Bending beams, a computer-aided approach

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Abstract: This paper presents a new general computer-aided (CA) way for optimization of bending beams that is an example of using the CA capabilities to obtain a large quantity of data to receive new qualitative information. Our method is obtained combining two new results based on the CA approach: (1) the computer-aided method of obtaining the influence coefficients for bending a beam and (2) the State Matrix Strategy- a quasi-optimization tool. The CA method of obtaining the influence coefficients stands for any statically determined or undetermined straight beam of a constant cross section under all the combinations of loading and boundary conditions. The extension to a non-constant cross section is easy to obtain. The adopted model for the bending beam is an n-lumped beam unrelated to how big is *n*. The flexibility of this mathematical model synergistically completed by the Mathematica® software symbolic calculus capabilities, allows us to determine the values of the design parameters that optimize the elasto-dynamic behavior of bending beams, according to predefined criteria, static and dynamic as well.

Key-Words: bending, lumped beam, influence coefficients, computer-aided optimization, Mathematica®.

1 Introduction

The interest for bending beams is self-evident: the bending beams are everywhere in our inside and/or outside vicinity, in constructions and industry, as well (fig. 1).

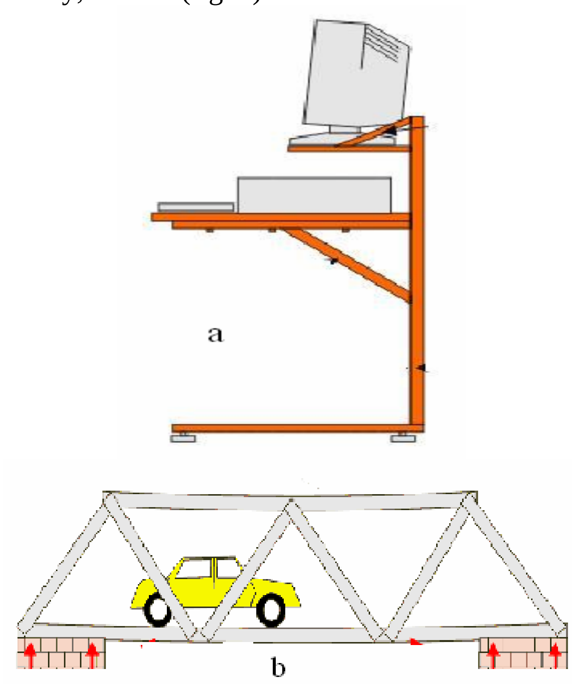

Fig.1 The bending beams around us.

My interest in the bending beams starts with a work on dynamics of the multi-bodies branched systems described as “a number of branches with a common driver” (fig. 2) where the main camshaft driver works as a bending beam with a number of forces acting on [4]. The lumped beam model gives a suitable description of the bending of the main camshaft driver. For a real system with n branches, we need (n+1)-lumped masses: one mass for each ramification and an additional one to describe the motor.

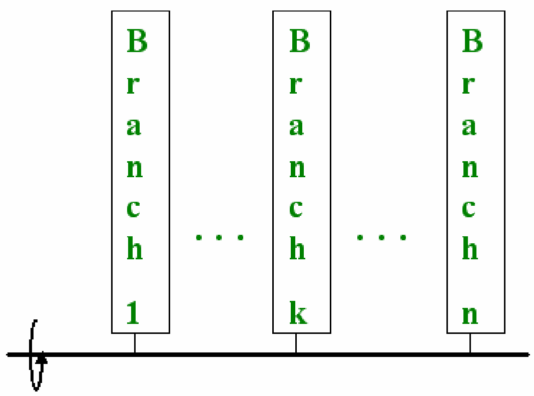

Fig.2 The main camshaft driver works as a bending beam.

Since the study of the bending behavior of a lumped beam uses the influence coefficients [2], for a real mechanism with n branches, a (n+1) square influence coefficients matrix is to be known and this is a problem. It is well known that the specialty literature

[3] offers computing methods for only a very small number, n, of concentrated masses and not for any type of boundary conditions. Thus, finding a way to compute influence coefficients matrix for a lumped beam with any finite number, n, of concentrated masses and in any boundary conditions, appears to be very challenging.

In this paper, our response to this challenge is presented. We chose to do this by using a computer-aided (CA) method starting from the initial parameters method [1, 3, 4 and n2].

2 Flexibility influence coefficients

In the literature, the concept of *influence coefficients* denotes both the stiffness influence coefficients and the flexibility influence coefficients, which are intimately related - they describe the manner in which the mechanical system deforms under the forces. We deal only with the *flexibility influence coefficients*, which will be named, *the influence coefficients* [6, 8, n2].

To define the influence coefficients, let us consider a simple discrete system, consisting of n masses m_i occupying the position $x_i, i=\overline{1,n}$ and being in equilibrium.

