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A Simple Equilibrated Homogenization Model for the Limit Analysis of Masonry Structures

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Abstract: - In this paper, an equilibrated homogenization model for the limit analysis of running bond masonry walls is revised. Assuming brickwork under plane stress condition and adopting a polynomial expansion for the 2D stress field, a linear optimization problem is derived on the elementary cell in order to recover the homogenized failure surfaces of the brickwork. Different models of higher accuracy can be obtained by increasing *ad libitum* the degree of the polynomial approximation. The homogenized failure surfaces so obtained are then implemented in a FE limit analysis code for the evaluation of collapse loads and failure mechanisms of entire masonry structures. Both upper and lower bound homogenized limit analysis approaches are used for treating a meaningful structural case consisting of a two storey panel seismically loaded. A comparison with a pushover analysis conducted with a standard commercial code shows the efficiency of the technique presented with respect to: 1) accuracy of the results; 2) reduced number of finite elements required; 3) reduced time required for the simulations.

Key-Words: Masonry, Homogenization, Limit analysis, FEM

1 Introduction

Masonry is a composite material constituted by bricks jointed together with mortar. The great amount of possible combinations generated by the geometry, the arrangement and the mechanical properties of the constituent materials complicates its study to a great extent, especially in the inelastic range.

From an experimental point of view, the difficulties in performing advanced testing are quite large, due to the innumerable variations of masonry, the large scatter of in situ material properties and the impossibility of reproducing it all in a specimen. Therefore, most of the advanced experimental research carried out in the last decades concentrated in regular textures of brick / block masonry, with particular emphasis on running bond pattern.

Another important aspect that deserves particular consideration in the study of brickwork structures is related to the fact that historical city centers are often constituted by ancient masonry buildings. Many efforts have been done by researches in order to better understand their behavior to horizontal seismic actions. For instance, earthquake surveys have demonstrated that the low tensile strength of masonry elements combined with an insufficient interlocking of perpendicular panels leads to overturning collapses of the perimeter walls under seismic horizontal acceleration and combined in- and out-of-plane failures.

Therefore, the evaluation of the ultimate load bearing capacity of masonry buildings subjected to horizontal loads is a fundamental task in their design and safety assessment. Simplified limit analysis methods are usually adopted by practitioners for safety analyses and design of strengthening, but codes of practice, as for instance the recent Italian O.P.C.M. 3431 [1], require a static non linear analysis for existing masonry buildings, in which a limited ductile behavior of the elements is taken into account, featuring failure mechanisms such as rocking, shear and diagonal cracking of the walls.

In this framework, many researchers tried to develop a number of different numerical approaches (see [2] for a comprehensive review), all based on micro-modeling, macro-modeling or homogenization, with the aim of obtaining reliable tools to predict masonry behavior at failure.

The present paper focuses exclusively on the analysis of running bond masonry structures making use of homogenization techniques, which has been receiving a growing interest from the scientific community.

In particular, the approach based on the use of averaged constitutive equations seems to be the only one suitable to be employed in a large scale finite element analysis [3]. In fact, heterogeneous approaches based on a distinct representation of bricks and joints seem to be limited to the study of panels of small dimensions, due to the large number of variables involved in a non linear finite element analysis. On the other hand, alternative strategies based on macro-modeling (see Lourenço et al. [4]) have the drawback of requiring a preliminary mechanical characterization of the model, which usually is obtained from experimental data fitting [4].

In this framework, homogenization techniques can be used for the analysis of large scale structures. Such techniques take into account at a cell level the mechanical properties of constituent materials and the geometry of the elementary cell, allowing the analysis of entire buildings through standard finite element codes. Furthermore, the application of homogenization theory to the rigid-plastic case [5] is particularly indicated for a simple but reliable structural analysis, requiring only a reduced number of material parameters and providing significant information at failure (e.g. limit multipliers, collapse mechanisms, stress distribution).

In this paper, the micro-mechanical model presented by the author in [5] for the limit analysis of masonry walls is reviewed and utilized for the analysis of a large scale structure subjected to seismic action. In the model, the elementary cell is subdivided in several sub-domains. For each sub-domain, fully equilibrated stress fields are assumed, adopting polynomial expressions for the stress tensor components. The continuity of the stress vector on the interfaces between adjacent sub-domains and suitable anti-periodicity conditions on the boundary surface are further imposed. In this way, linearized homogenized surfaces for masonry are obtained. Such surfaces are then implemented in a FE limit analysis code for the analysis at collapse of large scale walls.

