Spectral distribution of a star-shaped coupled network

XU GEN QI
Department of Mathematics
Tianjin University
Tianjin 300072
CHINA
NIKOS E. MASTORAKIS
Military Institutes of University Education
Hellenic Naval Academy
Terma Hatzikyriakou, Piraeus, GREECE

Abstract: In this paper we study a star-shaped coupled network of strings and Euler-Bernoulli beams with damping. Suppose that the exterior vertices of the coupled network are clamped and at common node there is a damping. In the present note, our attention concentrate on the spectral distribution of an operator determined by the system. Under certain condition we show that the spectrum of the operator distributes in a strip parallel the imaginary axis.

Key–Words: Star shape network stabilization string beam

1 Introduction

The structure, control and stabilization of the networks are important problem in the real world. There are many papers studying various properties of the networks, for instance, the papers [1], [2],[3], [4] [5], [6] and the references therein. In [7], a model of star-shaped coupled network of strings and Euler-Bernoulli beams with damping is given, in which a significant feature is that the global rotation of the elastic structure does not equal to any one of each branch. In the present paper, we will discuss the spectral distribution of the operator determined by the model in[7].

Let us recall the model introduced in[7]. Let $G = (V, E)$ be a graph, where $V = \{a, a_1, a_2, \cdots, a_n\}$ is the vertex set, $E = \{e_1, e_2, \cdots, e_n\}$ is the edge set. The graph has a common vertex a, the edge e_j connects a and a_j. There is a elastic structure on G, whose motion is governed by the partial equations

$$\begin{array}{l}\begin{pmatrix} \rho_i & 0 \\ 0 & \rho_i \end{pmatrix} \frac{\partial^2}{\partial t^2} \begin{pmatrix} u^i(s,t) \\ w^i(s,t) \end{pmatrix} \\ = \begin{pmatrix} h_i\partial_s^2 & 0 \\ 0 & -EI_i\partial_s^4 \end{pmatrix} \begin{pmatrix} u^i(s,t) \\ w^i(s,t) \end{pmatrix}, \ s \in (0, \ell_j)\end{array}$$

The exterior nodes are clamped,i.e,

$$\begin{pmatrix} u^i(\ell_i,t) \\ w^i(\ell_i,t) \end{pmatrix} = 0, \quad w^i_{ss}(\ell_i,t) = 0, \quad i = 1, 2, \cdots, n.$$

At common node a, the structure satisfy the condition

$$\begin{array}{l} A_i \begin{pmatrix} u^i(0,t) \\ w^i(0,t) \end{pmatrix} = \begin{pmatrix} u(a,t) \\ w(a,t) \end{pmatrix}, \\ \quad s_i w^i_s(0,t) = w_x(a,t), \quad i = 1, 2, \cdots, n, \\ -\sum_{j=1}^{n} (A_j^*)^{-1} \begin{pmatrix} h_i u^i_s(0,t) \\ -EI_i w^i_{sss}(0,t) \end{pmatrix} \\ +\begin{pmatrix} u(a,t) \\ w(a,t) \end{pmatrix} = -\alpha \begin{pmatrix} u_t(a,t) \\ w_t(a,t) \end{pmatrix} \\ -\sum_{j=1}^{n} \frac{EI_i}{s_i} w^i_{ss}(0,t) + w_x(a,t) = -\beta w_{xt}(a,t), \end{array}$$

The initial position and velocity of the network are

$$\begin{array}{l} \begin{pmatrix} u(x,0) \\ w(x,0) \end{pmatrix} = \begin{pmatrix} f_0(x) \\ g_0(x) \end{pmatrix}, \\ \begin{pmatrix} u_t(x,0) \\ w_t(x,0) \end{pmatrix} = \begin{pmatrix} f_1(x) \\ g_1(x) \end{pmatrix}, \quad x \in G, \end{array}$$

where $u(x,t)$ denotes the longitudinal displacement of elastic structure, and $w(x,t)$ denotes the transverse displacement of elastic structure. The functions $u^i(s,t)$ and $w^i(s,t)$ are the realization of $u(x,t)$ and $w(x,t)$ on edge e_j, respectively, i.e.,

$$\begin{array}{l} u^i(s,t) = u(\gamma_i(s),t), \\ w^i(s,t) = w(\gamma_i(s),t), \quad s \in (0, \ell_i), \end{array}$$

here $\gamma_i(s) = x \in e_j$ is the parameterizations mapping; the set $\{s_1, s_2, \cdots, s_n\}$ with $s_j \neq 0$ represents geometrical structure set.

Set

$$H_E^k(G) = \left\{ f \in L^2(G) \;\middle|\; \begin{array}{l} f \in H^k(\gamma_i), \\ f(\ell_i) = 0, \end{array} \; i \in [1,n], \right\}$$

and

$$\mathcal{W} = H_E^1(G) \times H_E^2(G) \times L^2(G) \times L^2(G).$$

Let $s_1, s_2, \cdots, s_n$ be given as in equation (1) with $s_j \neq 0$, and $A_j, j = 1, 2, \cdots, n$, be invertible 2×2 real matrices. Let the state space be

$$\mathcal{H} = \left\{ Y = (u,w,v,z) \in \mathcal{W} \;\middle|\; \begin{array}{l} w_x(a) = s_j w_s^j(0), \\ A_j \begin{pmatrix} u^j(0) \\ w^j(0) \end{pmatrix} \\ = \begin{pmatrix} u(a) \\ w(a) \end{pmatrix}, \\ j = 1, 2, \cdots, n; \end{array} \right\}$$

endowed the norm

$$\begin{aligned} \|Y\|_H^2 \;=\; & \sum_{j=1}^n \int_0^{\ell_j} \{h_j |u_s^j(s)|^2 + EI_j |w_{ss}^j(s)|^2|\} ds \\ & + \sum_{j=1}^n \int_0^{\ell_j} \{\rho_j [|v^j(s)|^2 + |z^j(s)|^2]\} ds \\ & + |u(a)|^2 + |w(a)|^2 + |w_x(a)|^2. \end{aligned} \quad (1)$$

It is easy to check that $\mathcal{H}$ is a Hilbert space.

Define the operator in $\mathcal{H}$ by

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{array}{l} (u,w,v,z) \in H_E^2 \times H_E^4 \times H_E^1 \times H_E^2 \mid \\ w_{ss}^j(\ell_j) = 0, \\ A_j \begin{pmatrix} v^j(0) \\ z^j(0) \end{pmatrix} = A_i \begin{pmatrix} v^i(0) \\ z^i(0) \end{pmatrix}, \\ z_x(a) = s_j z_s^j(0) = s_i z_s^i(0), \forall i,j \\ -\sum\limits_{j=1}^n (A_j^*)^{-1} \begin{pmatrix} h_j u_s^j(0) \\ -EI_j w_{sss}^j(0) \end{pmatrix} \\ + \begin{pmatrix} u(a) \\ w(a) \end{pmatrix} = -\alpha \begin{pmatrix} v(a) \\ z(a) \end{pmatrix} \\ -\sum\limits_{j=1}^n \frac{EI_j}{s_j} w_{ss}^j(0) + w_x(a) = -\beta z_x(a) \end{array} \right\}, \quad (2)$$

$$\mathcal{A}Y = \mathcal{A} \begin{pmatrix} u \\ w \\ v \\ z \end{pmatrix} = \begin{pmatrix} v \\ z \\ \left\{ \frac{h_i}{\rho_i} u_{ss}^i \right\} \\ \left\{ -\frac{EI_i}{\rho_i} w_{ssss}^i \right\} \end{pmatrix}, \; Y \in \mathcal{D}(\mathcal{A}) \quad (3)$$

With the help of these notation we can write the network equation into an evolutionary equation in $\mathcal{H}$

$$\left\{ \begin{array}{l} \frac{dY(t)}{dt} = \mathcal{A}Y(t), \quad t > 0 \\ Y(0) = Y_0 \end{array} \right. \quad (4)$$

where $Y(t) = [u(x,t), w(x,t), u_t(x,t), w_t(x,t)]^\tau$ and $Y_0 = [f_0(x), g_0(x), f_1(x), g_1(x)]^\tau \in \mathcal{H}$.