Forces F_j act upon each mass m_i (this can be assumed without loosing generality) so that the masses undergo displacements z_i. Thus, the flexibility influence coefficient e_{ij} *is the displacement of the point* x_i *due to a unit force,* $F_j = 1$*, applied at* x_j.

$$e_{ij}=z_i\big|_{F_j=1, F_k=0, k\neq j} \quad i, j=\overline{1,n}. \tag{1}$$

Note that the flexibility influence coefficients e_{ij} have the appropriate units corresponding to the type of loading: [LF] for torsion, and [LF^{-1}] for forces. For a linear system, using the principle of superposition, the flexibility influence coefficients e_{ij} allow to obtain the displacement at the point x_i due to all the forces F_j ($j=\overline{1,p}$) acting on the system, as:

$$z_i = e_{ij}F_j, \quad i, j=\overline{1,p}. \tag{2}$$

In (2) and everywhere else in text, the repeated index stands for summation.

3 Bending beam – equations, solution

In the classical notations from the bending beams theory [7]:

$T = T(x)$: the shear force;

$M = M(x)$: the bending moment;

$z = z(x)$: the elastic deflection;

$\theta = \theta(x) = \dfrac{dz}{dx}$: the slope (of the elastic deflection).

the generally accepted sign convention is shown in Fig.3. The moments were considered to be positive in a clockwise sense and the external forces in the descendent sense of the vertical axis.

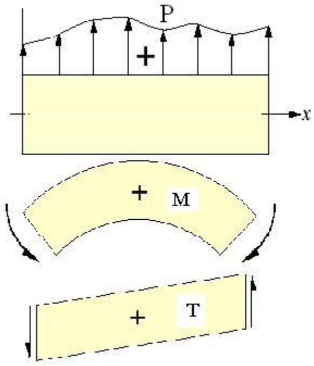

Fig.3 Sign convention

For a straight beam of constant cross section, hold the following equations between deflection, shear force, and bending moment:

$$f\ \frac{\mathrm{d}^4 z}{\mathrm{d}x^4} = \frac{1}{EI}, \tag{3}$$

$$\frac{\mathrm{d}^3 z}{\mathrm{d}x^3} = \frac{1}{EI}T, \tag{4}$$

$$\frac{\mathrm{d}M}{\mathrm{d}x} = T. \tag{5}$$

where

- E is Young's (elastic) module,
- I is the second moment of area (area moment of inertia or second moment of inertia), of the constant cross section; I must be calculated with respect to the neutral axis (the centroidal axis perpendicular to the applied loading),
- f is an externally applied load.

Eqs. (3) with the appropriate boundary conditions allows us to obtain the deflection $z = z(x)$ for any given external loading. The boundary conditions are to be written for each range: between any pair of externally concentrated forces/moments and for each portions of the beam on which the distributed forces are applied. The relationships (4) and (5) serve as continuity conditions.

Looking for the homogenous solution of (3) of the form

$$z(x) = A\frac{x^3}{3!} + B\frac{x^2}{2!} + Cx + D,$$

the slope, the bending moment and the shear force are:

$$\theta(x) = A\frac{x^2}{2!} + Bx + C,$$

$$M(x) = -EI(Ax + B), \quad T(x) = -EI\,A.$$

Denoting by index 0 the values at the point $x = 0$ (one of the ends of the beam), the values at $x = 0$ of the bending moment, the shear force, the elastic deflection and the slope are:

$$M_0 = M(0),\ T_0 = T(0), z_0 = z(0), \theta_0 = \theta(0),$$

we used these values to obtain the integration constants as:

$$A = -\frac{1}{EI}T_0, B = -\frac{1}{EI}M_0,\ C = \theta_0,\ D = z_0.$$

Therefore, the homogenous solution of (2) is:

$$z_H(x) = -\frac{1}{EI}T_0\frac{x^3}{3!} - \frac{1}{EI}M_0\frac{x^2}{2!} + \theta_0 x + z_0. \quad (6)$$

Since the relationship (6) connects the current values of the deflections $z(x)$ to the loading and deflection parameters at the origin ($x = 0$), the method is known as "the initial parameters method".

To obtain a general solution, the particular solutions, typical to each type of external loading, are to be added to the homogenous solution (5). To do this, let us consider the beam of length l ($0 \le x \le l$) shown in Fig.4, under the following four external loading (one single load from each type is considered):

(F1a)- a concentrated force, P

(F1b)- a constant uniform distributed force, p

(F1c)- a distributed force with linear variation,

$$\underline{p}(x-a)(x-b)$$

(F1d)- an external moment, Me.

If the concentrated force P is acting in the section $x = c$, the appropriate particular solution is:

$$z_c(x) = \begin{cases} 0, \text{ if } x < c \\ \dfrac{1}{EI}P\dfrac{(x-c)^3}{3!}, \text{ if } c \le x \end{cases} \quad (7)$$

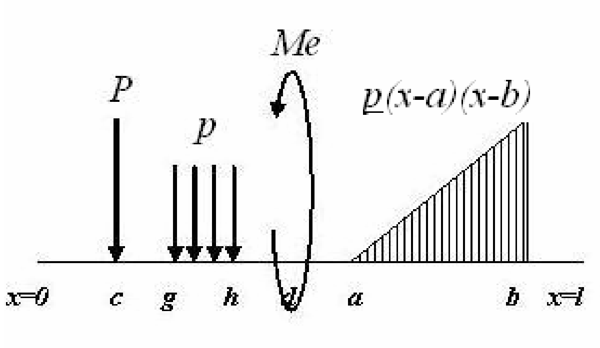

Fig.4 Typical external loading

If the moment Me is acting in the section $x = d$, the appropriate particular solution is:

$$z_d(x) = \begin{cases} 0, \text{ if } x < d \\ -\dfrac{1}{EI}Me\dfrac{(x-d)^2}{2!}, \text{ if } d \le x \end{cases} \quad (8)$$