In Section 2, the basic concepts of homogenization applied to running bond textures are briefly reviewed, with particular attention to the coupled in- and out-of-plane case.

In Section 3, the equilibrated micro-mechanical model employed for obtaining masonry homogenized surfaces is presented.

Finally, in Section 4, an entire two storey masonry wall subjected to seismic loads is numerically analyzed with the model proposed, making use of a specifically crafted limit analysis FE code. Both upper and lower bound homogenized limit analyses approaches are performed and compared with a pushover analysis conducted by means of a standard commercial package. The comparison demonstrates the reliability of the method presented.

2 Basic Concepts

Let a masonry wall Ω be considered, constituted by the periodic arrangement of bricks and mortar joints (Fig. 1). The periodicity allows to regard Ω as the repetition of a representative element of volume Y (REV or elementary cell). Let $\mathbf{x}=[x_1, x_2]$ be a frame of reference for the global description of Ω (macroscopic scale) and $\mathbf{y}=[y_1, y_2, y_3]$ a frame of reference for Y. We define (Caillerie [6]) the Y module as $Y = \omega \times \left]-\frac{t}{2};\frac{t}{2}\right[$ where $Y \in \Re^3$ is the elementary cell and $\omega \in \Re^2$ represents the middle plane of the plate. The ∂Y boundary surface of the elementary cell (see Fig. 1) is defined as $\partial Y = \partial Y_l \cup \partial Y_3^+ \cup \partial Y_3^-$, where $\partial Y_3^{\pm} = \omega \times \left[\pm\frac{t}{2}\right]$.

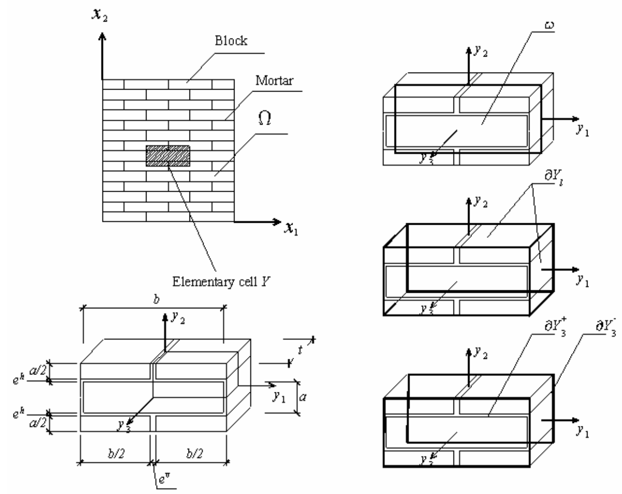


Fig. 1: Elementary cell used in finite element modeling.

The basic idea of homogenization consists in introducing averaged quantities representing the macroscopic stress and strain tensors (denoted respectively as $\mathbf{E}$ and $\mathbf{\Sigma}$), as follows:

$$\mathbf{E} = <\boldsymbol{\varepsilon}> = \frac{1}{V}\int_Y \boldsymbol{\varepsilon}(\mathbf{u})dY \; ; \; \mathbf{\Sigma} = <\boldsymbol{\sigma}> = \frac{1}{V}\int_Y \boldsymbol{\sigma} dY \qquad (1)$$

where V stands for the volume of the elementary cell, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ stand for the local quantities (stresses and strains respectively) and <*> is the average operator.

According to Anthoine [7] and Cecchi et al. [8], the homogenization problem in the linear elastic range, in presence of coupled membranal and flexural loads, under the assumption of the

Kirchhoff-Love plate theory, can be written as follows:

$$\begin{cases} div\boldsymbol{\sigma} = \mathbf{0} & (a) \\ \boldsymbol{\sigma} = \mathbf{a}(\mathbf{y})\boldsymbol{\varepsilon} & (b) \\ \boldsymbol{\varepsilon} = \mathbf{E} + y_3\boldsymbol{\chi} + sym(grad\mathbf{u}^{per}) & (c) \\ \boldsymbol{\sigma}\mathbf{e}_3 = 0 \text{ on } \partial Y_3^+ \text{ and } \partial Y_3^- & (d) \\ \boldsymbol{\sigma}\,\mathbf{n} \text{ antiperiodic on } \partial Y_l & (e) \\ \mathbf{u}^{per} \text{ periodic on } \partial Y_l & (f) \end{cases} \qquad (2)$$

Where $\boldsymbol{\sigma}$ is the microscopic stress tensor (micro-stress), $\mathbf{u}^{per}$ is a ω-periodic displacement field, $\mathbf{E}$ is the macroscopic in-plane strain tensor, $\boldsymbol{\chi}$ is the out-of-plane strain tensor (curvature tensor), $\mathbf{a}(\mathbf{y})$ represents a ω-periodic linear elastic constitutive law for the components (bricks and mortar), equation (2)(b). Equation (2)(a) represents the micro-equilibrium for the elementary cell with zero body forces, usually neglected in the framework of homogenization.

