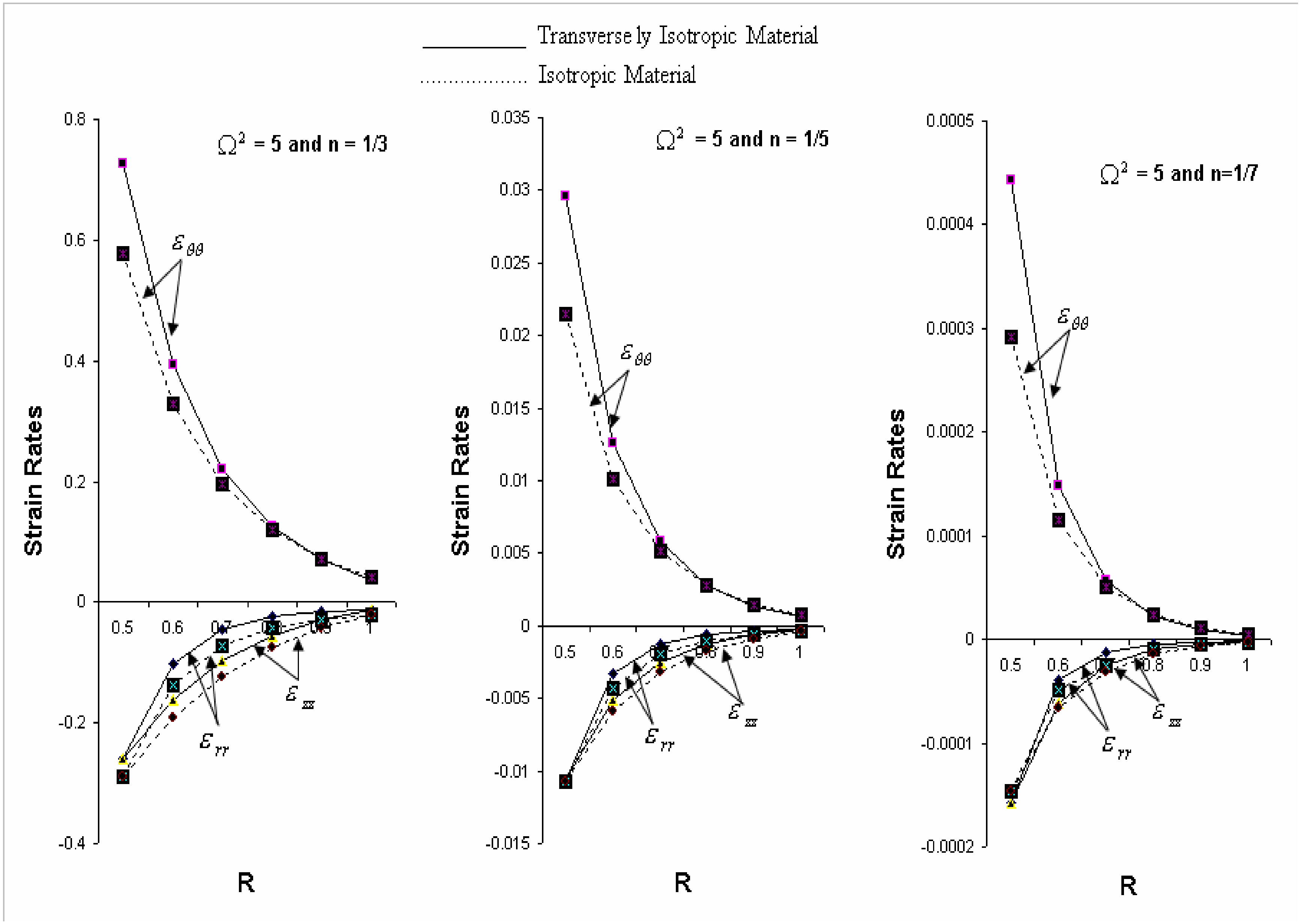


Figure 7: Strain Rates in a Thin Rotating Disc along the Radius R for Angular Speed $\Omega^2 = 5$

Figure 8: Strain Rates in a Thin Rotating Disc along the Radius R for Angular Speed $\Omega^2 = 10$