If the constant distributed force p is acting on the portion $x \in (g,h)$ of the beam section, the appropriate particular solution is:

$$z_g(x) = \begin{cases} 0, \quad \text{if } x < g \\ \dfrac{1}{EI}p\dfrac{(x-g)^4}{4!}, \text{ if } g \le x \le h \\ \dfrac{1}{EI}p\left[\dfrac{(x-g)^4}{4!} - \dfrac{(x-h)^4}{4!}\right], \text{ if } h < x \end{cases} \quad (9)$$

If the distributed force with linear variation, $\underline{p}(x-a)(x-b)$ is acting on the portion $x \in (a,b)$ of the beam section, the appropriate particular solution is:

$$z_a(x) = \begin{cases} 0, \text{ if } x < a \\ \dfrac{1}{EI}\underline{p}\dfrac{(x-a)^5}{5!}, \text{ if } a \le x \le b \\ \underline{p}\dfrac{1}{EI}\dfrac{1}{(b-a)} \times \left[\dfrac{(x-a)^5}{5!} - \dfrac{(x-b)^4}{4!} \times \left(\dfrac{x-b}{5} + 1\right)\right], \text{ if } b < x \end{cases} \quad (10)$$

For this non-homogeneous problem (with

particular solutions (7) – (10)) the general solution is:

$$z(x) = z_c(x) + z_d(x) + z_g(x) + z_a(x)\,. \qquad (11)$$

In the real cases, the loading is a combination of different numbers (some could be zero, too) of loading from each type (F1a) – (F1d). Let us consider the non-homogeneous problem that describes a beam loaded by the following ($n_1 + n_2 + n_3 + n_4$) external forces and/or moments:

(F2a) n_1 concentrated forces, P_i ($i = \overline{1, n_1}$), acting on $x = c_i$;

(F2b) n_2 constant distributed forces, p_i ($i = \overline{1, n_2}$), acting on the section $x \in (g_i, h_i)$ of the beam;

(F2c) n_3 distributed force with linear variation, $\underline{p}_i (x - a_i)(x - b_i)$ $\quad i = \overline{1, n_3}$, acting on the portion $x \in (a_i, b_i)$;

(F2d) n_4 external moments, Me_i ($i = \overline{1, n_4}$), acting on the section $x = d_i$ of the beam.

Using the relations (7) – (10), based on the superposition principle, the general solution is obtained combining the homogenous solution (6) with the particular solutions corresponding to each one of the external forces and/or moments (F2a) – (F2d).

$$\begin{aligned} z(x) = & -\frac{1}{EI} T_0 \frac{x^3}{3!} - \frac{1}{EI} M_0 \frac{x^2}{2!} + \theta_0 x + z_0 \\ & + \frac{1}{EI} \frac{1}{3!} \sum_{i=1}^{n_1} P_i < x - c_i >^3 \\ & + \frac{1}{EI} \frac{1}{4!} \sum_{i=1}^{n_2} p_i (< x - g_i >^4 - < x - h_i >^4) \\ & + \frac{1}{EI} \frac{1}{5!} \sum_{i=1}^{n_3} \underline{p}_i (< x - a_i >^5 - < x - b_i >^5) \\ & - \frac{1}{EI} \frac{1}{4!} \sum_{i=1}^{n_3} \underline{p}_i < x - b_i >^5 \\ & + \frac{1}{EI} \frac{1}{2!} \sum_{i=1}^{n_4} Me_i < x - d_i >^2 , \end{aligned} \qquad (12)$$

where the notation $< x - \alpha >^n$ is defined by:

$$< x - \alpha >^n = \begin{cases} (x - \alpha)^n, & \text{if} \quad x \geq \alpha, \\ 0, & \text{if} \quad x < \alpha. \end{cases} \qquad (13)$$

The relationship (12) establishes the elastic deflection, $z = z(x)$ for a straight beam of constant cross section under the external loading (F2a)-(F2d) with respect to the initial parameters z_0, θ_0, M_0, T_0.

It is important to mention some facts related to the elastic deflection formula (12) which is not sensitive about:
– the number of lumped masses;
– the type and / or number of external loadings;
– the number of intermediate supports;
– the kind of boundary conditions;
– the type of the studied case- statically determined or undetermined (the static determination of the reactions at the supports is not required as a separate step).

The elastic deflection formula (12) only depends on the initial parameters z_0, θ_0, M_0, T_0 that are to be established from the physical and geometrical aspects of each particular problem.
After analyzing all the possible combinations of the end conditions, we concluded that all the combinations of boundary conditions (free end, clamp end or supported end) can be grouped in five different categories described in Appendix 1.

We can use the relationship (12) to obtain the influence coefficients in any point $x_i, i = \overline{1, n}$ on the beam remembering that the flexibility influence coefficients are defined by formula (1) as: "e_{ij} is the displacement of the point x_i due to a unit force applied at x_j".