Furthermore, in equation (2)(c), the micro-strain tensor $\boldsymbol{\varepsilon}$ is obtained as a linear combination among macroscopic $\mathbf{E}$ and $\boldsymbol{\chi}$ tensors and a periodic strain field. $\mathbf{E}$ and $\boldsymbol{\chi}$ tensors are related to the macroscopic displacement field components $U_1(x_1,x_2)$, $U_2(x_1,x_2)$ and $U_3(x_1,x_2)$ by means of the classic relations $\mathbf{E}_{\alpha\beta} = \frac{1}{2}(U_{\alpha,\beta} + U_{\beta,\alpha})$, with $\mathbf{E}_{i3} = 0$, and $\chi_{\alpha\beta} = -U_{3,\alpha\beta}$ with $\chi_{i3} = 0$, $\alpha,\beta = 1,2$ and i =1,2,3.

Macroscopic homogenized membrane and bending constants can be obtained solving the elastostatic problem (2) and making use of the classic relations:

$$\begin{aligned} \mathbf{N} &= \langle \boldsymbol{\sigma} \rangle^* = \mathbf{A}\mathbf{E} + \mathbf{B}\boldsymbol{\chi} \\ \mathbf{M} &= \langle y_3\boldsymbol{\sigma} \rangle^* = \mathbf{B}^{\mathbf{T}}\mathbf{E} + \mathbf{D}\boldsymbol{\chi} \end{aligned} \qquad (3)$$

where **A**, **B** and **D** are the constitutive homogenized plate tensors. Usually, the elementary cell has a central symmetry, hence **B=0**. As a rule, a solution for problem (2) can be obtained using standard FE packages, as suggested for the in-plane case by Anthoine [7] and for the FRP reinforced and unreinforced in- and out-of-plane cases by Cecchi et al. [8] and Cecchi et al. [9][10].

Governing equations in the non-linear case are formally identical to equation (2), provided that a non linear stress-strain law for the constituent materials is assumed.

3 Homogenization in the rigid-plastic case

Limit analysis approaches (de Buhan and de Felice [11] and Milani et al. [5][12]) are based on the assumption of a perfectly plastic behavior with associated flow rule for the constituent materials. In this framework, Suquet [13] proved that both static and kinematic approaches can be used in order to obtain an upper or lower bound estimation of the homogenized failure surface of a periodic arrangement of rigid plastic materials.

In Fig. 2 (see also Fig. 1) a masonry wall Ω constituted by a periodic arrangement of bricks and mortar disposed in running bond texture is shown, together with a rectangular periodic R.V.E. As stated in [13], homogenization techniques combined with limit analysis can be applied for an estimation of the homogenized strength domain S^{hom} of masonry.

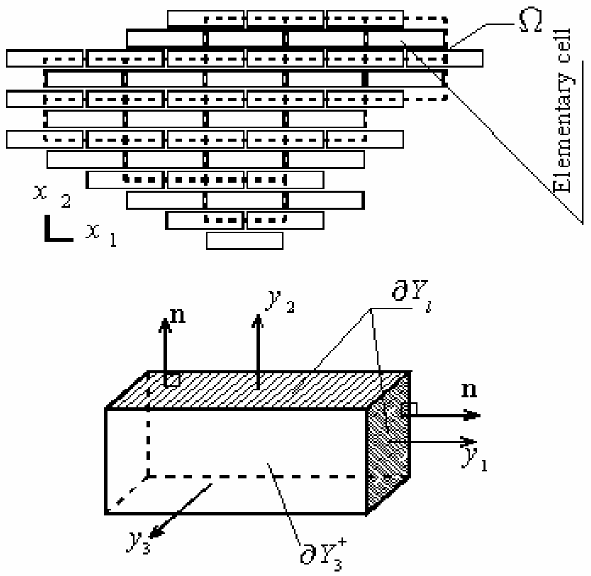


Fig. 2: Homogenized elementary cell for a running bond texture and meaning of outward versor $\mathbf{n}$.