In [7], we discussed the stability of the system (4). In present paper, we will discuss spectral distribution of operator $\mathcal{A}$. By a detail analysis, we show that the spectrum of A distributes in a strip parallel the imaginary axis.

2 Resolvent and spectral distribution

In this section we investigate the resolvent and spectral distribution of $\mathcal{A}$. From [7] we know that its spectrum composed of all eigenvalues of $\mathcal{A}$. So its resolvent has poles only. We are going to obtain the expression of the resolvent.

Let $\lambda \in \mathbb{C}$. For very $F \in \mathcal{H}$, we consider the resolvent problem $(\lambda I - \mathcal{A})Y = F$, where

$$Y = [u,w,v,z] \in \mathcal{D}(\mathcal{A}), \quad F = [f_1, g_1, f_2, g_2] \in \mathcal{H}$$

i.e.,

$$\left\{ \begin{array}{l} \lambda u(x) - v(x) = f_1(x), \quad x \in G, \\ \lambda w(x) - z(x) = g_1(x), \quad x \in G, \\ \lambda v^j(s) - \frac{h_j u_{ss}^j(s)}{\rho_j} = f_2^j(s), \quad s \in (0, \ell_j), \\ j = 1, 2, \cdots, n, \\ \lambda z^j(s) + \frac{EI_j}{\rho_j} w_{ssss}^j(s) = g_2^j(s), s \in (0, \ell_j), \\ j = 1, 2, \cdots, n, \\ u^j(\ell_j) = w^j(\ell_j) = 0, \; w_{ss}^j(\ell_j) = 0, \\ j = 1, 2, \cdots, n, \\ \begin{pmatrix} u(a) \\ w(a) \end{pmatrix} = A_j \begin{pmatrix} u^j(0) \\ w^j(0) \end{pmatrix}, \\ w_x(a) = s_j w_s^j(0), \quad j = 1, 2, \cdots, n, \\ -\sum\limits_{j=1}^n (A_j^*)^{-1} \begin{pmatrix} h_j u_s^j(0) \\ -EI_j w_{sss}^j(0) \end{pmatrix} \\ + \begin{pmatrix} u(a) \\ w(a) \end{pmatrix} = -\alpha \begin{pmatrix} v(a) \\ z(a) \end{pmatrix}, \\ -\sum\limits_{j=1}^n \frac{EI_j}{s_j} w_{ss}^j(0) + w_x(a) = -\beta z_x(a). \end{array} \right. \quad (5)$$

From the space condition we get $v(a) = \lambda u(a) - f_1(a)$, $z(a) = \lambda w(a) - g_1(a)$, $z_x(a) = \lambda w_x(a) - g_{1,x}(a)$. From the equation, we have

$$\begin{array}{l} v^j(s) = \lambda u^j(s) - f_1^j(s), \\ z^j(s) = \lambda w^j(s) - g_1^j(s), s \in (0, \ell_j) \end{array}$$

$$\begin{pmatrix} v(a) \\ z(a) \end{pmatrix} = A_j \begin{pmatrix} v^j(0) \\ z^j(0) \end{pmatrix} = \lambda \begin{pmatrix} u(a) \\ w(a) \end{pmatrix} + \begin{pmatrix} f_1(a) \\ g_1(a) \end{pmatrix},$$

and

$$z_x(a) = s_j z_s^j(0) = \lambda w_x(a) - g_{1,x}(a).$$

Thus u^j and w^j satisfy the following equations:

$$\begin{cases} \lambda^2 u^j(s) - \frac{h_j u^j_{ss}(s)}{\rho_j} = f^j_2(s) + \lambda f^j_1(s), \\ \lambda^2 w^j(s) + \frac{EI_j}{\rho_j} w^j_{ssss}(s) = g^j_2(s) + \lambda g^j_1(s), \\ s \in (0, \ell_j), j = 1, 2, \cdots, n, \\ u^j(\ell_j) = w^j(\ell_j) = 0,\ w^j_{ss}(\ell_j) = 0, \\ \begin{pmatrix} u(a) \\ w(a) \end{pmatrix} = A_j \begin{pmatrix} u^j(0) \\ w^j(0) \end{pmatrix}, \\ w_x(a) = s_j w^j_s(0), \quad j = 1, 2, \cdots, n, \\ -\sum_{j=1}^n (A^*_j)^{-1} \begin{pmatrix} h_j u^j_s(0) \\ -EI_j w^j_{sss}(0) \end{pmatrix} \\ +(1+\alpha\lambda) \begin{pmatrix} u(a) \\ w(a) \end{pmatrix} = \alpha \begin{pmatrix} f_1(a) \\ g_1(a) \end{pmatrix}, \\ -\sum_{j=1}^n \frac{EI_j}{s_j} w^j_{ss}(0) + (1+\beta\lambda) w_x(a) = \beta g_{1,x}(a) \end{cases} \tag{6}$$

which is equivalent to

$$\begin{cases} u^j_{ss}(s) - \frac{\rho_j}{h_j} \lambda^2 u^j(s) = -\frac{\rho_j}{h_j}[f^j_2(s) + \lambda f^j_1(s)], \\ w^j_{ssss}(s) + \frac{\rho_j}{EI_j} \lambda^2 w^j(s) = \frac{\rho_j}{EI_j}[g^j_2(s) + \lambda g^j_1(s)], \\ u^j(\ell_j) = w^j(\ell_j) = 0,\ w^j_{ss}(\ell_j) = 0, \\ \begin{pmatrix} u(a) \\ w(a) \end{pmatrix} = A_j \begin{pmatrix} u^j(0) \\ w^j(0) \end{pmatrix}, \\ w_x(a) = s_j w^j_s(0), \quad j = 1, 2, \cdots, n, \\ -\sum_{j=1}^n (A^*_j)^{-1} \begin{pmatrix} h_j u^j_s(0) \\ -EI_j w^j_{sss}(0) \end{pmatrix} \\ +(1+\alpha\lambda) \begin{pmatrix} u(a) \\ w(a) \end{pmatrix} = \alpha \begin{pmatrix} f_1(a) \\ g_1(a) \end{pmatrix}, \\ -\sum_{j=1}^n \frac{EI_j}{s_j} w^j_{ss}(0) + (1+\beta\lambda) w_x(a) = \beta g_{1,x}(a) \end{cases} \tag{7}$$