4 CA way to acquire the flexibility influence coefficients

The results obtained in the former section stay at the base of our CA way to obtain the matrix of the flexibility influence coefficients for a lumped beam with any finite number, n, of concentrated masses, under all the combinations of loading and boundary conditions, unrelated to how big is n.
This method has been created as a routine with the following important procedural steps:

(S1) Problem analysis in order to establish the loading conditions, the boundary conditions (at the ends of the beam), as well as the conditions at the intermediate supports;

(S2) Removal of the intermediate supports and their replacement by the intermediate reactions that will be treated as a part of the loading;

(S3) Computation of the terms appearing necessary in the conditions established for step S1 and writing the conditions. Thus, we obtain a set of algebraic equations in which the unknowns are the values of the initial parameters, z_0, θ_0, M_0, T_0, and the intermediate reactions;

(S4) Solving the equations established in step S3 and obtaining the intermediate reactions in terms of initial parameters;
(S5) Deflection determination in terms of values of the initial parameters (z_0, θ_0, M_0, T_0) from the eq. (12);
(S6) Determination of the shear force and the bending moment in terms of values of the initial parameters (z_0, θ_0, M_0, T_0) using the equations (3) and (4).
(S7) Determination of the matrix of the flexibility influence coefficients from the deflection computed at the step S5 using the equation (1).

Beside the influence coefficients this routine allow us to obtain the deflection, the shear force and the bending moment for both statically determined and undetermined beams with constant section (the extension to variable section beams, is reachable).

5 Use of the flexibility influence coefficients in bending beam design

Based of this routine, a MATHEMATICA®[10] software has been developed to obtain the matrix of the flexibility influence coefficients for a lumped beam with any finite number, n, of concentrated masses, under all the combinations of loading and boundary conditions, unrelated to how big is n. The flowchart is schematically described in Appendix 3.

In the Appendix 2 can be seen the calculated matrix of the flexibility influence coefficients for a 6-lumped masses beam shown in Fig. 5.

The easier way to obtain results stimulated us to extend this software to execute all the computations that are usually performed in the static and dynamic analysis of a bending beam and to verify the static or/and dynamic performance criteria.

So, after obtaining the displacement, the shear force and the bending moment, with appropriate fitting, the shear force and bending moment diagrams can be drawn. This way, all the static performance criteria can be verified. For example, in Appendix 2, the value of the area of the bending moment diagram is given.

Since the flexibility influence coefficients allow us to write the motion equation for the bending beam,

$$m_k \ddot{z}_k + \sum_{j=1}^{n} c_{kj} z_j = F_k, k = 1, n. \quad (14)$$

the dynamic analysis can be made. The stiffness influence coefficients, c_{jk}, $j, k = 1, n$, are intimately related to flexibility influence coefficients, e_{jk}, $j, k = 1, n$. The matrices are inverse one to other: $[c_{jk}] = [e_{jk}]^{-1}$.

Denoting by [**e**] the matrix of flexibility influence coefficients, e_{jk}, $j, k = 1, n$, and by [**m**] the diagonal matrix of the lumped masses $m_k, k = 1, n$, the product matrix [**e**] [**m**] represents the dynamic matrix of the *n*-lumped bending beam. Easy to compute, after the determination of the matrix of flexibility influence coefficients, the dynamic matrix allows obtaining the eigenvalues and the eigenvectors. So, any criteria based on the eigenfrequencies domain or on the gap between them can be also verified- see example in the following section.

Important to note that our MATHEMATICA®[10] software gives us, beside common numerical results, the symbolic ones, too. For example, in Appendix 2 can be seen the expression of the dynamic matrix for a 6-lumped masses beam shown in Fig. 5, depending on: Young's (elastic) module, E, second moment of area matrix, I, section area, AR and density, RO.

It is also important to note that this work can contribute to the engineering education development [13].

6 Study case

Let us consider, as an example, the beam with a fixed end at A, a free end at B, supported in the points b_1, b_2, b_3 and loaded by external concentrated forces F_1, F_6 that act in the points a_1, a_6 as shown in Fig. 5. The positions of the lumped masses m_1- m_6 are a_1-a_6.

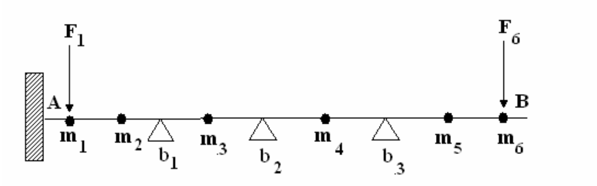

Fig.5 Beam example.

The numerical results computed for the particular bending beam of length 10 and

$\{b_1, b_2, b_3\} = \{2, 5, 8\}$

$\{a_1, a_2, a_3, a_4, a_5, a_6\} = \{1, 1.5, 3, 6, 8.5, 9\}$

are presented in Appendix 2.

These computed results are:

-Eij: the matrix of the flexibility influence coefficients;

-DEF: the dynamic matrix;

-Fr: the eigenfrequency vector.

A qualitative analysis was also done. We checked the sensibility of the eigenfrequencies to variation of b_1, b_2, b_3 (the supports positions) and a_1, a_6 (the external concentrated forces positions). The results are synthesized in Tables 2a and 2b.
From Table 2a it is easy to observe that the minimal eigenfrequency, w_1, is almost insensible to variation of b_1, b_2, a_1, a_6 and it is weakly sensible to variation of b_3.