In this framework, bricks and mortar are assumed rigid-perfectly plastic materials with associated flow rule. As the lower bound theorem of limit analysis states and under the hypotheses of homogenization, S^{hom} can be derived by means of the following (non-linear) optimization problem:

$$S^{\text{hom}} = \left\{ \Sigma \mid \begin{cases} \Sigma = <\boldsymbol{\sigma}> = \frac{1}{A}\int_{\omega} \boldsymbol{\sigma} d\omega & (a) \\ div\boldsymbol{\sigma} = \mathbf{0} & (b) \\ [[\boldsymbol{\sigma}]]\mathbf{n}^{\text{int}} = \mathbf{0} & (c) \\ \boldsymbol{\sigma}\mathbf{n} \quad \text{anti-periodic} \quad \text{on} \quad \partial Y_l & (d) \\ \boldsymbol{\sigma}(\mathbf{y}) \in S^m \quad \forall \mathbf{y} \in Y^m \text{ ; } \boldsymbol{\sigma}(\mathbf{y}) \in S^b \quad \forall \mathbf{y} \in Y^b & (e) \end{cases} \right\} \qquad (4)$$

where $[[\boldsymbol{\sigma}]]$ is the jump of micro-stresses across any discontinuity surface of normal $\mathbf{n}^{\text{int}}$. Conditions (a) and (d) are derived from periodicity, condition (b) imposes the micro-equilibrium and condition (e) represents the yield criteria for the components (brick and mortar). The averaged quantity representing the macroscopic stress tensors $\boldsymbol{\Sigma}$ is given by $\boldsymbol{\Sigma} = <\boldsymbol{\sigma}> = \frac{1}{A}\int_{\omega} \boldsymbol{\sigma} d\omega$, where A stands for the area of the elementary cell on $y_3 = 0$, $\boldsymbol{\sigma}$ stand for the local stress quantity and <*> is the averaging operator.

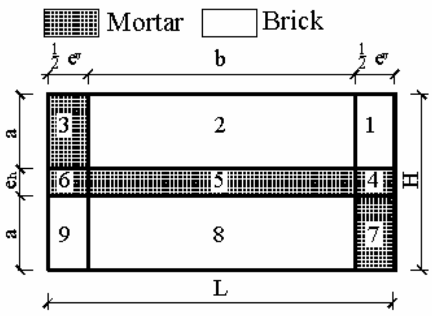


Fig. 3: Adopted division in sub-domains, subdivision and geometrical characteristics of one-fourth of the elementary cell.

The proposed solution approach involves a simple and numerically suitable model for solving the optimization problem. As shown in Fig. 3, one-fourth of the R.V.E. is sub-divided into nine geometrical elementary entities (sub-domains), so that all the cell is sub-divided into thirty-six sub-domains. The subdivision adopted is the coarser (for ¼ of the cell) that can be obtained using rectangular geometries for every sub-domain. The macroscopic behavior of masonry strongly depends on the mechanical and geometrical characteristics of both units and vertical/horizontal joints. For this reason, the subdivision adopted seems to be also particularly attractive, giving the possibility to characterize separately every component inside the elementary cell. For each sub-domain, polynomial distributions of degree m are a priori assumed for the stress components. Since stresses are polynomial expressions, the generic ij^{th} component can be written as follows:

$$\sigma_{ij}^{(k)} = \mathbf{X}(\mathbf{y})\mathbf{S}_{ij}^{T} \quad \mathbf{y} \in Y^{k} \tag{5}$$

Where $\mathbf{X}(\mathbf{y}) = [1 \quad y_1 \quad y_2 \quad y_1^2 \quad y_1 y_2 \quad y_2^2 \quad \ldots]$ and $\mathbf{S}_{ij} = [S_{ij}^{(1)} \quad S_{ij}^{(2)} \quad S_{ij}^{(3)} \quad S_{ij}^{(4)} \quad S_{ij}^{(5)} \quad S_{ij}^{(6)} \quad \ldots]$ is a vector of length ($\tilde{N}$) ($\tilde{N} = \frac{(m+1)(m+2)}{2}$) representing the unknown stress parameters (Y^k represents the k^{th} sub-domain).

The polynomial degree can be increased ad libitum, but at least a cubic interpolation is recommended for a reliable estimation of masonry homogenized failure surfaces. Details on equilibrium and anti-periodicity conditions, and validation of the approach are reported in Milani et al. [5]. Extensions of the formulation to out-of-plane behavior and combined in- and out-of-plane loads can be found in [12] and [3] respectively.