Set

$$\tau_j = \sqrt{\frac{\rho_j}{h_j}}, \quad \omega_j = \sqrt[4]{\frac{\rho_j}{EI_j}},$$

$$\widehat{f^j}(s, \lambda) = \tau^2_j [f^j_2(s) + \lambda f^j_1(s)],$$

$$\widehat{g^j}(s, \lambda) = \omega^4_j [g^j_2(s) + \lambda g^j_1(s)].$$

Then the general solution of differential equation in (7) are given by

$$\begin{cases} u^j(s) = b^j_1 \sinh \lambda\tau_j(\ell_j - s) + b^j_2 \cosh \lambda\tau_j(\ell_j - s) \\ +\frac{1}{\lambda\tau_j} \int_s^{\ell_j} \sinh \lambda\tau_j(s - r) \widehat{f^j}(r, \lambda) dr, \\ w^j(s) = c^j_1 \sinh \sqrt{i\lambda}\omega_j(\ell_j - s) \\ +c^j_2 \sin \sqrt{i\lambda}\omega_j(\ell_j - s) \\ +c^j_3 \cosh \sqrt{i\lambda}\omega_j(\ell_j - s) + c^j_4 \cos \sqrt{i\lambda}\omega_j(\ell_j - s) \\ -\frac{1}{2(\sqrt{i\lambda}\omega)^3} \int_s^{\ell_j} [\sinh \sqrt{i\lambda}\omega_j(s - r) \\ - \sin \sqrt{i\lambda}\omega(s - r)] \widehat{g^j}(r, \lambda) dr. \end{cases} \tag{8}$$

From the boundary conditions $u^j(\ell_j) = w^j(\ell_j) = w^j_{ss}(\ell_j) = 0$, we can deduce that $b^j_2 = 0, c^j_3 = c_4 = 0$. Therefore, we have

$$\begin{cases} u^j(s) = b^j_1 \sinh \lambda\tau_j(\ell_j - s) \\ +\frac{1}{\lambda\tau_j} \int_s^{\ell_j} \sinh \lambda\tau_j(s - r) \widehat{f^j}(r, \lambda) dr, \\ w^j(s) = c^j_1 \sinh \sqrt{i\lambda}\omega_j(\ell_j - s) \\ +c^j_2 \sin \sqrt{i\lambda}\omega_j(\ell_j - s) \\ -\frac{1}{2(\sqrt{i\lambda}\omega)^3} \int_s^{\ell_j} [\sinh \sqrt{i\lambda}\omega_j(s - r) \\ - \sin \sqrt{i\lambda}\omega(s - r)] \widehat{g^j}(r, \lambda) dr. \end{cases} \tag{9}$$

Now we are going to determine the coefficients in the expression of u^j and w^j. Observing that

$$\begin{cases} u^j_s(s) = -\lambda\tau_j b^j_1 \cosh \lambda\tau_j(\ell_j - s) \\ + \int_s^{\ell_j} \cosh \lambda\tau_j(s - r) \widehat{f^j}(r, \lambda) dr, \\ w^j_s(s) = (-\sqrt{i\lambda}\omega_j)[c^j_1 \cosh \sqrt{i\lambda}\omega_j(\ell_j - s) \\ +c^j_2 \cos \sqrt{i\lambda}\omega_j(\ell_j - s)] \\ -\frac{1}{2(\sqrt{i\lambda}\omega)^2} \int_s^{\ell_j} [\cosh \sqrt{i\lambda}\omega_j(s - r) \\ - \cos \sqrt{i\lambda}\omega_j(s - r)] \widehat{g^j}(r, \lambda) dr, \\ w^j_{ss}(s) = (-\sqrt{i\lambda}\omega_j)^2_j [c^j_1 \sinh \sqrt{i\lambda}\omega_j(\ell_j - s) \\ -c^j_2 \sin \sqrt{i\lambda}\omega_j(\ell_j - s)] \\ -\frac{1}{2(\sqrt{i\lambda}\omega)} \int_s^{\ell_j} [\sinh \sqrt{i\lambda}\omega_j(s - r) \\ + \sin \sqrt{i\lambda}\omega_j(s - r)] \widehat{g^j}(r, \lambda) dr, \\ w^j_{sss}(s) = (-\sqrt{i\lambda}\omega_j)^3 [c^j_1 \cosh \sqrt{i\lambda}\omega_j(\ell_j - s) \\ -c^j_2 \cos \sqrt{i\lambda}\omega_j(\ell_j - s)] \\ -\frac{1}{2} \int_s^{\ell_j} [\cosh \sqrt{i\lambda}\omega_j(s - r) \\ + \cos \sqrt{i\lambda}\omega_j(s - r)] \widehat{g^j}(r, \lambda) dr. \end{cases}$$

We have

$$\begin{cases} u^j_s(0) = -\lambda\tau_j b^j_1 \cosh \lambda\tau_j\ell_j + \int_0^{\ell_j} \cosh \lambda\tau_j r \widehat{f^j}(r, \lambda) dr, \\ w^j_s(0) = (-\sqrt{i\lambda}\omega_j)[c^j_1 \cosh \sqrt{i\lambda}\omega_j\ell_j + c^j_2 \cos \sqrt{i\lambda}\omega_j\ell_j] \\ -\frac{1}{2(\sqrt{i\lambda}\omega)^2} \int_0^{\ell_j} [\cosh \sqrt{i\lambda}\omega_j r - \cos \sqrt{i\lambda}\omega_j r] \widehat{g^j}(r, \lambda) dr, \\ w^j_{ss}(0) = (-\sqrt{i\lambda}\omega_j)^2_j [c^j_1 \sinh \sqrt{i\lambda}\omega_j\ell_j - c^j_2 \sin \sqrt{i\lambda}\omega_j\ell_j] \\ +\frac{1}{2(\sqrt{i\lambda}\omega)} \int_0^{\ell_j} [\sinh \sqrt{i\lambda}\omega_j r + \sin \sqrt{i\lambda}\omega_j r] \widehat{g^j}(r, \lambda) dr, \\ w^j_{sss}(0) = (-\sqrt{i\lambda}\omega_j)^3 [c^j_1 \cosh \sqrt{i\lambda}\omega_j\ell_j - c^j_2 \cos \sqrt{i\lambda}\omega_j\ell_j] \\ -\frac{1}{2} \int_0^{\ell_j} [\cosh \sqrt{i\lambda}\omega_j(r + \cos \sqrt{i\lambda}\omega_j r] \widehat{g^j}(r, \lambda) dr. \end{cases} \tag{10}$$

Set

$$\Phi(f)^j(s) = -\frac{1}{\lambda\tau_j} \int_s^{\ell_j} \sinh \lambda\tau_j(s - r) \widehat{f^j}(r, \lambda) dr, \tag{11}$$

$$\Psi(g)^j(s) = \frac{-1}{2(\sqrt{i\lambda}\omega_j)^3} \int_s^{\ell_j} [\sinh \sqrt{i\lambda}\omega_j(s - r) - \sin \sqrt{i\lambda}\omega_j(s - r)] \widehat{g^j}(r, \lambda) dr. \tag{12}$$