Table 2a: The sensibility of the minimal eigenfrequency to variation of supports/ forces positions

	w_1	
	Average	STDEV
b_1	1.29	0.001
b_2	1.29	0.023
b_3	1.76	0.496
a_1	1.66	6.11E-05
a_6	1.75	0.455

Table 2b: The sensibility of the maximal eigenfrequency to variation of supports/ forces positions

	w_6	
	Average	STDEV
b_1	10.32	1.84
b_2	10.66	0.073
b_3	10.65	0.0004
a_1	10.78	1.997
a_6	9.30	2.19E-05

From Table 2b is easy to observe that the maximal eigenfrequency, w_6, is insensible to variation of b_2, b_3 and a_6 but it is strongly sensible to variation of b_1 and a_1. The variation of the maximal eigenfrequency, w_6, in respect with b_1 and a_1 are shown in the Fig.6.

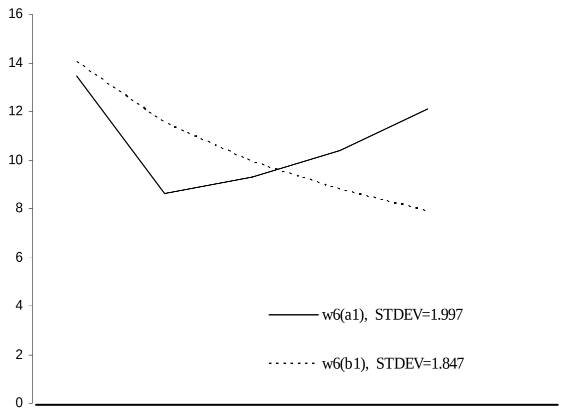

Fig.6 The variation of the maximal eigenfrequency in respect with a_1 and b_1

($a_1 \in [0.1 , 0.9]$, $b_1 \in [1.5 , 2.5]$)

More than this qualitative analysis, we can proceed to optimization.

7 CA quasi-optimization

Our CA quasi-optimization method is an example of using the CA capabilities to obtain a large quantity of data in order to receive new qualitative information.

The word "optimization" is not used in its strict mathematical meaning; for information about optimization in the classical meaning see [5, n3], for example.

In our view, the design optimization can be made comparing and choosing the “best fit” from a large enough set of calculated data. If the quantity of data and comparisons is large enough, we can get the combination of parameters that suits our needs: the fulfillment of predefined criteria.

The easy way to obtain the information allows us to repeat the calculation for a large number of *variants of the model* (the variants differ by design or value of one parameter at least). For each run, the defined measures are to be calculated and stored for further comparison. By measures we mean the output data that have been chosen to be stored. If the results are numerical solutions, a large storage memory is needed. In this case the statistical measures have to be used.

The increased capability of computers to obtain, to save and to compare a large number of different cases synergistically combined with the Mathematica®[10] software symbolic calculus capabilities and the statistical representation of the results, allowed us to developed a tool for design optimization: “The State Matrix Strategy, a quasi-optimization tool”[9].

For this work, several terms and definitions have been introduced.

The *state matrix* is a $N_m \times N_p$ matrix, where: N_m is the number of the chosen measures and N_p is the number of the critical points (the special points in which are to be computed different measures). Since the state matrix components could be: all kinds of numbers, matrix, functions and even messages, the word "matrix" is used here with an extended meaning.

A point in the $N_m \times N_p$ space is named *behavior point*. After each run, a behavior point is achieved.

The *behavior map* is a set of behavior points obtained when all the design parameters cover its utilization domains.

The *utilization domain of a design parameter* is a predefined domain in which the parameter may vary without the state matrix exceeding given admissible limits.

The influence of any parameter on the behavior of the studied model can be observed and analyzed.

Generally, we can consider:

- design changes and/or number of constructive parameters;
- changes in the mass distribution, the damping level, the mean frequency;
- supplementary design conditions;
- changes in the dynamic model;
- any particular parameter relevant for the studied case.

In our terms, to perform the optimization means to define the state matrix component, to create and store the behavior map and to select the solution- the combination of design parameters for which all predefined criteria are fulfilled.

As all the information used for further comparison is stored in the specially created state matrix, it gives the name of the method.

For a bending beam, the computer-aided quasi-optimization strategy follows the following steps:

i. define the *problem* – geometric and mechanic description of the bending beam;
ii. identify *all the parameters* that can change their value; determine their utilization domains; choose the appropriate *design parameters set*;
iii. establish the collection of the input data sets – each set describes a relevant combination of design parameters;
iv. define the *objective* - a set of static and/or dynamic criteria that have to be fulfilled;
v. define the *critical points* appropriate for the problem and for the objective; the results/measures obtained in these points have to be stored;
vi. obtain the *state matrix* that carries the information about each acceptable variant;
vii. obtain the *behavior map* by repeating step (vi.) for each input data set established at (iii.);
viii. get (automatically) the *optimal combination* of values of the design parameters for which the objective is reached**.**

Based on these eight procedural steps, we developed a MATHEMATICA® software.

We exemplify "The State Matrix Strategy, a quasi-optimization tool" in the case of the beam studied in section 6 with $F_1 = F_6 = 2000$N as the external loading.