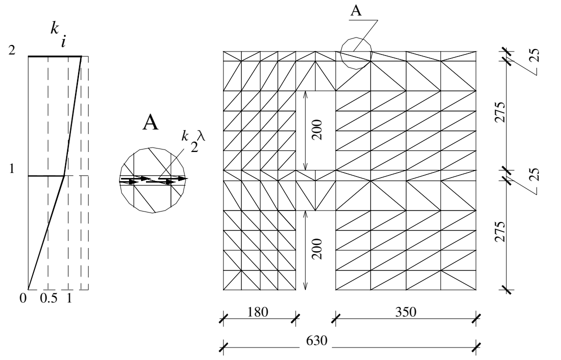
4 Structural examples

The homogenized failure surface obtained with the above approach has been coupled with finite element limit analysis and applied to an example of technical relevance. Both upper and lower bound approaches have been developed, with the aim to provide a complete set of numerical data for the design and/or the structural assessment of complex structures. The finite element lower bound analysis is based on the equilibrated triangular element by Sloan [14], while the upper bound is based on a modified version of the triangular element with discontinuities of the velocity field in the interfaces by Sloan and Kleeman [15]. The modification takes into account the actual shape of the yield surface for the homogenized material in the interfaces. Several numerical simulations have been carried out in Milani et al. [16] in order to test the accuracy of the results obtained using homogenized finite element limit analysis.

In this section, a comparison between the method proposed and a pushover analysis is presented for the two storey masonry wall shown in Fig. 4. The wall thickness is assumed to be $30\,cm$. Vertical loads (first floor 42 kN/m second floor 37 kN/m) are assumed acting in correspondence of the floors, whereas a distributed horizontal force simulating earthquake actions and depending on the λ load multiplier is applied in correspondence of the first and second level (see Fig. 4). Vertical distribution of horizontal actions is taken according to the the Italian Code requirements.

It is worth underlining that a comparison with a non-linear static pushover analysis is very attractive for a practical point of view, because the recent advent of performance based design has brought this approach to the forefront. Its importance has been already receipted by many national codes of practice, which can require, in specific cases, pushover analyses (see for details reports ATC-40 [17] and FEMA-273 [18]). Pushover analysis is a static, non-linear procedure in which the magnitude of the loads is increased in accordance with a predefined pattern. With the increase in the magnitude of the loading, weak links and failure modes of the structure are found, by means of the introduction of plastic/brittle hinges.

At present, standard pushover analyses specifically dedicated to masonry structures can be conducted by means of commercial codes.



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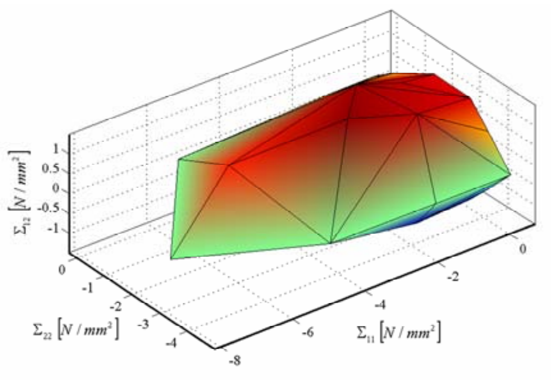

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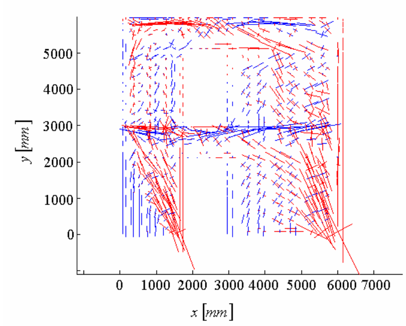
Fig. 4: -a: Two storey masonry wall, geometry (in cm), mesh utilized and seismic load applied for the lower bound analysis. –b: Homogenized masonry failure surface in the space of homogenized membrane stresses.

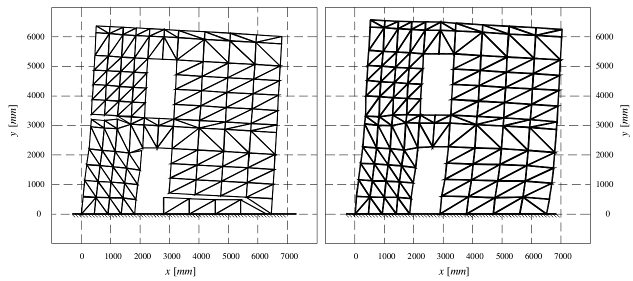
In this Section, the commercial code Aedes® is used for the comparison. In such a package, brickwork structural elements are substituted by means of the introduction of an equivalent frame, with mechanical non-linear behavior of the beams/columns very similar to that implemented in the SAM method [19]. Nevertheless, in comparison with SAM approach, the simplified assumption of perfectly plastic force-deformation criteria for hinges is adopted. Such simplification leads to treat a relatively simple and manageable incremental elastic-plastic equivalent frame.