And set

$$M_j(\lambda) = \begin{pmatrix} A_j & 0 \\ 0 & s_j \end{pmatrix} \times \begin{pmatrix} \sinh \lambda\tau_j\ell_j & 0 & 0 \\ 0 & \sinh \sqrt{i\lambda}\omega_j\ell_j & \sin \sqrt{i\lambda}\omega_j\ell_j \\ 0 & (-\sqrt{i\lambda}\omega_j)\cosh \sqrt{i\lambda}\omega_j\ell_j & (-\sqrt{i\lambda}\omega_j)\cos \sqrt{i\lambda}\omega_j\ell_j \end{pmatrix} \quad (13)$$

and

$$N_j(\lambda) = \begin{pmatrix} (A_j^*)^{-1} & 0 \\ 0 & s_j^{-1} \end{pmatrix} \times \begin{pmatrix} -\lambda\tau_j h_j \cosh \lambda\tau_j\ell_j & 0 & 0 \\ 0 & -EI_j(-\sqrt{i\lambda}\omega_j)^3 \cosh \sqrt{i\lambda}\omega_j\ell_j & EI_j(-\sqrt{i\lambda}\omega_j)^3 \cos \sqrt{i\lambda}\omega_j\ell_j \\ 0 & EI_j(-\sqrt{i\lambda}\omega_j)^2 \sinh \sqrt{i\lambda}\omega_j\ell_j & -EI_j(-\sqrt{i\lambda}\omega_j)^2 \sin \sqrt{i\lambda}\omega_j\ell_j \end{pmatrix} \quad (14)$$

and

$$Q(\lambda) = \begin{pmatrix} 1+\alpha\lambda & 0 & 0 \\ 0 & 1+\alpha\lambda & 0 \\ 0 & 0 & 1+\beta\lambda \end{pmatrix}. \quad (15)$$

Thus, we get equations

$$\begin{pmatrix} u(a) \\ w(a) \\ w_x(a) \end{pmatrix} = M_j(\lambda) \begin{pmatrix} b_1^j \\ c_1^j \\ c_2^j \end{pmatrix} + \begin{pmatrix} A_j \begin{pmatrix} \Phi(f)^j(0) \\ \Psi(g)^j(0) \end{pmatrix} \\ s_j \Psi(g)_s^j(0) \end{pmatrix}, j = 1, 2..n, \quad (16)$$

$$-\sum_{j=1}^{n} N_j(\lambda) \begin{pmatrix} b_1^j \\ c_1^j \\ c_2^j \end{pmatrix} + Q(\lambda) \begin{pmatrix} u(a) \\ w(a) \\ w_x(a) \end{pmatrix} = Q_0(F) \quad (17)$$

where

$$Q_0(F) = \begin{pmatrix} \alpha f_1(a) \\ \alpha g_1(a) \\ \beta g_{1,x}(a) \end{pmatrix} + \sum_{j=1}^{n} \begin{pmatrix} (A_j^*)^{-1} \begin{pmatrix} h_j \Phi(f)_s^j(0) \\ -EI_j \Psi(g)_{sss}^j(0) \end{pmatrix} \\ s_j^{-1} EI_j \Psi(g)_{ss}^j(0) \end{pmatrix} \quad (18)$$

We rewrite (16–17) as follows

$$\begin{pmatrix} M_1(\lambda) & 0 & 0 & 0 & 0 & -I \\ 0 & M_2(\lambda) & 0 & 0 & 0 & -I \\ \vdots & \dots & \ddots & \dots & \vdots & \\ 0 & 0 & 0 & 0 & M_n(\lambda) & -I \\ -N_1(\lambda) & -N_2(\lambda) & \cdots & 0 & -N_n(\lambda) & Q(\lambda) \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \\ W \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_n \\ Q_0 \end{pmatrix} \quad (19)$$

where

$$B_j = \begin{pmatrix} b_1^j \\ c_1^j \\ c_2^j \end{pmatrix}, \quad W = \begin{pmatrix} u(a) \\ w(a) \\ w_x(a) \end{pmatrix},$$

$$Q_j(F) = -\begin{pmatrix} A_j \begin{pmatrix} \Phi(f)^j(0) \\ \Psi(g)^j(0) \end{pmatrix} \\ s_j \Psi(g)_s^j(0) \end{pmatrix}.$$

Therefore, (19) is solvable if and only if its coefficients matrix is nonsingular, i.e.,

$$\Delta(\lambda) = \det \begin{vmatrix} M_1(\lambda) & 0 & 0 & 0 & 0 & -I \\ 0 & M_2(\lambda) & 0 & 0 & 0 & -I \\ \vdots & \dots & \ddots & \dots & \vdots & \\ 0 & 0 & 0 & 0 & M_n(\lambda) & -I \\ -N_1(\lambda) & -N_2(\lambda) & \cdots & 0 & -N_n(\lambda) & Q(\lambda) \end{vmatrix} \neq 0. \quad (20)$$

When $\Delta(\lambda) \neq 0$, in particular, when $\det|M_j(\lambda)| \neq 0$, we have

$$\begin{cases} W = \left[Q(\lambda) - \sum\limits_{j=1}^{n} N_j(\lambda) M_j^{-1}(\lambda)\right]^{-1} \left[Q_0(F) + \sum\limits_{j=1}^{n} N_j M_j^{-1} Q_j\right] \\ B_j = M_j^{-1} \left[Q - \sum\limits_{j=1}^{n} N_j M_j^{-1}\right]^{-1} \left[Q_0(F) + \sum\limits_{j=1}^{n} N_j M_j^{-1} Q_j\right] + M_j^{-1} Q_j. \end{cases} \quad (21)$$

By now we have gotten the expression of functions

$$\begin{pmatrix} u^j(s) \\ w^j(s) \end{pmatrix} = \begin{pmatrix} \sinh \lambda\tau_j(\ell_j - s) & 0 & 0 \\ 0 & \sinh \sqrt{i\lambda}\omega_j(\ell_j - s) & \sin \sqrt{i\lambda}\omega_j(\ell_j - s) \end{pmatrix} B_j + \begin{pmatrix} \Phi(f)^j(s) \\ \Psi(g)^j(s) \end{pmatrix} \quad (22)$$

$$\begin{pmatrix} u(a) \\ w(a) \\ w_x(a) \end{pmatrix} = W,$$

$$\begin{pmatrix} u_x(a) \\ w_{xxx}(a) \\ w_{xx}(a) \end{pmatrix} = -\sum_{j=1}^{n} \begin{pmatrix} (A_j^*)^{-1} & 0 \\ 0 & s_j^{-1} \end{pmatrix} \begin{pmatrix} h_j u_s^j(0) \\ -EI_j w_{sss}^j(0) \\ EI_j w_{ss}^j(0) \end{pmatrix}. \quad (23)$$

Therefore, we have

$$Y=\begin{pmatrix} u(x)\\ w(x)\\ v(x)\\ z(x)\end{pmatrix}=\begin{pmatrix} u(x)\\ w(x)\\ \lambda u(x)-f_1(x)\\ \lambda w(x)-g_1(x)\end{pmatrix}\in\mathcal{D}(\mathcal{A}) \tag{24}$$

and $Y=R(\lambda,\mathcal{A})F$. So we attain the following result.