The optimization problem was: "to find the supports positions (b_1, b_2, b_3) and the external concentrated forces positions (a_1, a_6) that minimize the value of the maximal eigenfrequency".

The definition of the *problem* – geometric and mechanic description of the bending beam- was made in section 6.

As design parameters we choose the positions: b_1, b_2, b_3, a_1, a_6; the utilization domains for these design parameters are:

$a_1 \in [0.1 , 0.9]$
$a_2 \in [9.1 , 9.9]$
$b_1 \in [1.5 , 2.5]$
$b_2 \in [4.5 , 5.5]$
$b_3 \in [7.5 , 8.5]$

The objective was: "to minimize the value of the maximal eigenfrequency**"**.

We found that the minimal value of the maximal eigenfrequency ($w_6 = 9.3$ Hz) is obtained for:

$b_1 = 2$, $b_2 = 5$, $b_3 = 8$, a1= 0.5, $a_6 = 9.5$.

8 Conclusions

This paper presents a CA way for the optimization of straight beams bending (statically determined or undetermined) with a constant cross section under all the combinations of loading and boundary conditions. The extension to a non-constant cross section is easy to obtain.

This CA optimization way was created based on two CA approaches: (1) the computer-aided method of obtaining the influence coefficients for a bending beam and (2) "The State Matrix Strategy- a quasi-optimization tool".

To obtain the influence coefficients, the beam is modeled by n- lumped masses that can be calculated even if the cross section is non-constant. Thus, the method can be used for bending beams with a constant or non-constant cross section. The software deals, at the moment, only with constant cross section beams.

Beside the influence coefficients, the deflection, the shear force and the bending moment are reachable. So, analysis can be done and the static or/and dynamic performance criteria can be verified.

The new terms used by "The State Matrix Strategy- a quasi-optimization tool" are explained here.

As a study case, a beam with a fixed end at A, a free end at B, three intermediate supports and being modeled by 6-lumped masses, is studied. The optimization is performed by determining the location of the intermediate supports and the external concentrated forces that minimize the value of the maximal eigenfrequency.

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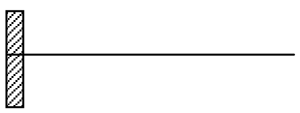
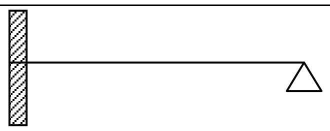
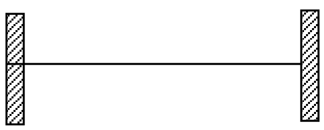
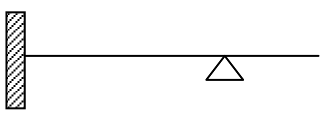
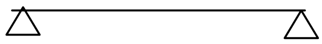
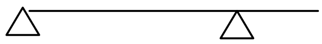
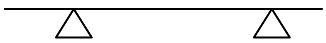
APPENDIX 1

Table 1:The type of the of the initial parameters z_0, θ_0, M_0, T_0

case		*The boundary conditions at*		Type
		A ($x=0$)	B ($x=l$)	
a		$\omega_A=0,\theta_A=0$ $M_A\neq 0,T_A\neq 0$	$\omega_B\neq 0,\theta_B\neq 0$ $M_B=0,T_B=0$	1
b		$\omega_A=0,\theta_A=0$ $M_A\neq 0,T_A\neq 0$	$\omega_B=0,\theta_B\neq 0$ $M_B=0,T_B\neq 0$	1
c		$\omega_A=0,\theta_A=0$ $M_A\neq 0,T_A\neq 0$	$\omega_B=0,\theta_B=0$ $M_B\neq 0,T_B\neq 0$	2
d		$\omega_A=0,\theta_A=0$ $M_A\neq 0,T_A\neq 0$	$\omega_B\neq 0,\theta_B\neq 0$ $M_B=0,T_B=0$	1

e		$\omega_A = 0, \theta_A \neq 0$ $M_A = 0, T_A \neq 0$	$\omega_B = 0, \theta_B \neq 0$ $M_B = 0, T_B \neq 0$	3
f		$\omega_A = 0, \theta_A \neq 0$ $M_A = 0, T_A \neq 0$	$\omega_B \neq 0, \theta_B \neq 0$ $M_B = 0, T_B = 0$	4
g		$\omega_A \neq 0, \theta_A \neq 0$ $M_A = 0, T_A = 0$	$\omega_B \neq 0, \theta_B \neq 0$ $M_B = 0, T_B = 0$	5

APPENDIX 2

Study case-numerical results

The results are obtained with our MATHEMATICA®[10] soft developed to obtain the matrix of the flexibility influence coefficients for a lumped beam with any finite number, n, of concentrated masses, under all the combinations of loading and boundary conditions, unrelated to how big is n.