On the other hand, for what concerns the homogenized limit analysis approach, the wall is supposed constituted only by masonry (for the sake of simplicity, no RC beams are introduced in correspondence of floors). Furthermore, mortar thickness is neglected, reducing the joints to interfaces. A frictional-type yield surface with cap in compression is chosen for joints, while for units a linear cut-off failure criterion in compression is adopted. Numerical values adopted for joints and bricks are reported in Table 1. The homogenized failure surface obtained using the micro-mechanical model proposed is shown in Fig. 4-b. Vertical loads and masonry self weight are applied in correspondence of the floors.



-a

Fig. 5: -a: Principal stress distribution at collapse, lower bound limit analysis. –b: Velocity fields at collapse from the upper bound analysis with and without jump of displacements on discontinuities between adjoining elements.



-b

Table 1: Comparison with a pushover analysis. Mechanical properties of joints and bricks. For joints, a Mohr-Coulomb failure criterion with tension cut-off and linearised cap in compression is adopted, see previous Chapter. Here, f_t is the tensile strength, f_c is the compressive strength, c is the cohesion, Φ_1 is the friction angle and Φ_2 represents the linearised shape of the compressive cap (see [16] for a detailed description of the model). For bricks f_c is the compressive strength (Rankine criterion in compression).

Joints	$c\ [N/mm^2]$	0.2
	$f_t\ [N/mm^2]$	$c/\tan(\Phi_1)$
	$f_c\ [N/mm^2]$	5
	Φ_1	37°
	Φ_2	90°
Bricks	$f_t\ [N/mm^2]$	12

In Fig. 5-a, the principal stress distribution at collapse from the lower bound analysis is reported. It is particularly evident the failure of the piers of the first-storey due to shear actions. On the other hand, in Fig. 5-b, fields of velocities at collapse from the upper bound analysis with and without discontinuities are reported. The failure mode of the structure is particularly evident in both cases.

The lower bound analysis gives a total shear at the base of $174\ kN$, whereas the upper bound approach gives a collapse load of $193.4\ kN$ and $209.1\ kN$ for the model with and without discontinuities, respectively.

The pushover analysis, in which averaged mechanical properties are assumed for masonry in order to fit as better as possible homogenized data of Fig. 4, gives a horizontal load near collapse of about $200\ kN$, in good agreement with limit analysis. Finally in Fig. 6, the total horizontal force/horizontal displacement obtained with the pushover analysis is reported. The agreement of the failure mechanism with respect to the homogenized limit analysis approach is worth noting.

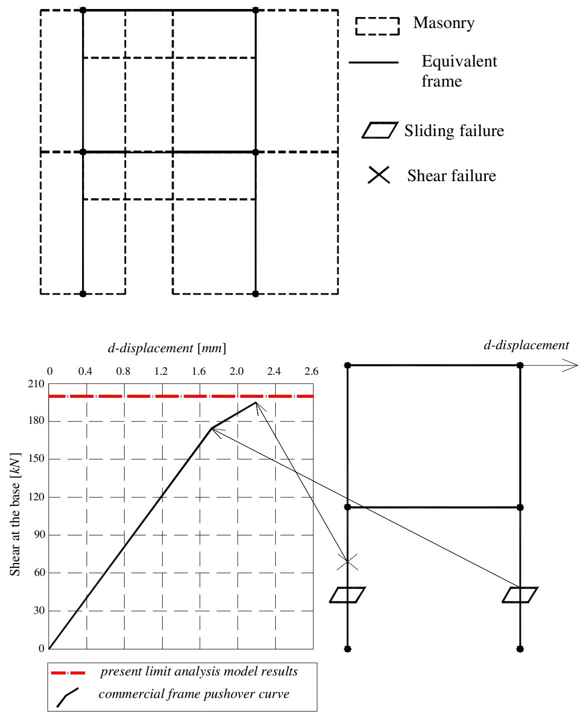


Fig. 6: Results of the pushover analysis (equivalent frame model) and comparison with the limit analysis model proposed.

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