Theorem 1 *For $\lambda\in\mathbb{C}$, when $\Delta(\lambda)\neq 0$, $\lambda\in\rho(\mathcal{A})$ and the resolvent is given by (24). In particular, $R(\lambda,\mathcal{A})F$ is a meromophic function of at most finite exponential type.*

Proof When $\Delta(\lambda)\neq 0$, $\lambda\in\rho(\mathcal{A})$, the resolvent of $\mathcal{A}$ has been given by (24). Clearly, $R(\lambda,A)F$ is a H-valued meromophic function. In particular, $\Delta(\lambda)R(\lambda,A)F$ is an entire function on $\mathbb{C}$. Note that the functions $\Phi(f)^j(s)$ and $\Psi(g)^j(s)$ consist of the integral of functions

$$\sinh\lambda\tau_j(\ell_j-s),\ \sinh\sqrt{i\lambda}\omega_j(\ell_j-s),\ \sin\sqrt{i\lambda}\omega_j(\ell_j-s)$$

and $\Delta(\lambda)$, $M_j(\lambda)$ and $N_j(\lambda)$ consist of the functions

$$\sinh\lambda\tau_j\ell_j,\ \sinh\sqrt{i\lambda}\omega_j\ell_j,\ \sin\sqrt{i\lambda}\omega_j\ell_j,$$

$$\cosh\lambda\tau_j\ell_j,\ \cosh\sqrt{i\lambda}\omega_j\ell_j,\ \cos\sqrt{i\lambda}\omega_j\ell_j$$

as well as the polynomial of λ and $\sqrt{i\lambda}$. So they are at most of finite exponential type in λ. Hence $B_j, j=1,2,\cdots,n$ are at most of finite exponential type. Therefore, $R(\lambda,\mathcal{A})F$ is a meromophic function of at most finite exponential type. □

From the representation we can get the following result.

Theorem 2 *Let $\mathcal{A}$ be defined as before. Then we have*

$$\sigma(\mathcal{A})=\{\lambda\in\mathbb{C}\mid\Delta(\lambda)=0\}. \tag{25}$$

In what following we shall discuss the distribution of spectrum by estimating $\Delta(\lambda)$.

Let $\lambda\in\mathbb{C}$, then we can write λ into the form

$$\lambda=r^2e^{i\theta},\quad \theta=\arg\lambda.$$

Since $i=e^{i\frac{\pi}{2}}$, we have

$$\rho=\sqrt{i\lambda}=\sqrt{r^2e^{i(\theta+\frac{\pi}{2})}}=re^{i\psi},\quad \psi=\frac{1}{2}(\theta+\frac{\pi}{2}).$$

So the transform translate λ-plane into ρ-plane as follows:

$$\lambda{-}plane:\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right]\to\rho{-}plane:\psi\in\left(0,\frac{\pi}{2}\right];$$

$$\lambda-plane:\theta=-\frac{\pi}{2}\to\rho-plane:\psi=0;$$

$$\lambda{-}plane:\theta\in\left(\frac{\pi}{2},\frac{3\pi}{2}\right)\to\rho{-}plane:\psi\in\left(\frac{\pi}{2},\pi\right]$$

$$\lambda-plane:\theta=\frac{3\pi}{2}\to\rho-plane:\psi=\pi.$$

Since $\Delta(\lambda)$ is mainly determined by the functions

$$\sinh\lambda\tau_j\ell_j,\ \cosh\lambda\tau_j\ell_j,\ \sinh\sqrt{i\lambda}\omega_j\ell_j,$$

$$\cosh\sqrt{i\lambda}\omega_j\ell_j,\ \sin\sqrt{i\lambda}\omega_j\ell_j,\ \cos\sqrt{i\lambda}\omega_j\ell_j,$$

we are going to study the asymptotic estimates of these functions.

Let $\lambda=r^2e^{i\theta}$. When $\theta\in\left(-\frac{\pi}{2}+\varepsilon,\frac{\pi}{2}-\varepsilon\right)$, we have

$$\Re\lambda>0,\quad \Re\sqrt{i\lambda}>0,\quad \Re(-i\sqrt{i\lambda})>0.$$

Therefore, for $\theta\in\left(-\frac{\pi}{2}+\varepsilon,\frac{\pi}{2}-\varepsilon\right)$, as $r\to+\infty$, we have

$$\begin{array}{l}
\sinh\lambda\tau_j\ell_j=\frac{1}{2}e^{\lambda\tau_j\ell_j}[1+o(r^2)],\\
\cosh\lambda\tau_j\ell_j=\frac{1}{2}e^{\lambda\tau_j\ell_j}[1+o(r^2)];\\
\sinh\sqrt{i\lambda}\omega_j\ell_j=\frac{1}{2}e^{\sqrt{i\lambda}\omega_j\ell_j}[1++o(r^2)],\\
\cosh\sqrt{i\lambda}\omega_j\ell_j=\frac{1}{2}e^{\sqrt{i\lambda}\omega_j\ell_j}[1+o(r^2)];\\
\sin\sqrt{i\lambda}\omega_j\ell_j=\frac{i}{2}e^{-i\sqrt{i\lambda}\omega_j\ell_j}[1++o(r^2)],\\
\cos\sqrt{i\lambda}\omega_j\ell_j=\frac{1}{2}e^{-i\sqrt{i\lambda}\omega_j\ell_j}[1+o(r^2)].
\end{array}$$

For $\theta\in\left(\frac{\pi}{2}+\varepsilon,\frac{3\pi}{2}-\varepsilon\right)$, we have

$$\Re(-\lambda)>0,\quad \Re(-\sqrt{i\lambda})>0,\quad \Re(-i\sqrt{i\lambda})>0.$$

Therefore, for $\theta\in\left(\frac{\pi}{2}+\varepsilon,\frac{3\pi}{2}-\varepsilon\right)$, as $r\to+\infty$, we have

$$\begin{array}{l}
\sinh\lambda\tau_j\ell_j=-\frac{1}{2}e^{-\lambda\tau_j\ell_j}[1+o(r^2)],\\
\cosh\lambda\tau_j\ell_j=\frac{1}{2}e^{-\lambda\tau_j\ell_j}[1+o(r^2)];\\
\sinh\sqrt{i\lambda}\omega_j\ell_j=-\frac{1}{2}e^{-\sqrt{i\lambda}\omega_j\ell_j}[1+o(r^2)],\\
\cosh\sqrt{i\lambda}\omega_j\ell_j=\frac{1}{2}e^{-\sqrt{i\lambda}\omega_j\ell_j}[1+o(r^2)];\\
\sin\sqrt{i\lambda}\omega_j\ell_j=\frac{i}{2}e^{-i\sqrt{i\lambda}\omega_j\ell_j}[1+o(r^2)],\\
\cos\sqrt{i\lambda}\omega_j\ell_j=\frac{1}{2}e^{-i\sqrt{i\lambda}\omega_j\ell_j}[1+o(r^2)].
\end{array}$$

Therefore, we have the following estimates:

A). When $\theta\in\left[-\frac{\pi}{2}+\varepsilon,\frac{\pi}{2}-\varepsilon\right]$, we have

$$M_j(\lambda)=\begin{pmatrix} A_j & 0\\ 0 & s_j\end{pmatrix}$$

$$\begin{pmatrix}
\frac{1}{2}e^{\lambda\tau_j\ell_j}[1] & 0 & 0\\
0 & \frac{1}{2}e^{\sqrt{i\lambda}\omega_j\ell_j}[1] & \frac{i}{2}e^{-i\sqrt{i\lambda}\omega_j\ell_j}[1]\\
0 & \frac{(-\sqrt{i\lambda}\omega_j)}{2}e^{\sqrt{i\lambda}\omega_j\ell_j}[1] & \frac{(-\sqrt{i\lambda}\omega_j)}{2}e^{-i\sqrt{i\lambda}\omega_j\ell_j}[1]
\end{pmatrix}$$

and

$$N_j(\lambda)=\begin{pmatrix}(A_j^*)^{-1} & 0\\ 0 & s_j^{-1}\end{pmatrix}\times$$
$$\begin{pmatrix} -\frac{\lambda\tau_j h_j}{2}e^{\lambda\tau_j\ell_j}[1] & 0 & 0\\ 0 & -\frac{EI_j(-\sqrt{i\lambda}\omega_j)^3}{2}e^{\sqrt{i\lambda}\omega_j\ell_j}[1] & \frac{EI_j(-\sqrt{i\lambda}\omega_j)^3}{2}e^{-i\sqrt{i\lambda}\omega_j\ell_j}[1]\\ 0 & \frac{EI_j(-\sqrt{i\lambda}\omega_j)^2}{2}e^{\sqrt{i\lambda}\omega_j\ell_j}[1] & -i\frac{EI_j(-\sqrt{i\lambda}\omega_j)^2}{2}e^{-i\sqrt{i\lambda}\omega_j\ell_j}[1]\end{pmatrix},$$

where $[1]$ means that $[1]=1+o(r^2)$ as r large enough. They are equivalently written as

$$M_j(\lambda)=\begin{pmatrix}A_j & 0\\ 0 & s_j\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & i\\ 0 & (-\sqrt{i\lambda}\omega_j) & (-\sqrt{i\lambda}\omega_j)\end{pmatrix}\times$$
$$\begin{pmatrix}\frac{1}{2}e^{\lambda\tau_j\ell_j}[1] & 0 & 0\\ 0 & \frac{1}{2}e^{\sqrt{i\lambda}\omega_j\ell_j}[1] & 0\\ 0 & 0 & \frac{1}{2}e^{-i\sqrt{i\lambda}\omega_j\ell_j}[1]\end{pmatrix}$$
$$=\widehat{M}_j(\lambda)\begin{pmatrix}\frac{1}{2}e^{\lambda\tau_j\ell_j}[1] & 0 & 0\\ 0 & \frac{1}{2}e^{\sqrt{i\lambda}\omega_j\ell_j}[1] & 0\\ 0 & 0 & \frac{1}{2}e^{-i\sqrt{i\lambda}\omega_j\ell_j}[1]\end{pmatrix} \tag{26}$$

and

$$N_j(\lambda)=\begin{pmatrix}(A_j^*)^{-1} & 0\\ 0 & s_j^{-1}\end{pmatrix}\times$$
$$\begin{pmatrix}-\lambda\tau_j h_j & 0 & 0\\ 0 & -EI_j(-\sqrt{i\lambda}\omega_j)^3 & EI_j(-\sqrt{i\lambda}\omega_j)^3\\ 0 & EI_j(-\sqrt{i\lambda}\omega_j)^2 & -iEI_j(-\sqrt{i\lambda}\omega_j)^2\end{pmatrix}\times$$
$$\begin{pmatrix}\frac{1}{2}e^{\lambda\tau_j\ell_j}[1] & 0 & 0\\ 0 & \frac{1}{2}e^{\sqrt{i\lambda}\omega_j\ell_j}[1] & 0\\ 0 & 0 & \frac{1}{2}e^{-i\sqrt{i\lambda}\omega_j\ell_j}[1]\end{pmatrix}$$
$$=\widehat{N}_j(\lambda)\begin{pmatrix}\frac{1}{2}e^{\lambda\tau_j\ell_j}[1] & 0 & 0\\ 0 & \frac{1}{2}e^{\sqrt{i\lambda}\omega_j\ell_j}[1] & 0\\ 0 & 0 & \frac{1}{2}e^{-i\sqrt{i\lambda}\omega_j\ell_j}[1]\end{pmatrix}. \tag{27}$$

Thus we have

$$\Delta(\lambda)=\frac{(1-i)^n}{2^{3n}}e^{\sum\limits_{j=1}^{n}[\lambda\tau_j\ell_j+(1-i)\sqrt{i\lambda}\omega_j\ell_j]}(-\sqrt{i\lambda})^n[1]$$
$$\prod_{j=1}^{n}|A_j|s_j\omega_j\det\left|Q(\lambda)-\sum_{j=1}^{n}\widehat{N}_j(\lambda)\widehat{M_j^{-1}}(\lambda)\right| \tag{28}$$

and

$$\widehat{N}_j(\lambda)\widehat{M_j^{-1}}(\lambda)=\begin{pmatrix}(A_j^*)^{-1} & 0\\ 0 & s_j^{-1}\end{pmatrix}$$
$$\begin{pmatrix}-\lambda\tau_j h_j & 0 & 0\\ 0 & -(1+i)EI_j(-\sqrt{i\lambda}\omega_j)^3 & iEI_j(-\sqrt{i\lambda}\omega_j)^2\\ 0 & iEI_j(-\sqrt{i\lambda}\omega_j)^2 & (1-i)EI_j(-\sqrt{i\lambda}\omega_j)\end{pmatrix}\times$$
$$\begin{pmatrix}A_j^{-1} & 0\\ 0 & \frac{1}{s_j}\end{pmatrix}.$$

Set

$$A_j=\begin{pmatrix}a_{11}^j & a_{12}^j\\ a_{21}^j & a_{22}^j\end{pmatrix}.$$

Then

$$A_j^{-1}=\begin{pmatrix}\frac{a_{22}^j}{|A_j|} & \frac{-a_{12}^j}{|A_j|}\\ \frac{-a_{21}^j}{|A_j|} & \frac{a_{11}^j}{|A_j|}\end{pmatrix},\ (A_j^*)^{-1}=\begin{pmatrix}\frac{a_{22}^j}{|A_j|} & \frac{-a_{21}^j}{|A_j|}\\ \frac{-a_{12}^j}{|A_j|} & \frac{a_{11}^j}{|A_j|}\end{pmatrix}$$

Thus, for $\rho=\sqrt{i\lambda}$, we have

$$\widehat{N}_j(\lambda)\widehat{M_j^{-1}}(\lambda)$$
$$=\begin{pmatrix} i\rho^2\frac{(a_{22}^j)^2}{|A_j|^2}\tau_j h_j+(1+i)\rho^3\frac{(a_{21}^j)^2}{|A_j|^2}EI_j\omega_j^3 & -i\rho^2\frac{a_{22}^ja_{12}^j}{|A_j|^2}\tau_j h_j-(1+i)\rho^3\frac{a_{21}^ja_{11}^j}{|A_j|^2}EI_j\omega_j^3 & -i\rho^2\frac{a_{21}^j}{|A_j|s_j}EI_j\omega_j^2\\ -i\rho\frac{a_{22}^ja_{12}^j}{|A_j|^2}\tau_j h_j-(1+i)\rho^3\frac{a_{11}^ja_{21}^j}{|A_j|^2}EI_j\omega_j^3 & i\rho^2\frac{(a_{12}^j)^2}{|A_j|^2}\tau_j h_j+(1+i)\rho^3\frac{(a_{11}^j)^2}{|A_j|^2}EI_j\omega_j^3 & i\rho^2\frac{a_{11}^j}{|A_j|s_j}EI_j\omega_j^2\\ -i\rho^2\frac{a_{21}^j}{|A_j|}\frac{EI_j}{s_j}\omega_j^2 & i\rho^2\frac{a_{11}^j}{|A_j|}\frac{EI_j}{s_j}\omega_j^2 & -(1-i)\rho\frac{EI_j\omega_j}{s_j^2}\end{pmatrix}.$$