The input data for this case are: n = 6 , typb=1
- b = {b_1, b_2, b_3}: the supports positions;
- a = {a_1, a_2, a_3, a_4, a_5, a_6}: the lumped masses positions;

```
In[18]:=
        b
Out[18]= {2., 5., 8.}
In[19]:= a
Out[19]= {1, 1.5, 3., 6., 8.5, 9.}
```

The output data:
- [OD1] Eij: the matrix of the flexibility influence coefficients;

```
In[20]:= N[Eij] // MatrixForm
```

Out[20]//MatrixForm=

$$\begin{pmatrix} 3.39382\times10^{-6} & 2.3822\times10^{-6} & -2.22774\times10^{-6} & 6.96168\times10^{-7} & -3.13276\times10^{-7} & -6.26551\times10^{-7} \\ 2.3822\times10^{-6} & 2.35691\times10^{-6} & -2.50621\times10^{-6} & 7.83189\times10^{-7} & -3.52435\times10^{-7} & -7.0487\times10^{-7} \\ -2.22774\times10^{-6} & -2.50621\times10^{-6} & 0.0000111387 & -4.33171\times10^{-6} & 1.94927\times10^{-6} & 3.89854\times10^{-6} \\ 6.96168\times10^{-7} & 7.83189\times10^{-7} & -4.33171\times10^{-6} & 0.0000152229 & -8.07555\times10^{-6} & -0.0000161511 \\ -3.13276\times10^{-7} & -3.52435\times10^{-7} & 1.94927\times10^{-6} & -8.07555\times10^{-6} & 0.0000142018 & 0.0000295523 \\ -6.26551\times10^{-7} & -7.0487\times10^{-7} & 3.89854\times10^{-6} & -0.0000161511 & 0.0000295523 & 0.0000659967 \end{pmatrix}$$

- [OD2] the elastic deflection as function of $x \in [1, 10]$:

```
Deflection[x]
```

Out[40]= $-0.000505912\,x^4 + 0.020475\,x^3 - 0.140646\,x^2 + 0.30735\,x - 0.188968$

- [OD3] the shear force as function of $x \in [1, 10]$;:

In[41]:= `ShearForce[x]`

Out[41]= $\text{ei}\,(0.12285 - 0.0121419\,x)$

- [OD4] the bending moment as function of $x \in [1, 10]$:

In[42]:= `Mom[x]`

Out[42]= $0.12285\,\text{ei}\,x - 0.00607094\,\text{ei}\,x^2$

- [OD5] the aria of the bending moment diagram:

`AriaMom = Integrate[ShearForce[x], {x, 0, 10}]`

Out[44]= $0.621404\,\text{ei}$

- [OD6] the dynamic matrix;

In[21]:= `N[DEF] // MatrixForm`

Out[21]//MatrixForm=

$$\begin{pmatrix} 0.00712702 & 0.00500262 & -0.00467825 & 0.00146195 & -0.000657879 & -0.00131576 \\ 0.00011911 & 0.000117845 & -0.00012531 & 0.0000391595 & -0.0000176218 & -0.0000352435 \\ -0.000334161 & -0.000375931 & 0.0016708 & -0.000649757 & 0.000292391 & 0.000584781 \\ 0.00020885 & 0.000234957 & -0.00129951 & 0.00456686 & -0.00242267 & -0.00484533 \\ -0.0000783189 & -0.0000881088 & 0.000487318 & -0.00201889 & 0.00355046 & 0.00738808 \\ -0.00128443 & -0.00144498 & 0.00799201 & -0.0331098 & 0.0605823 & 0.135293 \end{pmatrix}$$

- [OD7] Fr: the eigenfrequency vector.

In[22]:= `N[fr, 3]`

Out[22]= {1.64, 3.39, 4.12, 5.37, 8.21, 13.6}

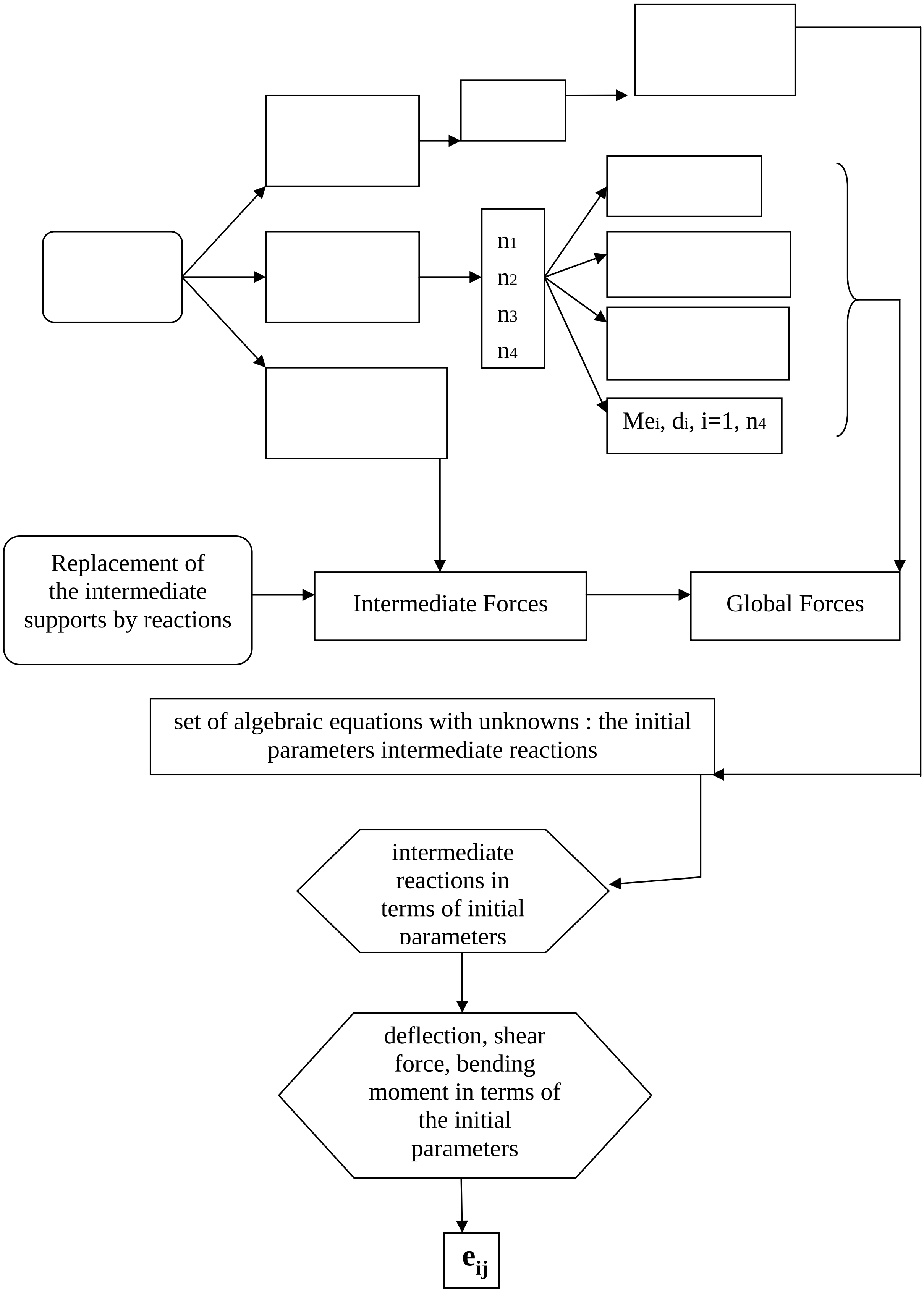
- [OD8] The expression of the dynamic matrix depending on:
  Young's (elastic) module, Ey,
  second moment of area matrix, Iz,
  section area, AR and
  density, RO.

In[39]:= `N[def, 3]`

Out[39]=

$$\begin{pmatrix} \frac{0.0102\,AR\,RO+40.7}{Ey\,Iz} & \frac{0.00843\,AR\,RO+33.7}{Ey\,Iz} & \frac{-0.00719\,AR\,RO-28.8}{Ey\,Iz} & \frac{0.0024\,AR\,RO+9.59}{Ey\,Iz} & \frac{-0.00107\,AR\,RO-4.26}{Ey\,Iz} & \frac{-0.0032\,AR\,RO-12.8}{Ey\,Iz} \\ \frac{0.0169\,AR\,RO}{Ey\,Iz} & \frac{0.0427\,AR\,RO}{Ey\,Iz} & -\frac{0.0431\,AR\,RO}{Ey\,Iz} & \frac{0.0144\,AR\,RO}{Ey\,Iz} & -\frac{0.00639\,AR\,RO}{Ey\,Iz} & -\frac{0.0192\,AR\,RO}{Ey\,Iz} \\ -\frac{0.0288\,AR\,RO}{Ey\,Iz} & -\frac{0.0863\,AR\,RO}{Ey\,Iz} & \frac{0.55\,AR\,RO}{Ey\,Iz} & -\frac{0.23\,AR\,RO}{Ey\,Iz} & \frac{0.102\,AR\,RO}{Ey\,Iz} & \frac{0.307\,AR\,RO}{Ey\,Iz} \\ \frac{0.0144\,AR\,RO}{Ey\,Iz} & \frac{0.0431\,AR\,RO}{Ey\,Iz} & -\frac{0.345\,AR\,RO}{Ey\,Iz} & \frac{1.17\,AR\,RO}{Ey\,Iz} & -\frac{0.614\,AR\,RO}{Ey\,Iz} & -\frac{1.84\,AR\,RO}{Ey\,Iz} \\ -\frac{0.00426\,AR\,RO}{Ey\,Iz} & -\frac{0.0128\,AR\,RO}{Ey\,Iz} & \frac{0.102\,AR\,RO}{Ey\,Iz} & -\frac{0.409\,AR\,RO}{Ey\,Iz} & \frac{0.515\,AR\,RO}{Ey\,Iz} & \frac{1.63\,AR\,RO}{Ey\,Iz} \\ \frac{-0.00639\,AR\,RO-12.8}{Ey\,Iz} & \frac{-0.0192\,AR\,RO-38.4}{Ey\,Iz} & \frac{0.153\,AR\,RO+307.}{Ey\,Iz} & \frac{-0.614\,AR\,RO-1230.}{Ey\,Iz} & \frac{0.814\,AR\,RO+1630.}{Ey\,Iz} & \frac{3.07\,AR\,RO+6140.}{Ey\,Iz} \end{pmatrix}$$

APPENDIX 3


n1
n2
n3
n4
Mei, di, i=1, n4
Replacement of the intermediate supports by reactions
Intermediate Forces
Global Forces
set of algebraic equations with unknowns : the initial parameters intermediate reactions
intermediate reactions in terms of initial parameters
deflection, shear force, bending moment in terms of the initial parameters
eij