Therefore we get estimate

$$\det\left|Q(\lambda)-\sum_{j=1}^{n}\widehat{N}_j(\lambda)\widehat{M}_j^{-1}(\lambda)\right|$$
$$=\ 2\beta\rho^8[1]\sum_{j=1}^{n}\frac{(a_{21}^j)^2}{|A_j|^2}EI_j\omega_j^3\sum_{j=1}^{n}\frac{(a_{11}^j)^2}{|A_j|^2}EI_j\omega_j^3$$
$$+2\beta\rho^8[1]\left(\sum_{j=1}^{n}\frac{a_{11}^ja_{21}^j}{|A_j|^2}EI_j\omega_j^3\right)^2$$

So, when $a_{21}^j\neq 0, j=1,2\cdots,n$, we have

$$\lim_{r\to\infty}\frac{\Delta(\lambda)}{(-\sqrt{i\lambda})^{n+8}e^{\sum\limits_{j=1}^{n}[\lambda\tau_j\ell_j+(1-i)\sqrt{i\lambda}\omega_j\ell_j]}}\neq 0.$$

When $a_{21}^j\equiv 0, j=1,2,\cdots,n$, we have

$$\widehat{N}_j(\lambda)\widehat{M_j^{-1}}(\lambda)=\begin{pmatrix} i\rho^2\frac{(a_{22}^j)^2}{|A_j|^2}\tau_j h_j & -i\rho^2\frac{a_{22}^ja_{12}^j}{|A_j|^2}\tau_j h_j & 0\\ -i\rho\frac{a_{22}^ja_{12}^j}{|A_j|^2}\tau_j h_j & i\rho^2\frac{(a_{12}^j)^2}{|A_j|^2}\tau_j h_j+(1+i)\rho^3\frac{(a_{11}^j)^2}{|A_j|^2}EI_j\omega_j^3 & i\rho^2\frac{a_{11}^j}{|A_j|s_j}EI_j\omega_j^2\\ 0 & i\rho^2\frac{a_{11}^j}{|A_j|}\frac{EI_j}{s_j}\omega_j^2 & -(1-i)\rho\frac{EI_j\omega_j}{s_j^2}\end{pmatrix}.$$

Hence,

$$\det\left|Q(\lambda)-\sum_{j=1}^{n}\widehat{N}_j(\lambda)\widehat{M}_j^{-1}(\lambda)\right|$$
$$=\ (1+i)\beta\rho^7[1]\left[\alpha+\sum_{j=1}^{n}\frac{(a_{22}^j)^2}{|A_j|^2}\tau_j h_j\right]\times$$
$$\sum_{j=1}^{n}\frac{(a_{11}^j)^2}{|A_j|^2}EI_j\omega_j^3.$$

Also we have

$$\lim_{r\to\infty}\frac{\Delta(\lambda)}{(-\sqrt{i\lambda})^{n+7}e^{\sum_{j=1}^{n}[\lambda\tau_j\ell_j-(1-i)\sqrt{i\lambda}\omega_j\ell_j]}}\neq 0.$$

B). When $\theta\in\left(-\frac{\pi}{2}+\varepsilon,\frac{3\pi}{2}-\varepsilon\right)$, we have

$$M_j(\lambda)=\widehat{M}_j(\lambda)\times\begin{pmatrix}\frac{1}{2}e^{-\lambda\tau_j\ell_j}[1] & 0 & 0\\ 0 & \frac{1}{2}e^{-\sqrt{i\lambda}\omega_j\ell_j}[1] & 0\\ 0 & 0 & \frac{1}{2}e^{-i\sqrt{i\lambda}\omega_j\ell_j}[1]\end{pmatrix}\tag{29}$$

and

$$N_j(\lambda)=\widehat{N}_j(\lambda)\times\begin{pmatrix}\frac{1}{2}e^{-\lambda\tau_j\ell_j}[1] & 0 & 0\\ 0 & \frac{1}{2}e^{-\sqrt{i\lambda}\omega_j\ell_j}[1] & 0\\ 0 & 0 & \frac{1}{2}e^{-i\sqrt{i\lambda}\omega_j\ell_j}[1]\end{pmatrix}\tag{30}$$

So we have

$$\Delta(\lambda)=\frac{1}{2^{3n}}e^{-\sum_{j=1}^{n}[\lambda\tau_j\ell_j+(1+i)\sqrt{i\lambda}\omega_j\ell_j]}[1]\times \det\begin{vmatrix}\widehat{M}_1(\lambda) & 0 & 0 & 0 & 0 & -I\\ 0 & \widehat{M}_2(\lambda) & 0 & 0 & 0 & -I\\ \vdots & \cdots & \ddots & \cdots & \vdots & \\ 0 & 0 & 0 & 0 & \widehat{M}_n(\lambda) & -I\\ \widehat{N}_1(\lambda) & \widehat{N}_2(\lambda) & \cdots & 0 & \widehat{N}_n(\lambda) & Q(\lambda)\end{vmatrix}\tag{31}$$

In this case,

$$\widehat{M}_j(\lambda)=\begin{pmatrix}A_j & 0\\ 0 & s_j\end{pmatrix}\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & i\\ 0 & (-\sqrt{i\lambda}\omega_j) & (-\sqrt{i\lambda}\omega_j)\end{pmatrix}\tag{32}$$

and

$$\widehat{N}_j(\lambda)=\begin{pmatrix}(A_j^*)^{-1} & 0\\ 0 & s_j^{-1}\end{pmatrix}\times\begin{pmatrix}-\lambda\tau_jh_j & 0 & 0\\ 0 & -EI_j(-\sqrt{i\lambda}\omega_j)^3 & EI_j(-\sqrt{i\lambda}\omega_j)^3\\ 0 & -EI_j(-\sqrt{i\lambda}\omega_j)^2 & -iEI_j(-\sqrt{i\lambda}\omega_j)^2\end{pmatrix}\tag{33}$$

Thus we have

$$\widehat{N}_j(\lambda)\widehat{M_j^{-1}}(\lambda)=\begin{pmatrix}(A_j^*)^{-1} & 0\\ 0 & s_j^{-1}\end{pmatrix}\begin{pmatrix}\lambda\tau_jh_j & 0 & 0\\ 0 & (1-i)EI_j(-\sqrt{i\lambda}\omega_j)^3 & -iEI_j(-\sqrt{i\lambda}\omega_j)^2\\ 0 & -iEI_j(-\sqrt{i\lambda}\omega_j)^2 & -(1+i)EI_j(-\sqrt{i\lambda}\omega_j)\end{pmatrix}\times\begin{pmatrix}A_j^{-1} & 0\\ 0 & \frac{1}{s_j}\end{pmatrix}.$$

For $\rho=\sqrt{i\lambda}$, we have

$$\widehat{N}_j(\lambda)\widehat{M_j^{-1}}(\lambda)=\begin{pmatrix}-i\rho^2\frac{(a_{22}^j)^2}{|A_j|^2}\tau_jh_j-(1-i)\rho^3\frac{(a_{21}^j)^2}{|A_j|^2}EI_j\omega_j^3 & i\rho^2\frac{a_{22}^ja_{12}^j}{|A_j|^2}\tau_jh_j+(1-i)\rho^3\frac{a_{21}^ja_{11}^j}{|A_j|^2}EI_j\omega_j^3 & i\rho^2\frac{a_{21}^j}{|A_j|s_j}EI_j\omega_j^2\\ i\rho\frac{a_{22}^ja_{12}^j}{|A_j|^2}\tau_jh_j+(1-i)\rho^3\frac{a_{11}^ja_{21}^j}{|A_j|^2}EI_j\omega_j^3 & -i\rho^2\frac{(a_{12}^j)^2}{|A_j|^2}\tau_jh_j-(1-i)\rho^3\frac{(a_{11}^j)^2}{|A_j|^2}EI_j\omega_j^3 & -i\rho^2\frac{a_{11}^j}{|A_j|s_j}EI_j\omega_j^2\\ i\rho^2\frac{a_{21}^j}{|A_j|}\frac{EI_j}{s_j}\omega_j^2 & -i\rho^2\frac{a_{11}^j}{|A_j|}\frac{EI_j}{s_j}\omega_j^2 & (1+i)\rho\frac{EI_j\omega_j}{s_j^2}\end{pmatrix}$$

Therefore we have estimate

$$\begin{aligned}&\det\left|Q(\lambda)-\sum_{j=1}^{n}\widehat{N_j}(\lambda)\widehat{M_j}^{-1}(\lambda)\right|\\ =\;&2i\beta\rho^8[1]\sum_{j=1}^{n}\frac{(a_{21}^j)^2}{|A_j|^2}EI_j\omega_j^3\sum_{j=1}^{n}\frac{(a_{11}^j)^2}{|A_j|^2}EI_j\omega_j^3\\ &+2i\beta\rho^8[1]\left(\sum_{j=1}^{n}\frac{a_{11}^ja_{21}^j}{|A_j|^2}EI_j\omega_j^3\right)^2.\end{aligned}$$

So, when $a_{21}^j\neq 0, j=1,2\cdots,n$, we have

$$\lim_{r\to\infty}\frac{\Delta(\lambda)}{(-\sqrt{i\lambda})^{n+8}e^{\sum_{j=1}^{n}[\lambda\tau_j\ell_j+(1-i)\sqrt{i\lambda}\omega_j\ell_j]}}\neq 0.$$

When $a_{21}^j\equiv 0, j=1,2,\cdots,n$, we have

$$\widehat{N}_j(\lambda)\widehat{M_j^{-1}}(\lambda)=\begin{pmatrix}-i\rho^2\frac{(a_{22}^j)^2}{|A_j|^2}\tau_jh_j & i\rho^2\frac{a_{22}^ja_{12}^j}{|A_j|^2}\tau_jh_j & 0\\ i\rho\frac{a_{22}^ja_{12}^j}{|A_j|^2}\tau_jh_j & -i\rho^2\frac{(a_{12}^j)^2}{|A_j|^2}\tau_jh_j-(1-i)\rho^3\frac{(a_{11}^j)^2}{|A_j|^2}EI_j\omega_j^3 & -i\rho^2\frac{a_{11}^j}{|A_j|s_j}EI_j\omega_j^2\\ 0 & -i\rho^2\frac{a_{11}^j}{|A_j|}\frac{EI_j}{s_j}\omega_j^2 & (1+i)\rho\frac{EI_j\omega_j}{s_j^2}\end{pmatrix}.$$

Thus,

$$\begin{aligned}&\det\left|Q(\lambda)-\sum_{j=1}^{n}\widehat{N}_j(\lambda)\widehat{M}_j^{-1}(\lambda)\right|\\ =\;&(1-i)\beta\rho^7[1]\left[\alpha-\sum_{j=1}^{n}\frac{(a_{22}^j)^2}{|A_j|^2}\tau_jh_j\right]\times\\ &\sum_{j=1}^{n}\frac{(a_{11}^j)^2}{|A_j|^2}EI_j\omega_j^3.\end{aligned}$$

When $\alpha-\sum_{j=1}^{n}\frac{(a_{22}^j)^2}{|A_j|^2}\tau_jh_j\neq 0$, we have

$$\lim_{r\to\infty}\frac{\Delta(\lambda)}{(-\sqrt{i\lambda})^{n+7}e^{\sum_{j=1}^{n}[\lambda\tau_j\ell_j-(1-i)\sqrt{i\lambda}\omega_j\ell_j]}}\neq 0.$$

The calculation above shows that there is no eigenvalue in the domain

$$\lambda=r^2e^{i\theta},\ \theta\in\left[-\frac{\pi}{2}+\varepsilon,\frac{\pi}{2}-\varepsilon\right]\cup\left[\frac{\pi}{2}+\varepsilon,\frac{3\pi}{2}-\varepsilon\right]$$

with $|\lambda|$ large enough. So the spectrum of $\mathcal{A}$ distributes in a strip parallel the imaginary axis in λ plane. Therefore, we have the following result.

Theorem 3 *Let $\mathcal{A}$ be defined as before. Then when $a_{21}^j \neq 0$, or $a_{21}^j \equiv 0, j = 1, 2, \cdots, n$ and $\alpha - \sum_{j=1}^{n} \frac{(a_{22}^j)^2}{|A_j|^2}\tau_j h_j \neq 0$, there exists positive constants M_1, M_2 and h such that for $\forall\lambda \in \mathbb{C}$ with $|\Re\lambda| > h$,*

$$M_1 \leq \left| \frac{\Delta(\lambda)}{(-\sqrt{i\lambda})^{n+k} e^{\sum_{j=1}^{n}[\lambda\tau_j\ell_j + (1-i)\sqrt{i\lambda}\omega_j\ell_j]}} \right| \leq M_2, \tag{34}$$

where $k = 8$ as $a_{21}^j \neq 0$, or $k = 7$ as $a_{21}^j \equiv 0, j = 1, 2, \cdots, n$ and $\alpha - \sum_{j=1}^{n} \frac{(a_{22}^j)^2}{|A_j|^2}\tau_j h_j \neq 0$. Hance, we have

$$\sigma(\mathcal{A}) \subset \{\lambda \in \mathbb{C} \mid -h \leq \Re\lambda \leq 0\}. \tag{35}$$

Remark 4 *In Theorem 3, the condition $a_{21}^j \equiv 0, j = 1, 2, \cdots, n$ means that the strings and beams are weakly coupled. In this case, the condition $\alpha - \sum_{j=1}^{n} \frac{(a_{22}^j)^2}{|A_j|^2}\tau_j h_j \neq 0$ says that the damping constant is not equal to the velocity of wave. This result coincides with the case $n = 1$.*

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